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# Representations of Virasoro-Heisenberg Algebras and Virasoro-Toroidal Algebras

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*Abstract.* Virasoro-toroidal algebras, $\tilde{\mathcal{T}}_{[n]}$, are semi-direct products of toroidal algebras $\mathcal{T}_{[n]}$ and the Virasoro algebra. The toroidal algebras are, in turn, multi-loop versions of affine Kac-Moody algebras. Let $\Gamma$ be an extension of a simply laced lattice $\dot{Q}$ by a hyperbolic lattice of rank two. There is a Fock space $V(\Gamma)$ corresponding to $\Gamma$ with a decomposition as a complex vector space: $V(\Gamma) = \coprod_{m\in\mathbf{Z}} K(m)$. Fabbri and Moody have shown that when $m \neq 0$, $K(m)$ is an irreducible representation of $\tilde{\mathcal{T}}_{[2]}$. In this paper we produce a filtration of $\tilde{\mathcal{T}}_{[2]}$-submodules of $K(0)$. When $L$ is an arbitrary geometric lattice and $n$ is a positive integer, we construct a Virasoro-Heisenberg algebra $\tilde{\mathcal{H}}(L, n)$. Let $Q$ be an extension of $\dot{Q}$ by a degenerate rank one lattice. We determine the components of $V(\Gamma)$ that are irreducible $\tilde{\mathcal{H}}(Q, 1)$-modules and we show that the reducible components have a filtration of $\tilde{\mathcal{H}}(Q, 1)$-submodules with completely reducible quotients. Analogous results are obtained for $\tilde{\mathcal{H}}(\dot{Q}, 2)$. These results complement and extend results of Fabbri and Moody.

## 0 Introduction

Toroidal algebras, $\mathcal{T}_{[n]}$, are the universal central extensions of the iterated loop algebra $\dot{\mathcal{G}} \otimes_{\mathbf{C}} \mathbf{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ where $\dot{\mathcal{G}}$ is a simple finite-dimensional complex Lie algebra. They were introduced by R. Moody, Eswara Rao, and T. Yokonuma in [MEY]. They also produced indecomposable representations of $\mathcal{T}_{[2]}$. The results in [MEY] were extended to arbitrary $n$ in [EM]. The authors in [MEY] remark on the difficulty of producing irreducible representations of $\mathcal{T}_{[n]}$ in a natural way. It is implicit in [MEY] that the authors consider an irreducible of $\mathcal{T}_{[n]}$ to be natural if it is a direct summand of some Fock space. Let us call an irreducible representation of $\mathcal{T}_{[n]}$ *good* if a subspace of the centre of $\mathcal{T}_{[n]}$ does not act as multiplication by a scalar. See p. 284 of [MEY] for comments on good representations. Until [E1] there were no known good representations of $\mathcal{T}_{[n]}$.

Starting with tensor products of highest weight modules, Eswara Rao constructs in [E1] a family of completely reducible representations of $\mathcal{T}_{[n]}$. He also shows that the indecomposable $\mathcal{T}_{[n]}$-modules constructed in [MEY] and [EM] admit a filtration of submodules such that the successive irreducible quotient modules are isomorphic to the irreducible modules in [E1] up to an automorphism of the toroidal algebra. Note that $\tilde{\mathcal{T}}_{[n]}$ in [E1] is $\mathcal{T}_{[n]} \oplus D$ where $D$ is the linear span of $n$ derivations on $\mathcal{T}_{[n]}$ and so is entirely different from $\tilde{\mathcal{T}}_{[n]}$ in this paper. We refer to [E2] for comments and results on good representations of affine algebras.

A different tack is taken in [BC]. They factor out all but a finite-dimensional piece of the centre of $\mathcal{T}_{[n]}$. This enables them to establish an irreducibility criterion for Verma-type modules for the resulting algebra. Results and references on connections between toroidal

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algebras and other classes of Lie algebras, for instance P. Slodowy's GIM algebras, can also be found in [BC].

Fabbri and Moody initiated a third approach in [FM]. They enlarged the algebra $\mathcal{T}_{[2]}$ to the semi-direct algebra $\tilde{\mathcal{T}}_{[2]} = \mathrm{Vir} \propto \mathcal{T}_{[2]}$. This is the route we shall follow in this paper. We extend the toroidal algebra in two directions to obtain Virasoro-Heisenberg algebras and Virasoro-toroidal algebras. We shall be more precise after we develop the requisite notation. Here is a summary of the sections of the paper.

In Section 1 we recall the definition of the toroidal algebra, Virasoro algebra, the oscillator operators, and the generalized Heisenberg algebras. The construction of the generalized Heisenberg algebras requires three ingredients: a free $\mathbf{Z}$-module, $\mathbf{Z}^n$ of finite rank $n$, where $\mathbf{Z}$ is the ring of integers, $\mathbf{C}^n$, the $n$-dimensional complex vector space, and a geometric lattice $L$, *i.e.*; a free $\mathbf{Z}$-module of finite rank, not necessarily $n$, together with a non-trivial symmetric $\mathbf{Z}$-bilinear form. The notation $\mathcal{H}(L, n)$ for generalized Heisenberg algebras attempts to capture these ingredients. The Fock spaces crucial for this paper are obtained from the generalized Heisenberg algebras with $n = 1$. We now define the lattice $\Gamma$ that gives the most pervasive Fock space, $V(\Gamma)$.

In this paper $\dot{Q}$ will denote a lattice of type $A_m$, $D_m$ or $E_m$ with root lengths normalized to two. Let

$$Q = \dot{Q} \oplus \mathbf{Z}\delta \tag{1}$$

$$\Gamma = Q \oplus \mathbf{Z}\mu \tag{2}$$

$$\Lambda = \mathbf{Z}\delta \oplus \mathbf{Z}\mu \tag{3}$$

where $(Q \mid \delta) = (\dot{Q} \mid \mu) = (\mu \mid \mu) = 0$ and $(\delta \mid \mu) = 1$.

In Section 2 we obtain simpler expressions for the oscillator operators for the hyperbolic lattice $\Lambda$ in (3). We then obtain a family of completely reducible representations of the Virasoro algebra. The results on $V(\Lambda)$ are used in Sections 4 and 5 of the paper where we deal with reducible representations of a Virasoro-Heisenberg algebra and a Virasoro-toroidal algebra.

In Section 3 we use the algebras from Section 1 to construct the Virasoro-Heisenberg algebras, $\tilde{\mathcal{H}}(L, n)$, and the Virasoro-toroidal algebras, $\tilde{\mathcal{T}}_{[n]}$. We then show that the Fock space $V(\Gamma)$ from Section 1 are representations of $\tilde{\mathcal{T}}_{[2]}$ and $\tilde{\mathcal{H}}(L, n)$ for some restricted choices of $L$ and $n \leq 2$.

Let $\dot{Q}$ and $Q$ be the lattices in (1). We give decompositions of $V(\Gamma)$ as representations of $\tilde{\mathcal{H}}(Q, 1)$ and $\tilde{\mathcal{H}}(\dot{Q}, 2)$. In [FM] the components of $V(\Gamma)$ that afford irreducible representations of $\tilde{\mathcal{H}}(Q, 1)$ are identified. Using the irreducible representations of the Virasoro algebra from Section 2, we show in Section 4 that the reducible components have a filtration of $\tilde{\mathcal{H}}(Q, 1)$-submodules with completely reducible quotients.

In Section 4 we also identify the components of the Fock space that afford irreducible representations of $\tilde{\mathcal{H}}(\dot{Q}, 2)$. The components that are reducible as representations of $\tilde{\mathcal{H}}(\dot{Q}, 2)$ are shown in Section 5 to have a filtration of subrepresentations.

As a $\tilde{\mathcal{T}}_{[2]}$-representation the Fock space $V(\Gamma)$ decomposes as $\prod_{m \in Z} K(m)$, for some subrepresentations $K(m)$. In [FM] it is shown that $K(m)$ is an irreducible representation of $\tilde{\mathcal{T}}_{[2]}$ when $m \neq 0$. In Section 5 we show that $K(0)$ has a filtration of subrepresentations of $\tilde{\mathcal{T}}_{[2]}$.

The introduction ends with a list of the main objects of the paper. The object is defined in or near ($n$). Other objects are defined as they occur. All vector spaces are over $\mathbf{C}$, the field of complex numbers.

• $\tilde{\mathcal{T}}_{[n]}$ is the Virasoro-toroidal algebra, where $\mathcal{T}_{[n]}$ is the toroidal algebra, *i.e.*, the universal central extension of $\dot{\mathcal{G}} \otimes_{\mathbf{C}} \mathbf{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, while $\dot{\mathcal{G}}$ is a simple finite-dimensional complex Lie algebra. (4) and (58).

• $\tilde{\mathcal{H}}(L, n)$ is the Virasoro-Heisenberg algebra attached to the lattice $L$ and $\mathbf{Z}^n$, where $\mathcal{H}(L, n)$ is the corresponding generalized Heisenberg algebra. (9) and (59).

• $\tilde{\mathcal{A}}(L)$ is $\tilde{\mathcal{H}}(L, 1)$. (12) and (33).

• $S\big(\mathcal{A}(L)_-\big)$ is the symmetric algebra of $\mathcal{A}(L)_-$, where $\mathcal{A}(L)_-$ is the lower subalgebra in a triangular decomposition of $\mathcal{A}(L)$. (15) and (24).

• $V_L(\lambda) = \mathbf{C}e^{\lambda} \otimes_{\mathbf{C}} S\big(\mathcal{A}(L)_-\big)$ is a canonical representation of $\mathcal{A}(L)$, where $\lambda$ is an element in the complexification of a nondegenerate lattice containing $L$. (20).

• $V(\Gamma) = \mathbf{C}[\Gamma] \otimes_{\mathbf{C}} S\big(\mathcal{A}(\Gamma)_-\big)$ is the *full Fock space* corresponding to $\Gamma$ in (3). (38).

## 1 The Canonical Representations

We begin by recalling the construction of the toroidal algebras $\mathcal{T}_{[n]}$.

Let $A$ be any commutative $\mathbf{C}$-algebra with identity element. Let $\dot{\mathcal{G}}$ be a simple finite-dimensional complex Lie algebra. The structure of the universal covering algebra of $\dot{\mathcal{G}} \otimes_{\mathbf{C}} A$ has been determined by Kassel in [KS]. Let $\Omega_A$ be the $A$-module of differentials of $A$. Let $d\colon A \to \Omega_A$ be the differential map. Let $-\colon \Omega_A \to \Omega_A/dA$ be the canonical map.

***Theorem 1.1*** **([KS, Proposition 2.2], [MEY])** *The Lie algebra $\mathcal{G} = \dot{\mathcal{G}} \otimes_{\mathbf{C}} A \oplus \Omega_A/dA$ with $\Omega_A/dA$ central and multiplication given by*

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab + \langle x, y \rangle \overline{(da)b} \tag{4}$$

*where $\langle\, ,\, \rangle$ is the Killing form, is the universal covering algebra of $\dot{\mathcal{G}} \otimes_{\mathbf{C}} A$.*

When $A = \mathbf{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ in Theorem 1.1, the algebra $\mathcal{G}$ is the *toroidal* algebra of *rank* $n$ or the *n-toroidal* algebra. We denote it by $\mathcal{T}_{[n]}$. In this case a basis of $\Omega_A$ is $\{t_1^{r_1} t_2^{r_2} \cdots t_{i-1}^{r_{i-1}} t_i^{r_i - 1} t_{i+1}^{r_{i+1}} \cdots t_n^{r_n}\, dt_i : 1 \leq i \leq n, r = (r_1, \ldots, r_n) \in \mathbf{Z}^n\}$.

It is noted in [MEY] that the toroidal Lie algebra contains a generalized Heisenberg algebra. To introduce the latter, let $\big(L, (\,|\,)\big)$ be a geometric lattice, that is, a free $\mathbf{Z}$-module $L$ of finite rank together with a non-trivial symmetric $\mathbf{Z}$-bilinear form $(\,|\,)\colon L \times L \to \mathbf{Z}$. Let $\mathcal{L} = \mathbf{C} \otimes_{\mathbf{Z}} L$, the *complexification of* $L$. Extend $(\,|\,)$ to a symmetric bilinear form on $\mathcal{L}$ also denoted by $(\,|\,)$. We say that $L$ is *nondegenerate* if $(\,|\,)$ is nondegenerate on $\mathcal{L}$.

For each $r \in \mathbf{Z}^n \subset \mathbf{C}^n$, let $\mathcal{L}(r)$ be an isomorphic copy of $\mathcal{L}$ while $\mathbf{C}^n(r)$ is an isomorphic copy of $\mathbf{C}^n$. The isomorphism is given by $x \mapsto x(r)$. If $x \in \mathbf{C}^n$, $z_x(r)$ will denote the element $x(r)$ to distinguish it from elements of $\mathcal{L}(r)$. For $r \in \mathbf{Z}^n$, $\gamma, \gamma' \in \mathcal{L}$, $s, s' \in \mathbf{C}^n$ and $\alpha \in \mathbf{C}$ we have

$$z_s(r) + z_{s'}(r) = z_{s+s'}(r) \tag{5}$$

$$\alpha z_s(r) = z_{\alpha s}(r). \tag{6}$$

$$\gamma(r) + \gamma'(r) = (\gamma + \gamma')(r) \tag{7}$$

$$\alpha\gamma(r) = (\alpha\gamma)(r). \tag{8}$$

Now, let $\mathcal{C}_n = \bigoplus_{r\in\mathbf{Z}^n} \mathbf{C}^n(r)$, $\mathcal{D}_n = \bigoplus_{r\in\mathbf{Z}^n} \mathbf{C}z_r(r)$, where $\mathbf{C}z_r(r)$ is the one-dimensional complex vector space with basis $z_r(r)$. Let $\mathcal{Z}_n = \mathcal{C}_n/\mathcal{D}_n$. Consider the $\mathbf{C}$-space

$$\mathcal{H}(L,n) = \Big(\bigoplus_{r\in\mathbf{Z}^n} \mathcal{L}(r)\Big) \oplus \mathcal{Z}_n. \tag{9}$$

Introduce a bracket operation on $\mathcal{H}(L,n)$ as follows

$$[\gamma(r_1), \eta(r_2)] = (\gamma \mid \eta) z_{r_1}(r_1 + r_2) \tag{10}$$

$$\mathcal{Z}_n \text{ central.} \tag{11}$$

By (10) and (11), $\mathcal{H}(L,n)$ is a two-step nilpotent algebra and hence the multiplication satisfies the Jacobi identity. From (5), (6), (10), and (11) we deduce that $\mathcal{H}(L,n)$ is a Lie algebra. We call it the *generalized Heisenberg algebra associated to $L$ and $n$.*

The proofs of the next two propositions rely on (5) to (11). Denote vector space dimension by dim.

***Proposition 1.2***

*(a)*

$$\dim \mathcal{Z}_n = \begin{cases} 1 & \text{if } n = 1 \\ \infty & \text{if } n \geq 2. \end{cases}$$

*(b) Let $n = 2$. Then the collection of elements $\{z_{(0,1)}(m,0), z_{(1,0)}(0,0) : m \in \mathbf{Z}\} \cup \{z_{(1,0)}(m,n) : m \in \mathbf{Z}, n \in \mathbf{Z} \setminus \{0\}\}$ is a basis for $\mathcal{Z}_2$ over $\mathbf{C}$.*

***Proposition 1.3*** *The centre of $\mathcal{H}(L,n)$ is $\mathcal{L}(0) \oplus \mathcal{Z}_n \oplus \big(\bigoplus_{r\in\mathbf{Z}^n\setminus\{0\}\gamma\in \mathrm{rad}(\,|\,)} \mathbf{C}\gamma(r)\big)$, where* rad *is radical.*

Proposition 1.4 gives a realisation of $\mathcal{H}(L,n)$ when $L$ is the root lattice of a simple finite-dimensional complex Lie algebra, see Section 3 of [MEY].

***Proposition 1.4*** *Let $\dot{\mathcal{G}}$ be a simple finite-dimensional Lie algebra with root lattice $\dot{Q}$. Let $\dot{\mathcal{H}}$ be a fixed Cartan subalgebra of $\dot{\mathcal{G}}$. Let $\mathcal{X}$ be the subalgebra of $\mathcal{T}_{[n]}$ generated by the subspace $\dot{\mathcal{H}} \otimes_{\mathbf{C}} \mathbf{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Then $\mathcal{H}(\dot{Q}, n)$ and $\mathcal{X}$ are isomorphic Lie algebras.*

The Heisenberg algebra $\mathcal{H}(L,1)$ is the linchpin of most of the representations in this paper. We use the following simpler notation for it.

$$\mathcal{H}(L,1) = \mathcal{A}(L). \tag{12}$$

By Proposition 1.2(a), $\mathcal{Z}_1$ is one-dimensional. Let $c$ denote a fixed generator of $\mathcal{Z}_1$. Then $\mathcal{A}(L) = \big(\bigoplus_{k\in\mathbf{Z}} \mathcal{L}(k)\big) \oplus \mathbf{C}c$. In $\mathcal{A}(L)$ Equations (10) and (11) assume the more familiar form

$$[a(k_1), b(k_2)] = k_1 \delta_{k_1+k_2,0} (a \mid b) c \tag{13}$$

$$c \text{ central} \tag{14}$$

where $\delta$ denotes Kronecker delta.

Observe that $\mathcal{L}(0)$ is an abelian subalgebra of $\mathcal{A}(L)$. It has a complement $= \big(\bigoplus_{n\in\mathbf{Z}\setminus\{0\}} \mathcal{L}(n)\big) \oplus \mathbf{C}c$ satisfying $\mathcal{A}(L) = \Big(\big(\bigoplus_{n\in\mathbf{Z}\setminus\{0\}} \mathcal{L}(n)\big) \oplus \mathbf{C}c\Big) \times \mathcal{L}(0)$ where $\times$ denotes the direct product of Lie algebras. We shall construct a canonical representation of $\mathcal{A}(L)$ by first defining a representation of the subalgebra $\big(\bigoplus_{n\in\mathbf{Z}\setminus\{0\}} \mathcal{L}(n)\big) \oplus \mathbf{C}c$. Let

$$\mathcal{A}(L)_- = \coprod_{n>0} \mathcal{L}(-n) \tag{15}$$

with corresponding symmetric algebra $S\big(\mathcal{A}(L)_-\big)$. We may think of $S\big(\mathcal{A}(L)_-\big)$ as the polynomial ring in the indeterminates $\{a_i(-n) : 1 \le i \le m, n > 0\}$, where $\{a_i\}_{i=1}^m$ is an orthonormal basis of $\mathcal{L} = \mathbf{C} \otimes_{\mathbf{Z}} L$. By replacing $n > 0$ with $n < 0$ in (15) we get $\mathcal{A}(L)_+$ with corresponding symmetric algebra $S\big(\mathcal{A}(L)_+\big)$.

Let $a, b \in \mathcal{L}$. Let $m, n$ be positive integers. Denote by $\partial_{a(n)}$ the unique derivation of $S\big(\mathcal{A}(L)_-\big)$ satisfying

$$\partial_{a(n)}\big(b(-m)\big) = n\delta_{n,m}(a \mid b) \tag{16}$$

where $\delta_{n,m}$ is Kronecker delta. Let $l_{a(-n)}$ be the map on $S\big(\mathcal{A}(L)_-\big)$ defined by $f \mapsto a(-n)f$, multiplication by $a(-n)$. We then get the following representation on $S\big(\mathcal{A}(L)_-\big)$ of the Lie algebra $\big(\bigoplus_{n\in\mathbf{Z}\setminus\{0\}} \mathcal{L}(n)\big) \oplus \mathbf{C}c$.

$$cf = f \tag{17}$$

$$a(-n)f = l_{a(-n)}f \tag{18}$$

$$a(n)f = \partial_{a(n)}f. \tag{19}$$

We see from (16) that the derivation $\partial_{a_i(n)}$ corresponds to the partial differentiation operator $n\frac{\partial}{\partial_{a_i(n)}}$ on $S\big(\mathcal{A}(L)_-\big)$.

Let $M$ be any nondegenerate lattice containing $L$. Let $\mathcal{M}$ be the complexification of $M$. Fix $\lambda \in \mathcal{M}$ and let $\mathbf{C}e^\lambda$ be the one-dimensional $\mathbf{C}$-space. Consider the $\mathbf{C}$-space

$$V_L(\lambda) = \mathbf{C}e^\lambda \otimes_{\mathbf{C}} S\big(\mathcal{A}(L)_-\big). \tag{20}$$

We make $V_L(\lambda)$ an $\mathcal{A}(L)$-module by defining

$$c(e^\lambda \otimes f) = e^\lambda \otimes f \tag{21}$$

$$a(n)(e^\lambda \otimes f) = e^\lambda \otimes a(n)f, \quad n \neq 0 \tag{22}$$

$$a(0)(e^\lambda \otimes f) = (a \mid \lambda)e^\lambda \otimes f. \tag{23}$$

where $a(n)f, n \neq 0$ is given by (18) and (19). As in Section 2 of [KR] one proves the next proposition.

***Proposition 1.5*** *$V_L(\lambda)$ is affords a representation of $\mathcal{A}(L)$ which is irreducible if and only if $L$ is a nondegenerate lattice.*

Since by (21) and (23) $(a \mid \lambda)c - a(0)(e^{\lambda} \otimes f) = 0$, $V_L(\lambda)$ is never a faithful $\mathcal{A}(L)$-module. The module $V_L(\lambda)$ is called a *canonical* representation of $\mathcal{A}(L)$.

We shall now realise $V_L(\lambda)$ as an induced module relative to a triangular decomposition of $\mathcal{A}(L)$ in the sense of [MP2]. To that end, let $L$ be a nondegenerate geometric lattice with complexification, $\mathcal{L}$. Define $\mathcal{A}(L)_- = \coprod_{n>0} \mathcal{L}(-n)$, $\mathcal{A}(L)_+ = \coprod_{n>0} \mathcal{L}(n)$, and $\mathcal{A}(L)_0 = \mathcal{L}(0) \oplus \mathbf{C}c$. Then we have

$$\mathcal{A}(L) = \mathcal{A}(L)_- \oplus \mathcal{A}(L)_0 \oplus \mathcal{A}(L)_+. \tag{24}$$

Next let $\sigma\colon \mathcal{A}(L) \to \mathcal{A}(L)$ be the unique linear map satisfying $\sigma\big(a(n)\big) = a(-n)$, $a \in \mathcal{L}$, $n \in \mathbf{Z}$, and $\sigma(c) = c$. Then $\sigma$ fixes $\mathcal{A}(L)_0$, and interchanges $\mathcal{A}(L)_+$ and $\mathcal{A}(L)_-$. So $\sigma$ is an involution. This makes (24) a triangular decomposition of $\mathcal{A}(L)$ in the sense of [MP2].

Let $\alpha$ be a linear functional on $\mathcal{A}(L)_0$ and consider the one-dimensional vector space $\mathbf{C}v_+$. Let $\mathcal{B} = \mathcal{A}(L)_0 \oplus \mathcal{A}(L)_+$. We make $\mathbf{C}v_+$ into a $\mathcal{B}$-module by setting

$$\mathcal{A}(L)_+ v_+ = 0 \tag{25}$$

$$\mathcal{A}(L)_0 v_+ = \alpha\big(a(0)\big) v_+ \tag{26}$$

$$c v_+ = v_+. \tag{27}$$

Finally, we define the induced $\mathcal{A}(L)$-module $M(\alpha) = \mathcal{U}\big(\mathcal{A}(L)\big) \otimes_{\mathcal{U}(\mathcal{B})} \mathbf{C}v_+$ where $\mathcal{U}(X)$ denotes the universal enveloping algebra of the Lie algebra $X$. Let $\lambda \in \mathcal{L}$ and let $\alpha$ be the linear functional on $\mathcal{A}(L)_0$ defined by

$$\alpha\big(a(0)\big) = (\lambda \mid a) \tag{28}$$

$$\alpha(c) = 1. \tag{29}$$

The map $e^{\lambda} \otimes u \mapsto u \otimes v_+$ gives the the isomorphism of the next proposition.

***Proposition 1.6*** *Let $\alpha$ be the linear functional in (28) and (29). Then $M(\alpha)$ and $V_L(\lambda)$ are isomorphic as $\mathcal{A}(L)$-modules.*

Vir ***and its oscillator operators*** The Virasoro algebra Vir is an infinite-dimensional Lie algebra with generators $\{d_k : k \in \mathbf{Z}\}$ and bracket relations

$$[d_k, d_l] = (k - l)d_{k+l} + \frac{1}{12}\delta_{k+l,0}(k^3 - k)\zeta \tag{30}$$

where $\zeta$ is a central symbol.

Let $L$ be a geometric lattice of rank $m$. Define a representation of Vir on $\mathcal{A}(L)$ as follows. For every $k \in \mathbf{Z}$, let

$$d_k\big(a(n)\big) = -na(n + k) \tag{31}$$

$$d_k(c) = 0 = \zeta\big(\mathcal{A}(L)\big). \tag{32}$$

One checks that $(\zeta d_k - d_k\zeta)\big(a(n)\big) = 0 = [d_k, \zeta]\big(a(n)\big)$ and $[d_k, d_l]\big(a(n)\big) = (d_k d_l - d_l d_k)\big(a(n)\big)$. This means that $\mathcal{A}(L)$ affords a representation of Vir. This representation is a special case of a class of well-known representations of Vir. It is a direct sum of $m$ copies of $V_{0,0}$ in the notation of Proposition 1.1 of [KR]. See also [Z]. We now construct a new Lie algebra, $\tilde{\mathcal{A}}(L)$, from this representation. As a $\mathbf{C}$-space,

$$\tilde{\mathcal{A}}(L) = \mathrm{Vir} \oplus \mathcal{A}(L). \tag{33}$$

We use (31) and (32) to make $\tilde{\mathcal{A}}(L)$ a Lie algebra. For instance,

$$[d_k, a(n)] = d_k\big(a(n)\big) = -na(n+k). \tag{34}$$

With $Q$ as the lattice in (1), let $\varepsilon\colon Q \times Q \to \{\pm 1\}$ be a bimultiplicative map satisfying, for $\alpha, \beta \in Q$,

$$\varepsilon(\alpha, \alpha) = (-1)^{(\alpha|\alpha)/2} \tag{35}$$

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)} \tag{36}$$

$$\varepsilon(\alpha, \delta) = 1. \tag{37}$$

Extend $\varepsilon$ to a bimultiplicative map $\varepsilon\colon Q \times \Gamma \to \{\pm 1\}$. For $\gamma \in \Gamma$, let $e^\gamma$ be a symbol. Let $\mathbf{C}[\Gamma]$ be the complex vector space with $\mathbf{C}$-basis $\{e^\gamma : \gamma \in \Gamma\}$. Then $\mathbf{C}[\Gamma]$ contains the subspace $\mathbf{C}[Q] = \coprod_{\gamma \in Q} \mathbf{C}e^\gamma$. We equip $\mathbf{C}[Q]$, as in [BO] and [MEY], with a twisted group algebra structure by defining $e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta}$, $\alpha, \beta \in Q$. Then $\mathbf{C}[\Gamma]$ becomes a $\mathbf{C}[Q]$-module in such a way that $e^\alpha e^\gamma = \varepsilon(\alpha, \gamma)e^{\alpha+\gamma}$, $\alpha \in Q$, $\gamma \in \Gamma$. Here now is the *full Fock space* associated to $\Gamma$.

$$V(\Gamma) = \mathbf{C}[\Gamma] \otimes_{\mathbf{C}} S\big(\mathcal{A}(\Gamma)_-\big). \tag{38}$$

As $\mathbf{C}$-spaces, $V(\Gamma) = \coprod_{\lambda \in \Gamma} \mathbf{C}e^\lambda \otimes_{\mathbf{C}} S\big(\mathcal{A}(\Gamma)_-\big) = \coprod_{\lambda \in \Gamma} V_\Gamma(\lambda)$, where $V_\Gamma(\lambda)$ is a canonical representation of $\mathcal{A}(\Gamma)$.

By Proposition 1.5, $V_\Gamma(\lambda)$ affords a representation of $\mathcal{A}(\Gamma)$. Componentwise action makes $V(\Gamma)$ an $\mathcal{A}(\Gamma)$-module. Since $Q \subset \Gamma$, $V_\Gamma(\lambda)$ also affords a representation of $\mathcal{A}(Q)$. Hence we have:

***Proposition 1.7*** *$V(\Gamma) = \mathbf{C}[\Gamma] \otimes_{\mathbf{C}} S\big(\mathcal{A}(\Gamma)_-\big)$ affords a representation of $\mathcal{A}(\Gamma)$, hence of $\mathcal{A}(Q)$.*

In order to make $V(\Gamma)$ a representation of the algebras in Section 3 we recall the oscillator representation of Vir.

Let $L$ be an arbitrary non-degenerate geometric lattice of rank $m$ with complexification $\mathcal{L} = \mathbf{C} \otimes_{\mathbf{Z}} L$. Let $\{a_i\}_{i=1}^m$ be an orthonormal basis for $\mathcal{L}$ over $\mathbf{C}$. We want to define a representation of Vir on $V_L(\lambda)$. For $r, s \in \mathbf{Z}$ we define a *normal ordering* $:\quad:$ of $a_i(r)a_i(s)$, as in [KR], by

$$:a_i(r)a_i(s): = a_i(r)a_i(s) \quad \text{if } r \le s \tag{39}$$

$$:a_i(r)a_i(s): = a_i(s)a_i(r) \quad \text{if } r > s. \tag{40}$$

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Now for $k \in \mathbf{Z}$ consider the infinite quadratic expression, $L_k$, defined as follows

(41) $$L_k = \frac{1}{2}\sum_{j\in\mathbf{Z}}\sum_{i=1}^{m} : a_i(-j)a_i(j+k) :.$$

Due to the normal ordering each $L_k$ is an operator of $V_L(\lambda)$ using (21) to (23). The operator $L_k$ is called a *Virasoro operator* or *oscillator operator*. A proof of Proposition 1.8 can be obtained along similar lines as the proof of Proposition 2.3 of [KR]. The following formula is obtained along the way, see Lemma 2.2 of [KR].

(42) $$[L_k, a(n)] = -na(n+k)$$

where $k$ and $n$ are integers and $a$ is an arbitrary element of $\mathcal{L}$.

***Proposition 1.8*** *The assignment $d_k \mapsto L_k$, $\zeta \mapsto mI$, where $m$ is the rank of $L$ and $I$ is the identity operator, gives a representation of* Vir *on $V_L(\lambda)$.*

## 2 Oscillator Representations of Vir Over $\Lambda$

In order to facilitate the computations we shall need notations specific to the hyperbolic lattice $\Lambda$ in (3). Recalling (15), let

(43) $$\mathcal{S} = S\big(\mathcal{A}(\Lambda)_-\big).$$

The set $\{\alpha_1, \alpha_2\}$, where $\alpha_1 = \frac{\delta}{2} + \mu$ and $\alpha_2 = i(\frac{\delta}{2} - \mu)$, $i^2 = -1$, is an orthonormal basis for $\mathbf{C} \otimes_{\mathbf{Z}} \Lambda$. We use the notation $H_k$, $k \in \mathbf{Z}$, for the corresponding oscillator operators. So (41) becomes

(44) $$H_k = \frac{1}{2}\sum_{j\in\mathbf{Z}} : \alpha_1(-j)\alpha_1(j+k) : + : \alpha_2(-j)\alpha_2(j+k) :$$

***Proposition 2.1*** *For every $n \in \mathbf{Z}$ we have that*

*(i)* $H_n = \frac{1}{2}\sum_{j\in\mathbf{Z}}\big( : \mu(-j)\delta(j+n) : + : \delta(-j)\mu(j+n) : \big)$

*(ii)* $H_n = H_n^- + H_n^+$ *where*

$$H_n^- = \frac{\epsilon}{2}\mu(n/2)\delta(n/2) + \sum_{j>-n/2}\mu(-j)\delta(j+n),$$

$$H_n^+ = \frac{\epsilon}{2}\delta(n/2)\mu(n/2) + \sum_{j>-n/2}\delta(-j)\mu(j+n),$$

*where*

$$\epsilon = \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases}$$

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**Proof** For any $j \in \mathbf{Z}$, $:\alpha_1(-j)\alpha_1(j+n):+:\alpha_2(-j)\alpha_2(j+n): = \alpha_1(-j)\alpha_1(j+n)+\alpha_2(-j)\alpha_2(j+n)$ if $-j \le j+n$ or $\alpha_1(j+n)\alpha_1(-j)+\alpha_2(j+n)\alpha_2(-j)$ if $-j > j+n$. Thus for (i), it suffices to show that $\alpha_1(-j)\alpha_1(j+n)+\alpha_2(-j)\alpha_2(j+n) = \mu(-j)\delta(j+n)+\delta(-j)\mu(j+n)$ and $\alpha_1(j+n)\alpha_1(-j)+\alpha_2(j+n)\alpha_2(-j) = \mu(j+n)\delta(-j)+\delta(j+n)\mu(-j)$. We show only the first since the second is similar. Since $(a+b)(n) = a(n)+b(n)$, we have $\alpha_1(-j)\alpha_1(j+n)+\alpha_2(-j)\alpha_2(j+n) = \big(\frac{\delta(-j)}{2}+\mu(-j)\big)\big(\frac{\delta(j+n)}{2}+\mu(j+n)\big) - \big(\frac{\delta(-j)}{2}-\mu(-j)\big)\big(\frac{\delta(j+n)}{2}-\mu(j+n)\big) = \frac{1}{4}\delta(-j)\delta(j+n)+\frac{1}{2}\delta(-j)\mu(j+n)+\frac{1}{2}\mu(-j)\delta(j+n)+\mu(-j)\mu(j+n) - \frac{1}{4}\delta(-j)\delta(j+n)+\frac{1}{2}\delta(-j)\mu(j+n)+\frac{1}{2}\mu(-j)\delta(j+n)-\mu(-j)\mu(j+n) = \delta(-j)\mu(j+n)+\mu(-j)\delta(j+n)$. This proves (i).

For (ii), we first use (i) and then use the definition of normal ordering. Hence $H_n = \frac{1}{2}\sum_{-j\le j+n}\big(\mu(-j)\delta(j+n)+\delta(-j)\mu(j+n)\big)+\frac{1}{2}\sum_{-j>j+n}\big(\delta(j+n)\mu(-j)+\mu(j+n)\delta(-j)\big) = \frac{1}{2}\sum_{j>-n/2}\big(\mu(-j)\delta(j+n)+\delta(-j)\mu(j+n)\big)+\frac{1}{2}\sum_{j>-n/2}\big(\delta(-j)\mu(j+n)+\mu(-j)\delta(j+n)\big)+\frac{\epsilon}{2}\big(\mu(n/2)\delta(n/2)+\delta(n/2)\mu(n/2)\big)$, where we have split the first sum into $j=-n/2$, $j>-n/2$ and replaced $j$ by $-j-n$ in the second sum. Regrouping we have $H_n = \sum_{j>-n/2}\mu(-j)\delta(j+n)+\frac{\epsilon}{2}\mu(n/2)\delta(n/2)+\sum_{j>-n/2}\delta(-j)\mu(j+n)+\frac{\epsilon}{2}\delta(n/2)\mu(n/2)$. ∎

If we replace $-j$ by $i$ and $j+n$ by $j$ then we get the following alternative way of expressing $H_n^{\pm}$

(45) $$H_n^- = \sum_{\substack{i<j\\ i+j=n}} \mu(i)\delta(j) + \frac{\epsilon}{2}\mu(n/2)\delta(n/2)$$

(46) $$H_n^+ = \sum_{\substack{i<j\\ i+j=n}} \delta(i)\mu(j) + \frac{\epsilon}{2}\delta(n/2)\mu(n/2).$$

Since $\mathbf{C}\otimes_{\mathbf{Z}}\Lambda = \mathbf{C}\delta \oplus \mathbf{C}\mu$, we have $(\mathbf{C}\otimes_{\mathbf{Z}}\Lambda)(n) = \mathbf{C}\delta(n)\oplus\mathbf{C}\mu(n)$. The algebra $\mathcal{S}$ in (43) contains the following $\mathbf{C}$-subspaces

(47) $$M = S\Big(\coprod_{n>0}\mathbf{C}\mu(-n)\Big), \quad D = S\Big(\coprod_{n>0}\mathbf{C}\delta(-n)\Big)$$

We have that $\mathcal{S} = MD$ and hence for $\lambda \in \mathbf{C}\otimes_{\mathbf{Z}}\Lambda$, we have the following canonical representation of $\mathcal{A}(\Lambda)$.

(48) $$V_\Lambda(\lambda) = \mathbf{C}e^\lambda \otimes_{\mathbf{C}} MD.$$

By Proposition 1.8, $V_\Lambda(\lambda)$ is a Vir-module via $H_n$ in Proposition 2.1. We shall now show that it has a filtration of Vir-submodules. To that end we note that $M = S\big(\coprod_{n>0}\mathbf{C}\mu(-n)\big)$ has the following $\mathbf{C}$-basis:

(49) $$\{\mu(-\mathbf{n}) : \mathbf{n}\in\mathbf{Z}_+^s, s\ge 1, n_1\le\cdots\le n_s\}\cup\{1\},$$

where $\mathbf{Z}_+$ is the set of natural numbers, $\mathbf{n} = (n_1,n_2,\ldots,n_s)$, and $\mu(-\mathbf{n}) = \mu(-n_1)\mu(-n_2)\cdots\mu(-n_s)$.

We say that $\mu(-\mathbf{n})$ has *$\mu$-length s*. By replacing $\mu$ by $\delta$ we get *$\delta$-length*. The length of the zero polynomial is taken to be $-\infty <$ the length of every nonzero polynomial.

Let $M_j = 0$, if $j < 0$, $M_0 = \mathbf{C}$. For $j > 0$, let $M_j =$ the $\mathbf{C}$-span of all monomials in $M$ of *$\mu$-length $j$*. Set $M_{\leq j} = \coprod_{k \leq j} M_k$. Then $M = \coprod_{j \geq 0} M_j$.

We use $D_j$ to denote the analogous $\mathbf{C}$-spaces with $\mu$ replaced by $\delta$. Then $D = \coprod_{j \geq 0} D_j$. With $l$ an arbitrary integer, let

$$\mathcal{S}_l = \coprod_{j=0}^{\infty} (M_{\leq j+l} D_j) \subset \mathcal{S}. \tag{50}$$

For $\lambda \in \mathbf{Z}\delta$, we let

$$V_\Lambda(\lambda)_l = \mathbf{C}e^\lambda \otimes_{\mathbf{C}} \mathcal{S}_l. \tag{51}$$

Sections 4 and 5 pivot around $V_\Lambda(\lambda)_l$ and $\mathcal{S}_l$. So we are going to develop their properties in detail. First we note that

$$M_{\leq j+l} \subseteq M_{\leq j+l+1}, \quad \mathcal{S}_l \neq \mathcal{S}, \quad \mathcal{S}_l \subseteq \mathcal{S}_{l+1}. \tag{52}$$

***Lemma 2.2***

*(a) Let $x \in \mathcal{S}_l$ and let $n$ be a positive integer. Then $\delta(-n)x \in \mathcal{S}_l$.*
*(b) Let $f \in S\big(\mathcal{A}(\dot{Q})_-\big)S\big(\mathcal{A}(\mathbf{Z}\delta)_-\big)$. Then $fx \in S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ for every $x \in S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$*
*(c) Let $(\alpha + n\delta)(m) \in S\big(\mathcal{A}(Q)_+\big)$, $\alpha \in \dot{Q}$, $f \in S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$. Then $(\alpha + n\delta)(m)f \in S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$.*

**Proof** (a) For some positive integer $t$, $x = x_0 + \cdots + x_t$, where $x_j \in M_{\leq j+l}D_j$. Then $\delta(-n)x = \sum_{j=0}^{t} \delta(-n)x_j$. Since $\delta(-n)x_j \in M_{\leq j+l}D_{j+1} \subseteq M_{\leq j+l+1}D_{j+1} \subseteq \mathcal{S}_l$ we get that $\delta(-n)x \in \mathcal{S}_l$.

(b) The ring $S\big(\mathcal{A}(\dot{Q})_-\big)S\big(\mathcal{A}(\mathbf{Z}\delta)_-\big)$ is commutative. Hence (b) follows from (a).

(c) Since $m > 0$, $(\alpha+n\delta)(m) = \alpha(m)+n\delta(m)$ acts as differentiation, see the remark after (19). The ring $S\big(\mathcal{A}(\dot{Q})_-\big)$ is closed under differentiation. So it is sufficient to show that $M_{\leq j+l}D_j$ is invariant under $\alpha(m)+n\delta(m)$. Every element in $M_{\leq j+l}D_j$ is a sum of scalar multiples of elements of the form $x = \mu(-n_1)\cdots\mu(-n_s)\delta(\mathbf{k})$ where $\delta(\mathbf{k}) = \delta(-k_1)\cdots\delta(-k_j)$, $s \leq j + l$, and $n_1, \ldots, n_s, k_1, \ldots, k_j$ are positive integers. From (16) and the line after (3) we get that $(\alpha + n\delta)(m)x = mn\sum_{t=1}^{i} \delta_{m,n_t}\mu(-1)\cdots\overline{\mu(-t)}\cdots\mu(-i)\delta(\mathbf{k})$, where overbar denotes omission. So $(\alpha + n\delta)(m)x$ is in $M_{\leq j-1+l}D_j \subseteq M_{\leq j+l}D_j$. ■

Recall the definition of $V_\Lambda(n\delta)$ and $V_\Lambda(n\delta)_l$ from (48) and (51) with $\lambda = n\delta$.

***Theorem 2.3*** *For any integers $n$ and $l$, $V_\Lambda(n\delta)_l$ is a proper* Vir*-submodule of $V_\Lambda(n\delta)$ and $V_\Lambda(n\delta)_l \subseteq V_\Lambda(n\delta)_{l+1}$.*

**Proof** Since $\mathcal{S}_l \neq \mathcal{S}$, we have that $V_\Lambda(n\delta)_l \neq V_\Lambda(n\delta)$. The inclusion follows from the definition. We now have to show that $V_\Lambda(n\delta)_l$ is closed under the action of $H_n^{\pm}$. Using (45) and (46) we need only check closure under (a) $\mu(i)\delta(j)$, $i < j$, (b) $\delta(i)\mu(j)$, $i < j$,

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(c) $\mu(n/2)\delta(n/2)$, and (d) $\delta(n/2)\mu(n/2)$. We proceed as in the proof of Lemma 2.2(c). Let $f = e^{n\delta} \otimes x$, where $x$ is as in the proof of Lemma 2.2(c). Let

$$z = \mu(i)\delta(j), \quad i < j.$$

The element $z$ acts on $f$ as outlined in (22) and (23). We shall be using (16) to (23) in the proof below. If $j > 0$ then $\delta(j)(e^{n\delta} \otimes x) = je^{n\delta} \otimes \sum_{t=1}^{s} x_t$, where $x_t = \delta_{j,n_t}\mu(-n_1)\cdots \overline{\mu(-n_t)}\cdots\mu(-n_s)\delta(\mathbf{k})$, and overbar denotes omission. Each summand is either zero or its $\mu$-length is one less than that of $x$. If $i < 0$, then the $\mu$-length of $\mu(i)x_t$ is restored to that of $x$. If $i > 0$, then the effect of $\mu(i)$ on each summand, $x_t$, is to break it into summands that are 0 or have $\delta$-length one less than the $\delta$-length of $x_t$. Either way $zx$ remains in $\mathcal{S}_l$. So $\mu(i)\delta(j)f \in V_\Lambda(n\delta)_l$.

Suppose $j < 0$. Then $\delta(j)x$ has $\delta$-length one more than that of $x$. Since $i < j$ we have that $i < 0$. In that case, the $\mu$-length of $\mu(i)\delta(j)x$ is one more than that of $x$. So $\mu(i)\delta(j)x$ is in $M_{\leq j+1+l}D_{j+1} \subseteq \mathcal{S}_l$.

Suppose $j = 0$. Since $(\delta \mid n\delta) = 0$ we get from (23) that $\mu(i)\delta(j)f = 0$. Cases (b), (c), and (d) are handled in a similar fashion. ∎

The next goal is to show that $\overline{V_\Lambda(n\delta)_l} = V_\Lambda(n\delta)_l/V_\Lambda(n\delta)_{l-1}$ is a completely reducible representation of the Virasoro algebra. Even though our representations are more complicated than those in [KR] we can still rely on Lectures 2 and 3 of [KR].

Denote the quotient $\mathcal{S}_l/\mathcal{S}_{l-1}$ by $\overline{\mathcal{S}_l}$ and $V_\Lambda(\lambda)_l/V_\Lambda(\lambda)_{l-1}$ by $\overline{V_\Lambda(\lambda)_l}$. We have that $\overline{\mathcal{S}_l} \cong \prod_{j=0}^{\infty} M_{j+l}D_j$.

***Proposition 2.4*** *Let $n$ be any integer. The* Vir*-modules $\overline{V_\Lambda(n\delta)_l}$ and $\overline{V_\Lambda(0)_l}$ are isomorphic.*

**Proof** Let $f_0 = \sum_{k=0}^{r} c_k(e^0 \otimes x_k) \in V_\Lambda(0)_l$, $c_k \in \mathbf{C}$. One checks using the method in the proof of Theorem 2.3 that $f_0 \mapsto f_{n\delta} = \sum_{k=0}^{r} c_k(e^{n\delta} \otimes x_k) \in V_\Lambda(n\delta)_l$ induces a Vir-module isomorphism between $\overline{V_\Lambda(n\delta)_l}$ and $\overline{V_\Lambda(0)_l}$. ∎

We now define a positive definite Hermitian form $\langle\,|\,\rangle$ on $V_\Lambda(\lambda)$ by extending the original $\mathbf{Z}$-bilinear form $(\,|\,)$ on $\Lambda$ to a Hermitian form on $\mathcal{S}$: for $a_i, b_i \in \{\delta, \mu\}$, let

$$\big(a_1(-n_1)\cdots a_s(-n_s) \mid b_1(-m_1)\cdots b_r(-m_r)\big) = \delta_{r,s} \sum_{\sigma\in P(r)} \prod_{k=1}^{r} n_k \delta_{n_k, m_{\sigma(k)}}(a_k \mid b_{\sigma(k)}) \tag{53}$$

where $\delta_{x,y}, x, y \in \mathbf{Z}$, denotes the usual Kronecker delta and $P(r)$ denotes the symmetric group on $r$ symbols.

We use below the notation in (49) for tuples of integers.

Let $\iota\colon \mathcal{S} \to \mathcal{S}$ be the unique anti-linear map satisfying $\iota\big(\mu(-\mathbf{n})\delta(-\mathbf{m})\big) = \mu(-\mathbf{m})\delta(-\mathbf{n})$, $\iota(1) = 1$, where $\mathbf{n} \in \mathbf{Z}_+^s$, $\mathbf{m} \in \mathbf{Z}_+^r$ $r, s \geq 1$. The map $\iota$ is an involution.

Next we define a Hermitian form on $V_\Lambda(\lambda)$ using (53). Let $x, x' \in \mathcal{S}$, $z = e^\lambda \otimes x$, $z' = e^\lambda \otimes x'$. Set $\langle z \mid z'\rangle = \big(x \mid \iota(x')\big)$.

The proof of Proposition 2.2 in [KR] works for the next proposition.

***Proposition 2.5*** *(a) The set $\{z = e^\lambda \otimes \mu(-\mathbf{n})\delta(-\mathbf{m}) : \mathbf{n} \in \mathbf{Z}_+^s, \mathbf{m} \in \mathbf{Z}_+^r, n_1 \leq n_2 \cdots \leq n_s, m_1 \leq m_2 \cdots \leq m_r\} \cup \{e^\lambda \otimes 1\}$ is an orthogonal basis of $V_\Lambda(\lambda)$ with respect to $\langle\,|\,\rangle$.*

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*(b) The form* $\langle\,|\,\rangle$ *is positive definite on* $V_\Lambda(\lambda)$ *and* $\|z\|^2 = c(\mathbf{n})c(\mathbf{m})\prod_{i=1}^{s} n_i \prod_{j=1}^{r} m_j$*, where* $\|z\|$ *is the norm of* $z$ *and* $c(\mathbf{n})$ *is the cardinality of the set* $\{\sigma \in P(s) : \sigma(\mathbf{n}) = \mathbf{n}\}$ *(replace* $s$ *by* $r$ *for the definition of* $c(\mathbf{m})$*.)*

The *degree* of $z$ in Proposition 2.5 is defined as $\sum_{i=1}^{s} n_i + \sum_{j=1}^{r} m_j$.

Let $\overline{V_\Lambda(0)_l}(j)$ denote the subspace of $\overline{V_\Lambda(0)_l}$ spanned by elements of degree $j$. This is a finite-dimensional vector space. One checks that this finite-dimensional space is the eigenspace of the eigenvalue $j$ of the oscillator operator $H_0$ in Proposition 2.1. In fact $\overline{V_\Lambda(0)_l} = \coprod_{j\geq 0} \overline{V_\Lambda(0)_l}(j)$ is a weight space decomposition of $\overline{V_\Lambda(0)_l}$ with respect to the commutative subalgebra of the Virasoro algebra generated by $d_0$ and the central element $\zeta$. The material above starting from (53) allows us to use Lectures 2 and 3 of [KR], in particular Proposition 3.1 of [KR], as a proof of the next theorem.

***Theorem 2.6*** *Let* $l$ *be any integer. Then the* Vir*-module* $\overline{V_\Lambda(0)_l}$ *is completely reducible.*

By Proposition 2.4 and Theorem 2.6 we have

***Corollary 2.7*** *For every pair of integers* $(n, l)$*, the* Vir*-module* $\overline{V_\Lambda(n\delta)_l}$ *is completely reducible.*

## 3 Virasoro-Heisenberg and Virasoro-Toroidal Algebras

It is well-known that one often gets a more satisfactory representation theory by enlarging the algebra, see for instance the introduction of [MEY]. We shall accomplish our enlargement through semi-direct products. The use of semi-direct products in the representation theory of Lie algebras can be traced back to E. Cartan's thesis. See [COL]. We now recall the essentials from the theory of vertex operators that we need and refer to [MEY], [MP1], and [FLM] for more details.

Let $z$ be a complex variable. Let $\Gamma$ and $Q$ be as in (1) and (2). Let $\alpha \in Q$. So for $n \in \mathbf{Z}$, $\alpha(n)$ is the operator on $V_\Gamma(\lambda)$ defined in (22) and (23). Define

(54)
$$T_+(\alpha, z) = -\sum_{n>0} \frac{1}{n}\alpha(n)z^{-n}.$$

(55)
$$T_-(\alpha, z) = -\sum_{n<0} \frac{1}{n}\alpha(n)z^{-n}.$$

The vertex operator, $X(\alpha, z)$, for $\alpha$ on $V(\Gamma)$ is defined by

(56)
$$X(\alpha, z) = z^{(\alpha|\alpha)/2} \exp T(\alpha, z)$$

where $\exp T(\alpha, z) = \exp T_-(\alpha, z) e^\alpha z^{\alpha(0)} \exp T_+(\alpha, z)$ and $z^{\alpha(0)}(e^\lambda \otimes f) = z^{(\alpha|\lambda)}(e^\lambda \otimes f)$, $f \in S\big(\mathcal{A}(\Gamma)_-\big)$. It is also shown in [MP1] that $X(\alpha, z)$ can be formally expanded in powers of $z$ to give $X(\alpha, z) = \sum_{n\in\mathbf{Z}} X_n(\alpha) z^{-n}$. The coefficients $X_n(\alpha)$ are called *moments* and are operators on $V(\Gamma)$.

***Proposition 3.1*** **([MP1])** *Let* $f \in S\big(\mathcal{A}(\Gamma)_-\big)$*. Suppose* $\alpha \in Q$*,* $\lambda \in \Gamma$*. Then*

(57)
$$X_n(\alpha)(e^\lambda \otimes f) = e^{\lambda+\alpha} \otimes f_1$$

*where* $f_1 \in S\big(\mathcal{A}(\Gamma)_-\big)$*.*

Thus in the decomposition of the full Fock space $V(\Gamma) = \coprod_{\lambda \in \Gamma} V_\Gamma(\lambda)$ one can view the moments as operators which move an element in the $\lambda$-stalk to an element in the $(\lambda + \alpha)$-stalk.

The next theorem summarizes the key commutation relations between the moments.

***Theorem 3.2*** **([FK] and [GO], [MP1])** *Let $\alpha$ and $\beta$ be elements of the lattice $Q$ in (1).*

*CR0* $[\alpha(k), X_n(\beta)] = (\alpha \mid \beta)X_{n+k}(\beta)$.
*CR1* *If* $(\alpha \mid \beta) \geq 0$ *then* $[X_m(\alpha), X_n(\beta)] = 0$.
*CR2* *If* $(\alpha \mid \beta) = -1$ *then* $[X_m(\alpha), X_n(\beta)] = \varepsilon(\alpha, \beta)X_{m+n}(\alpha + \beta)$.
*CR3* *If* $(\alpha \mid \alpha) = (\beta \mid \beta) = -(\alpha \mid \beta) = 2$ *then* $[X_m(\alpha), X_n(\beta)] = \varepsilon(\alpha, \beta)\{mX_{n+m}(\alpha + \beta) + \sum_{k \in \mathbf{Z}} : \alpha(k)X_{m+n-k}(\alpha + \beta) :\}$ *where* $: \alpha(k)X_{m+n-k}(\beta) : = \alpha(k)X_{m+n-k}(\beta)$ *if* $k \leq m + n - k$ *and* $X_{m+n-k}(\beta)\alpha(k)$ *if* $k > m + n - k$.
*CR4* $[L_k, X(\alpha, z)] = z^k\{\frac{k}{2}(\alpha \mid \alpha) + z\frac{d}{dz}\}X(\alpha, z)$.

Let $\{e_{\pm\alpha_i}, h_i : 1 \leq i \leq l\}$ be a Chevalley basis of $\dot{\mathcal{G}}$ in Proposition 1.4. As an addendum to Proposition 1.4 we note that $\mathcal{T}_{[2]}$ contains an affine Kac-Moody algebra $\dot{\mathcal{G}} \otimes_{\mathbf{C}} \mathbf{C}[t_1, t_1^{-1}] \oplus \mathbf{C}c$. Denote its root system by $\Delta$, its set of real roots by $\Delta^{\mathrm{re}}$ and its root lattice by $Q$ in (1). Now we can state the main result from [MEY] which gives vertex representations of $\mathcal{T}_{[2]}$.

***Theorem 3.3*** **([MEY])** *The assignment*

$e_{\alpha_i} \otimes \pm s^m t^n \mapsto X_m(\alpha_i + n\delta)$, $n, m \in \mathbf{Z}$
$-e_{-\alpha_i} \otimes \pm s^m t^n \mapsto X_m(\pm\alpha_i + n\delta)$, $n, m \in \mathbf{Z}$, $1 \leq i \leq l$
$z_{(1,0)}(m, n) \mapsto X_m(n\delta)$, $n \neq 0$
$z_{(0,1)}(m, 0) \mapsto \delta(m)$
$z_{(1,0)}(0, 0) \mapsto I$ *where $I$ is the identity map on* $V(\Gamma)$

*gives an isomorphism $\phi$ between the Lie algebra of operators $\mathcal{T}$ on $V(\Gamma)$ generated by the moments $X_m(\alpha), \alpha \in \Delta^{\mathrm{re}}, m \in \mathbf{Z}$, and the toroidal algebra $\mathcal{T}_{[2]}$.*

Let $\{a_i\}_{i=1}^l$ be an orthonormal basis for $\mathbf{C} \otimes_{\mathbf{Z}} \dot{Q}$. Let $\{\alpha_1, \alpha_2\}$ be an orthonormal basis for $\mathbf{C} \otimes_{\mathbf{Z}} \Lambda$ where $\Lambda = \mathbf{Z}\delta \oplus \mathbf{Z}\mu$. Let $u_i = a_i$, $i = 1, \ldots, l$ and $u_{l+1} = \alpha_1$ and $u_{l+2} = \alpha_2$. Then $\{u_i\}_{i=1}^{l+2}$ is an orthonormal basis over $\mathbf{C}$ for $\mathbf{C} \otimes_{\mathbf{Z}} \Gamma$. Therefore by Proposition 1.8, the oscillator operators given by this basis affords a representation of Vir on $V_\Gamma(\lambda)$ where the centre $\zeta$ acts as $(l+2)I$. So $V_\Gamma(\lambda)$ affords a representation of both Vir and $\mathcal{A}(Q)$. However from (42), $[L_k, a(n)] = -na(n + k)$. Therefore, by (34) we have proved that $V_\Gamma(\lambda)$ is an $\tilde{\mathcal{A}}(\Gamma)$-module and consequently we have the next proposition.

***Proposition 3.4*** $V(\Gamma) = \coprod_{\lambda \in \Gamma} V_\Gamma(\lambda)$ *is an $\tilde{\mathcal{A}}(\Gamma)$-module decomposition.*

Let $\tilde{\mathcal{T}}_{[2]}$ be the Lie algebra of operators on $V(\Gamma)$ generated by $X_m(\alpha)$ and $L_k$ where $m, k \in \mathbf{Z}$ and $\alpha \in \Delta^{\mathrm{re}} \subset Q$.

***Proposition 3.5*** **([FM, Section 4])** *$\tilde{\mathcal{T}}_{[2]}$ is the semi-direct product of* Vir *and* $\mathcal{T}_{[2]}$.

**Proof** As $\mathbf{C}$-spaces,

$$\tilde{\mathcal{T}}_{[2]} = \mathrm{Vir} \oplus \mathcal{T}_{[2]} \tag{58}$$

after using Theorem 3.3. A Lie algebra is a semi-direct product $A \propto B$ if $A$ is a subalgebra and $B$ is an ideal. We will show that $\mathcal{T}_{[2]}$ is an ideal in (58). It suffices to show that $[L_k, X_m(\alpha)] \in \mathcal{T}_{[2]}$, where $k, m \in \mathbf{Z}$, and $\alpha \in \Delta^{\mathrm{re}}$. Indeed, by CR4, $\sum_{m\in\mathbf{Z}}[L_k, X_m(\alpha)]z^{-m} = [L_k, X(\alpha, z)] = z^k\{\frac{k}{2}(\alpha, \alpha) + z\frac{d}{dz}\}X(\alpha, z) = \sum_{n\in\mathbf{Z}} z^k\{\frac{k}{2}(\alpha, \alpha) + z\frac{d}{dz}\}X_n(\alpha)z^{-n} = \sum_{n\in\mathbf{Z}}\big(\frac{k}{2}(\alpha, \alpha) - n\big)X_n(\alpha)z^{-n+k}$. Now replacing $n$ by $m + k$ and equating coefficients of $z^{-m}$, we get

$$[L_k, X_m(\alpha)] = \Big\{\frac{k}{2}(\alpha \mid \alpha) - (m + k)\Big\}X_{m+k}(\alpha) \tag{59}$$

which is in $\mathcal{T}_{[2]}$. ■

The Lie algebra $\tilde{\mathcal{T}}_{[2]}$ is called the *Virasoro-toroidal algebra* of *rank two*.

A generalization of this situation occurs when we replace $\Gamma$ by $\dot{Q} \perp \Lambda_{n-1}$ with $\{\delta_1, \dots, \delta_{n-1}, \mu_1, \dots, \mu_{n-1}\}$ a basis for $\Lambda_{n-1}$ and $(\delta_i \mid \dot{Q}) = (\mu_j \mid \dot{Q}) = (\delta_i \mid \delta_j) = (\mu_i \mid \mu_j) = 0$ and $(\delta_i \mid \mu_j) = \delta_{i,j}$ ($\delta_{i,j}$ is Kronecker delta) for all pairs $(i, j)$.

Using this new lattice an analogue of Theorem 3.3 is proved in Theorem 3.14 of [EM] for an arbitrary positive integer $n$. We can now define the Virasoro-toroidal algebra, $\tilde{\mathcal{T}}_{[n]}$, of rank $n$ for an arbitrary positive integer $n$, as the algebra of operators on $V(\dot{Q} \perp \Lambda_{n-1})$ generated by the moments $X_m(\alpha + \delta)$ in Theorem 3.14 of [EM] and the Virasoro operators on $V(\dot{Q} \perp \Lambda_n)$. The subalgebra of $\tilde{\mathcal{T}}_{[n]}$ generated by the Virasoro operators $L_k$ on $V(\dot{Q} \perp \Lambda_n)$ and the subalgebra $\mathcal{X} \subset \mathcal{T}_{[n]}$ in Proposition 1.4 is the Virasoro-Heisenberg algebra $\tilde{\mathcal{H}}(\dot{Q}, n)$.

We can be explicit when $n = 2$. To that end we use the basis of $\mathcal{Z}_2$ given in Proposition 1.2. We let generators of Vir act on $\mathcal{H}(\dot{Q}, 2)$ as follows:

$$d_k\big(\gamma(m, n)\big) = -m\gamma(m + k, n) \tag{60}$$

$$d_k\big(z_{(1,0)}(m, n)\big) = -(m + k)z_{(1,0)}(m + k, n), \quad n \neq 0 \tag{61}$$

$$d_k\big(z_{(0,1)}(m, 0)\big) = -mz_{(0,1)}(m + k, 0) \tag{62}$$

$$d_k\big(z_{(1,0)}(0, 0)\big) = 0 \tag{63}$$

$$\zeta\big(\mathcal{H}(\dot{Q}, 2)\big) = 0. \tag{64}$$

We use the above equations to get the Lie bracket in $\tilde{\mathcal{H}}(\dot{Q}, 2)$. So, for instance, $[d_k, \gamma(m, n)] = -m\gamma(m + k, n)$.

Let $V(\lambda) = e^{\lambda+Q} \otimes_{\mathbf{C}} S\big(\mathcal{A}(\Gamma)_-\big)$, where $\lambda \in \Gamma$. (This is not to be confused with $V_L(\lambda)$ in (20).) To see how moments act on $V(\lambda)$, set $\tau = X_{-N}(\delta)$ where $N = (\lambda \mid \delta)$.

***Proposition 3.6*** **([MEY, Proposition 5.3])** *Let $k, m \in \mathbf{Z}$.*

*(a) The operator $X_{-kN}(k\delta)$ acts on $V(\lambda)$ as multiplication by $\varepsilon(\delta, \lambda)^k e^{k\delta}$. In particular $\tau$ acts as multiplication by $\varepsilon(\delta, \lambda)e^{\delta}$ and $X_{-kN}(k\delta)$ acts as $\tau^k$ on $V(\lambda)$.*

*(b) $X_m(k\delta)$ annihilates $V(\lambda)$ if and only if $m + kN > 0$.*

For $\gamma \in \mathbf{C} \otimes_{\mathbf{Z}} \dot{Q}$, $m, n \in \mathbf{Z}$, define

$$T^{\gamma}_m(n\delta) = \sum_{k\in\mathbf{Z}} : \gamma(k)X_{-k+m}(n\delta) : \tag{65}$$

where the normal ordering is defined as in CR3 of Theorem 3.2. It follows from Proposition 3.6 and (16) that only finitely many terms of the infinite sum act non-trivially on any fixed $v \in V(\Gamma)$. We note that $T^{\gamma}_m(n\delta)$ is linear in its superscript.

In the next computation we use Theorem 3.3 with $e_{\alpha_i}$ denoted by $e_i$, the CR relations, and the properties of the map $\varepsilon$ in (35) to (37). We have $[e_i \otimes 1, e_{-i} \otimes s^m t^n] = -[X_0(\alpha_i + 0\delta), X_m(-\alpha_i + n\delta)] = -\varepsilon(\alpha_i, -\alpha_i) \sum_{k \in \mathbf{Z}} : \alpha_i(k) X_{-k+m}(n\delta) : = T^{\alpha_i}_m(n\delta)$. Let $\{h_1, \ldots, h_l\}$ be the basis of $\dot{\mathcal{H}}$, the Cartan subalgebra in Proposition 1.4. Let $\gamma \in \dot{\mathcal{H}}$. Then for some complex numbers $c_1, \ldots, c_l$, $\gamma = \sum_{i=1}^{l} c_i h_i$. Then in Proposition 1.4, $\gamma(m, n) \mapsto \gamma \otimes s^m t^n = \sum_{i=1}^{l} c_i (h_i \otimes s^m t^n) \in \mathcal{T}_{[2]}$. Since $T^{\gamma}_m$ is linear in its superscript, the proof of the next proposition follows from Theorem 3.3 and the above calculation.

***Proposition 3.7*** *$V(\Gamma)$ is an $\mathcal{H}(\dot{Q}, 2)$-module under the following correspondences.*

$$\gamma(m, n) \mapsto T^{\gamma}_m(n\delta) \tag{66}$$

$$z_{(1,0)}(m, n) \mapsto X_m(n\delta), n \neq 0 \tag{67}$$

$$z_{(0,1)}(m, 0) \mapsto \delta(m) \tag{68}$$

$$z_{(1,0)}(0, 0) \mapsto I \tag{69}$$

*where $I$ is the identity operator on $V(\Gamma)$ and $\delta(m)$ acts on $V(\Gamma)$ as specified in (22) and (23).*

We extend the representation of $\mathcal{H}(\dot{Q}, 2)$ in Proposition 3.7 to a representation of $\tilde{\mathcal{H}}(\dot{Q}, 2)$ on $V(\Gamma)$ by letting the Virasoro generator $d_k$ act on each $V_\Gamma(\lambda)$ in Proposition 3.4 by the oscillator operator $L_k$ defined just before Proposition 3.4, and extending the action linearly. On each $V_\Gamma(\lambda)$, $\zeta$ acts as $(l+2)I$.

***Proposition 3.8*** *$V(\Gamma)$ is an $\tilde{\mathcal{H}}(\dot{Q}, 2)$-module.*

**Proof** We know from Proposition 1.8 and Proposition 3.7 that $V(\Gamma)$ is both a Vir-module and an $\mathcal{H}(\dot{Q}, 2)$-module. So we need only check that the operations from both algebras are compatible with the bracket operations in $\tilde{\mathcal{H}}(\dot{Q}, 2)$. By (42), $[L_k, \delta(m)] = -m\delta(m+k)$. By Proposition 3.7 and the remark following it $[L_k, \delta(m)]$ corresponds to $[d_k, z_{(0,1)}(m, 0)] \in \tilde{\mathcal{H}}(\dot{Q}, 2)$. By (61), $[d_k, z_{(0,1)}(m, 0)] = -m z_{(0,1)}(m + k, 0)$, as required.

Since $(n\delta \mid n\delta) = 0$, we get by (59) that $[L_k, X_m(n\delta)] = -(m + k) X_{m+k}(n\delta)$. By (61), $[d_k, z_{(1,0)}(m, n)] = -(m + k) z_{(1,0)}(m + k, n)$. By (67), the latter element gives the operator $-(m + k) X_{m+k}(n\delta)$ as required.

For the next computation, we first justify (70), which will permit us to remove $:\ :$ in (65).

$$\sum_{l \in \mathbf{Z}} : a_l a_{-l+m} : = \sum_{l \in \mathbf{Z}} a_l a_{-l+m} - \sum_{l > \frac{m}{2}} [a_l, a_{-l+m}]. \tag{70}$$

As $-l + m < l \Leftrightarrow l > \frac{m}{2}$, the left hand side of (70) is $\sum_{l \leq \frac{m}{2}} a_l a_{-l+m} + \sum_{l > \frac{m}{2}} a_{-l+m} a_l$, while the right hand side is $\sum_{l \in \mathbf{Z}} a_l a_{-l+m} - \sum_{l > \frac{m}{2}} (a_l a_{-l+m} - a_{-l+m} a_l)$. Simplifying this expression gives the above form of the left hand side of (70).

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By (65) $[L_k, T_m^\gamma(n\delta)] = [L_k, \sum_{l\in\mathbf{Z}} : \gamma(l)X_{-l+m}(n\delta) :]$. By (70) the latter is equal to $[L_k, \sum_{l\in\mathbf{Z}} \gamma(l)X_{-l+m}(n\delta)] - \sum_{l>\frac{m}{2}} \big[L_k, [\gamma(l), X_{-l+m}(n\delta)]\big] = \sum_{l\in\mathbf{Z}}[L_k, \gamma(l)X_{-l+m}(n\delta)] - \sum_{l>\frac{m}{2}}[L_k, (\gamma \mid n\delta)X_m(n\delta)]$ where the last equality follows from CRO in Theorem 3.2. But $(\gamma \mid n\delta) = 0$ because $\gamma \in Q$ in (3). Hence $[L_k, T_m^\gamma(n\delta)] = \sum_{l\in\mathbf{Z}}[L_k, \gamma(l)X_{-l+m}(n\delta)] = \sum_{l\in\mathbf{Z}}\{[L_k, \gamma(l)]X_{-l+m}(n\delta) + \gamma(l)[L_k, X_{-l+m}(n\delta)]\}$, which by (42) and (59) is equal to $\sum_{l\in\mathbf{Z}}\{-l\gamma(k+l)X_{-l+m}(n\delta) + (l-m-k)\gamma(l)X_{k+m-l}(n\delta)\} = \sum_{l\in\mathbf{Z}}\{-l\gamma(k+l)X_{-l+m}(n\delta) + (l-m)\gamma(l+k)X_{-l+m}(n\delta)\} = -m\sum_{l\in\mathbf{Z}} \gamma(k+l)X_{-l+m}(n\delta) = -mT_{m+k}^\gamma(n\delta)$.

By Proposition 3.7 the operator $[L_k, T_m^\gamma(n\delta)]$ comes from $[d_k, \gamma(m,n)]$, which by (60), is $-m\gamma(m+k,n)$. By (66) this gives the operator $-mT_{m+k}^\gamma(n\delta)$, as required. ■

For $\lambda \in \Gamma$, let $H(\lambda)$ be the $\mathbf{C}$-subspace of $V(\Gamma)$ spanned by $\mathbf{C}[\lambda + \mathbf{Z}\delta] \otimes_{\mathbf{C}} S\big(\mathcal{A}(\Gamma)_-\big)$. In multiplicative notation, $\mathbf{C}[\lambda + \mathbf{Z}\delta]$ has $\mathbf{C}$-basis $\{e^{\lambda+n\delta} : n \in \mathbf{Z}\}$. As $\mathbf{C}$-spaces,

$$V(\Gamma) = \coprod_{\lambda} H(\lambda) \tag{71}$$

where $\lambda$ ranges over a complete set of representatives of $\Gamma/\mathbf{Z}\delta$. The oscillator operator $L_k$ is a sum of compositions of the operator in (21) to (23). So it follows from Proposition 3.7 and Proposition 3.1 with $\alpha = n\delta$ that $H(\lambda)$ is an $\tilde{\mathcal{H}}(\dot{Q}, 2)$-submodule of $V(\Gamma)$. We shall study its structure in the next two sections.

## 4 Irreducible Representations

One of the main result of this section is that $H(\lambda)$ in (71) affords an irreducible representation of $\mathcal{H}(\dot{Q}, 2)$ if $\lambda \notin Q$. We begin by stating the results that we need for its proof.

We use (31) and (32) to extend the action of Vir on $\mathcal{A}(Q)$ to an action on $S\big(\mathcal{A}(Q)\big)$ so that each $d_k$ acts as a derivation and $\zeta$ acts trivially. Then for any homogeneous polynomial $f \in S\big(\mathcal{A}(Q)_-\big)$, $d_0(f) = (\deg f)f$. It is shown in (16) of [FM] that

$$d_n(e^\lambda \otimes f) = \left(\delta_{n,0}\frac{(\lambda \mid \lambda)}{2} f + d_n(f)\right)(e^\lambda \otimes 1) \tag{72}$$

where $\delta_{n,0}$ is Kronecker delta.

***Proposition 4.1*** *The set* $A = \big(\coprod_{m\in\mathbf{Z}} \mathcal{L}(m,0)\big) \oplus \big(\coprod_{m\in\mathbf{Z}} \mathbf{C}z_{(0,1)}(m,0)\big) \oplus \mathbf{C}z_{(1,0)}(0,0)$ *is a Lie-subalgebra of* $\mathcal{H}(\dot{Q}, 2)$ *isomorphic to* $\mathcal{A}(Q)$.

**Proof** We use Proposition 1.2. First, $A$ is a subalgebra of $\mathcal{H}(\dot{Q}, 2)$: For $\gamma, \eta \in \mathcal{L}$, and $m, n \in \mathbf{Z}$ we have, by (10) and (6), that $[\gamma(m,0), \eta(n,0)] = (\gamma \mid \eta)z_{(m,0)}(m+n,0) = m(\gamma \mid \eta)z_{(1,0)}(m+n,0) = m\delta_{m+n,0}(\gamma \mid \eta)z_{(1,0)}(0,0)$, where the last equality follows from the fact that if $m+n \neq 0$ then $z_{(1,0)}(m+n,0) = \frac{1}{m+n}z_{(m+n,0)}(m+n,0) \in \mathcal{D}_2$. In that case $z_{(1,0)}(m+n,0) = 0$ in $\mathcal{Z}_2 = \mathcal{C}_2/\mathcal{D}_2$. If $m+n = 0$ then $z_{(1,0)}(m+n,0) = z_{(1,0)}(0,0)$, which is in $A$. Now recall the Heisenberg algebra $\mathcal{A}(Q) = \mathcal{H}(Q,1)$ in (12) with $L = Q$. Under the correspondences $\gamma(m,0) \mapsto \gamma(m)$, $z_{(0,1)}(m,0) \mapsto \delta(m)$, $z_{(1,0)}(0,0) \mapsto c$, one checks using (10), (11), (13), and (14) that this yields an isomorphism between $A$ and $\mathcal{A}(Q)$. ■

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By Propositions 4.1 and 1.4 we have the following inclusions of Lie algebras

$$\tilde{\mathcal{A}}(Q) \subseteq \tilde{\mathcal{H}}(\dot{Q}, 2) \subseteq \tilde{\mathfrak{T}}_{[2]}.$$

Any representation of $\tilde{\mathfrak{T}}_{[2]}$ is automatically a representation of its subalgebras. This will be useful in the establishment of the irreducibility of some modules.

***Proposition 4.2*** *Let $\lambda \in \Gamma \setminus Q$. Then $H(\lambda)$ is an irreducible $\tilde{\mathcal{H}}(\dot{Q}, 2)$-module.*

**Proof** We will show that (i) $e^\lambda \otimes 1$ generates $H(\lambda)$ and (ii) every non-zero submodule of $H(\lambda)$ contains $e^\lambda \otimes 1$. First note that as a **C**-space

$H(\lambda) = \coprod_{n\in\mathbf{Z}} \mathbf{C}e^{\lambda+n\delta} \otimes_{\mathbf{C}} S\big(\mathcal{A}(\Gamma)_-\big) = \coprod_{n\in\mathbf{Z}} V_\Gamma(\lambda + n\delta)$.

Since $\lambda \in \Gamma \setminus Q$ it follows that for each $n \in \mathbf{Z}$, $\lambda + n\delta \in \Gamma \setminus Q$. Thus by Proposition 9 of [FM], $\mathbf{C}e^{\lambda+n\delta} \otimes_{\mathbf{C}} S\big(\mathcal{A}(\Gamma)_-\big)$ is an irreducible $\tilde{\mathcal{A}}(Q)$-module generated by $e^{\lambda+n\delta} \otimes 1$. To show (i) it suffices, by Proposition 4.1, to check that for every $n \in \mathbf{Z}$, $e^{\lambda+n\delta} \otimes 1$ is in the $\tilde{\mathcal{H}}(\dot{Q}, 2)$-submodule generated by $e^\lambda \otimes 1$. Indeed given $n \in \mathbf{Z}$ choose $m \in \mathbf{Z}$ so that $m + nN = 0$ where $N = (\lambda \mid \delta)$. Then by Proposition 3.6(a) we have $X_m(n\delta)(e^\lambda \otimes 1) = \pm e^{\lambda+n\delta} \otimes 1$.

Next let $R$ be a non-zero submodule of $H(\lambda)$ and let $0 \neq z = \sum_{i=1}^{s} e^{\lambda+k_i\delta} \otimes f_i \in R$, $k_i \in \mathbf{Z}$, $f_i \in S\big(\mathcal{A}(\Gamma)_-\big)$ and $s \geq 1$. We may assume that the $k_i$'s are distinct. We use (16) to differentiate out all indeterminates of the form $\mu(-n)$, $n > 0$ using $\delta(n)$ and those of the form $\alpha(-m)$, $\alpha \in \dot{Q}$, $m > 0$ using $\alpha(m)$. Thus we may assume that $f_i \in S\big(\coprod_{m>0} \mathbf{C}\delta(-m)\big)$. Now using (31) and (72) as in the proof of Proposition 7 of [FM] we can further reduce $z$ to a non-zero element $x = \sum_{i=1}^{r} c_i e^{\lambda+k_i\delta} \otimes 1$ in $R$ where $c_i$'s are non-zero complex numbers and $r \leq s$. We say that $x$ has *length* $r$ if it has $r$ distinct $k_i$'s. If $r = 1$ then by Proposition 3.6, $X_{k_1N}(-k_1\delta)(c_1 e^{\lambda+k_1\delta} \otimes 1) = \pm c_1(e^\lambda \otimes 1)$ as required. If $r \geq 2$ then by induction it suffices to show that we can shorten the length of $x$ by exactly one.

Let $m = \frac{(\lambda|\lambda)}{2} \in \mathbf{Z}$ and choose an integer $k$ so that $n_1 = m + (k - k_1)N < 0$, where $N = (\lambda \mid \delta)$. Write $n_1 = -n$, $n > 0$. Let $y = L_0\delta(-n)X_{kN}(-k\delta)x$. We claim that $y$ has length $r - 1$. Write $x = (c_1 e^{\lambda+k_1\delta} \otimes 1) + x'$, where $x' = \sum_{i=2}^{r} c_i e^{\lambda+k_i\delta} \otimes 1$. So $y = c_1 L_0\delta(-n)X_{kN}(-k\delta)(e^{\lambda+k_1\delta} \otimes 1) + L_0\delta(-n)X_{kN}(-k\delta)x'$. It suffices to show that the first term is zero and no other term is zero. Indeed, for $1 \leq i \leq r$ and $\epsilon_i = \varepsilon(-k\delta, \lambda + k_i\delta) = \pm 1$, and using Proposition 3.6 and (72), we get

$$\begin{aligned} L_0\delta(-n)X_{kN}(-k\delta)(e^{\lambda+k_i\delta} \otimes 1) &= \epsilon_i L_0\big(e^{\lambda+(k_i-k)\delta} \otimes \delta(-n)\big) \\ &= \epsilon_i\Big(\frac{(\lambda \mid \lambda)}{2} + (k_i - k)(\lambda \mid \delta) + n\Big)\big(e^{\lambda+(k_i-k)\delta} \otimes \delta(-n)\big) \\ &= \epsilon_i\big(m + (k_i - k)N + n\big)\big(e^{\lambda+(k_i-k)\delta} \otimes \delta(-n)\big). \end{aligned}$$

Now the coefficient $m + (k_i - k)N + n = 0 \Leftrightarrow k = k_1$ by the choice of $n$. So the length of $x$ has been shortened by one as required. ■

When $\lambda \in Q$ we shall see in the next section that $H(\lambda)$ is a reducible $\tilde{\mathcal{H}}(\dot{Q}, 2)$-module.

Our next batch of irreducible modules will be $\tilde{\mathcal{A}}(Q)$-modules and will come from the completely reducible modules in Theorem 2.6.

Every element $\lambda$ in the lattice $Q$ of (1) is of the form $\alpha + n\delta$, $\alpha \in \dot{Q}$, $n\delta \in \mathbf{Z}\delta$. Define $\phi\colon V_{\dot{Q}}(\alpha) \otimes_{\mathbf{C}} V_{\Lambda}(n\delta) \to V_{\Gamma}(\lambda)$ to be the unique linear map satisfying

$$\phi\big((e^{\alpha} \otimes f) \otimes (e^{n\delta} \otimes g)\big) = e^{\lambda} \otimes fg. \tag{73}$$

Since $\Gamma = \dot{Q} \perp \Lambda$, we have that $S\big(\mathcal{A}(\dot{Q}_-)\big)S\big(\mathcal{A}(\Lambda)_-\big)$. Let

$$\dot{L}_k = \frac{1}{2}\sum_{j\in\mathbf{Z}}\sum_{1=1}^{l} : u_i(-j)u_i(j+k) :$$

where $\{u\}_{i=1}^{l}$ is an orthonormal basis for $\mathbf{C} \otimes_{\mathbf{Z}} \dot{Q}$. Let

$$H_k = \frac{1}{2}\sum_{j\in\mathbf{Z}}\sum_{i=1}^{2} : \alpha_i(-j)\alpha_i(j+k) :$$

where $\{\alpha_1, \alpha_2\}$ is an orthonormal basis for $\mathbf{C} \otimes_{\mathbf{Z}} \Lambda$. Now, let $m \in \mathbf{Z}$, $a \in \mathbf{C} \otimes_{\mathbf{Z}} \dot{Q}$.

We make $V_{\dot{Q}}(\alpha) \otimes_{\mathbf{C}} V_{\Lambda}(n\delta)$ an $\tilde{\mathcal{A}}(Q)$-module as follows. We set

$$a(m)\big((e^{\alpha} \otimes f) \otimes (e^{n\delta} \otimes g)\big) = \big(a(m)(e^{\alpha} \otimes f)\big) \otimes (e^{n\delta} \otimes g)$$
$$\delta(m)\big((e^{\alpha} \otimes f) \otimes (e^{n\delta} \otimes g)\big) = (e^{\alpha} \otimes f) \otimes \big(\delta(m)(e^{n\delta} \otimes g)\big)$$
$$d_m\big((e^{\alpha} \otimes f) \otimes (e^{n\delta} \otimes g)\big) = \big((\dot{L}_m \otimes I) + (I \otimes H_m)\big)\big((e^{\alpha} \otimes f) \otimes (e^{n\delta} \otimes g)\big).$$

Use $\phi$ to make both sides of (73) $\tilde{\mathcal{A}}(Q)$-modules.

***Proposition 4.3*** *The map in (73) is an $\tilde{\mathcal{A}}(Q)$-module isomorphism between $V_{\dot{Q}}(\alpha) \otimes_{\mathbf{C}} V_{\Lambda}(n\delta)$ and $V_{\Gamma}(\lambda)$.*

Denote $V_{\dot{Q}}(\alpha) \otimes_{\mathbf{C}} V_{\Lambda}(n\delta)_l$ by $V_{\Gamma}(\lambda)_l$.

***Proposition 4.4*** *If $\lambda \in Q$ then the family $\{V_{\Gamma}(\lambda)_l\}_{l\in\mathbf{Z}}$ is a filtration of $\tilde{\mathcal{A}}(Q)$-submodules of $V_{\Gamma}(\lambda)$.*

**Proof** By Theorem 2.3, $V_{\Lambda}(n\delta)_l$ is a Vir-submodule of $V_{\Lambda}(n\delta)$. We get from (21) to (23), with $L = \Lambda$, and Lemma 2.2 that it is also an $\mathcal{A}(\Lambda)$-submodule. Hence $V_{\Lambda}(n\delta)_l$ is an $\tilde{\mathcal{A}}(\Lambda)$-submodule of $V_{\Lambda}(n\delta)$. We get from Proposition 1.7 that $V_{\dot{Q}}(\alpha)$ is an $\tilde{\mathcal{A}}(\dot{Q})$-module. So Proposition 4.4 follows from Proposition 4.3 and Theorem 2.3. ■

For each $l \in \mathbf{Z}$, let $\overline{V_{\Gamma}(\lambda)_l}$ denote $V_{\Gamma}(\lambda)_l / V_{\Gamma}(\lambda)_{l-1}$.

***Theorem 4.5*** *If $\lambda \in Q$ then $\{\overline{V_{\Gamma}(\lambda)_l}\}_{l\in\mathbf{Z}}$ is a family of $\tilde{\mathcal{A}}(Q)$-completely reducible modules.*

**Proof** As $\mathbf{C}$-spaces, the map $\phi$ in (73) induces a vector space isomorphism $\overline{\phi}\colon \overline{V_{\Gamma}(\lambda)_l} \to V_{\dot{Q}}(\alpha) \otimes \overline{V_{\Lambda}(n\delta)_l}$.

As in (73) we make $\overline{\phi}$ an $\tilde{\mathcal{A}}(Q)$-module isomorphism. By Corollary 2.7, $X = \overline{V_{\Lambda}(n\delta)_l}$ is a completely reducible Vir-module. Say $X = \coprod_{j\in J} X_j$ with $X_j$ irreducible as a Vir-module. By Proposition 1.5, $V_{\dot{Q}}(\alpha)$ is an irreducible $\mathcal{A}(\dot{Q})$-module. Using these one shows

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that $\coprod_{j\in J}(V_{\dot Q}(\alpha)\otimes X_j)$ is a completely reducible decomposition of $V_{\dot Q}(\alpha)\otimes\overline{V_\Lambda(n\delta)_l}$ as an $\tilde{\mathcal{A}}(Q)$-module. ■

***Remarks*** In Proposition 9 of [FM] it is shown that if $\lambda\notin Q$ then $V_\Gamma(\lambda)$ is an irreducible $\tilde{\mathcal{A}}(Q)$-module. Our proof of complete reducibility in Theorem 4.5 depends on the factorisation in (73). The modules in the next section do not have such a factorisation.

## 5 Reducible Modules

For $m$ an integer let $K(m)$ be the $\mathbf{C}$-subspace of $V(\Gamma)$ spanned by $\{\mathbf{C}[m\mu+\lambda]\otimes_{\mathbf{C}} S(\mathcal{A}(\Gamma)_-) : \lambda\in Q\}$. From Proposition 1.8, Theorem 3.3, and Proposition 3.1, we deduce that $K(m)$ is a $\tilde{\mathfrak{T}}_{[2]}$-submodule of $V(\Gamma)$. We have the decomposition of $\tilde{\mathfrak{T}}_{[2]}$-submodules

(74) $$V(\Gamma)=\coprod_{m\in\mathbf{Z}}K(m).$$

In [FM] it was shown that if $m\neq 0$, then $K(m)$ is irreducible as a $\tilde{\mathfrak{T}}_{[2]}$-module. We shall show that both $K(0)$ and $H(\lambda)$ in (71), $\lambda\in Q$, have filtrations of submodules as modules over $\tilde{\mathfrak{T}}_{[2]}$ and $\tilde{\mathcal{H}}(\dot Q,2)$ respectively. In order to do that we shall need an explicit expression for $X_m(\alpha+n\delta)(e^{\sigma+p\delta}\otimes f)$, where $X_m$ is a moment as in Proposition 3.1 and

(75) $$\{\alpha,\sigma\}\subset\dot Q,\quad \{m,n,p\}\subset\mathbf{Z},$$

and $f\in\mathcal{S}_l$, $l$ an arbitrary but fixed integer.

The notation in (75) above will be in force for the rest of the paper.

From the definition of $\mathcal{S}_l$ in (50), $f$ is a finite sum of scalar multiples of elements of the form $\mu(-q_1)^{a_1}\cdots\mu(-q_s)^{a_s}\delta(\mathbf{k})$, for various integers $j$ and $l$, where $a=\sum_{i=1}^s a_i\le j+l$, $\delta(\mathbf{k})=\delta(-k_1)\cdots\delta(-k_j)$, $a_1,\dots,a_s,k_1,\dots,k_j$ are positive integers, while $q_1,\dots,q_s$ are distinct positive integers. Distributivity of $\otimes$ allows us to take $f$ to be one such summand. So let

(76) $$f=\mu(-q_1)^{a_1}\cdots\mu(-q_s)^{a_s}\delta(\mathbf{k}).$$

For $\beta\in\Gamma$, define the elementary Schur polynomials $S_r(\beta)$, $r\in\mathbf{Z}$, by the expressions

$$\exp T_-(\beta,z)=\sum_{r=0}^{\infty}S_r(\beta)z^r.$$

If $r<0$, put $S_r(\beta)=0$.

***Example 5.1*** Let $x\in\Gamma$. The general formula for $S_r(x)$ can be read off from p. 59 of [KR]. For instance $S_4(x)=\frac{1}{24}\big(x(-1)\big)^4+\frac12\big(x(-1)\big)^2x(-2)+x(-1)x(-3)+x(-4)$. The actual coefficients are irrelevant in our computations. We shall be working with $x=\alpha+n\delta$, $\alpha\in\dot Q$, $n\in\mathbf{Z}$. Using (7), $\big((\alpha+n\delta)(-1)\big)^4=\big(\alpha(-1)+n\delta(-1)\big)^4$. Since $S(\mathcal{A}(\Gamma)_-)$ is commutative we see that, for all integers $r$, $\alpha$ and $n$ as in (75), we have that

(77) $$S_r(\alpha+n\delta)\in S\big(\mathcal{A}(\dot Q)_-\big)S\big(\mathcal{A}(\mathbf{Z}\delta)_-\big).$$

Let $f$ be as in (76).

We want to compute $X_m(\alpha+n\delta)(e^{\sigma+p\delta}\otimes f)$. Let $\epsilon=\varepsilon(\alpha+n\delta,\sigma+p\delta)=\pm 1$. Suppose $f=1$. Then by (19) and (54) $\exp T_+(\alpha+n\delta,z)(1)=1$. So by (56), $\sum_{m\in\mathbf{Z}}X_m(\alpha+n\delta)z^{-m}(e^{\sigma+p\delta}\otimes 1)=z^{\frac{1}{2}(\alpha+n\delta|\alpha+n\delta)}\exp T_-(\alpha+n\delta,z)e^{\sigma+n\delta}z^{(\alpha+n\delta)(0)}(e^{\sigma+p\delta}\otimes 1)=\epsilon\sum_{r=0}^{\infty}\big(e^{\sigma+\alpha+(p+n)\delta}\otimes S_r(\alpha+n\delta)z^{r+\frac{1}{2}(\alpha|\alpha)+(\alpha|\sigma)}\big)$. Matching powers of $z$ by putting $-m=r+\frac{1}{2}(\alpha\mid\alpha)+(\alpha\mid\sigma)$ and solving for $r$, we get from equating coefficients of $z^{-m}$ that

$$X_m(\alpha+n\delta)(e^{\sigma+p\delta}\otimes 1)=\epsilon e^{\sigma+\alpha+(p+n)\delta}\otimes S_{-m-N}(\alpha+n\delta) \tag{78}$$

where $N=\frac{1}{2}(\alpha\mid\alpha)+(\alpha\mid\sigma)$.

***Lemma 5.2*** *If $e^{\sigma+p\delta}\otimes 1$ is in $e^{\sigma+Z\delta}\otimes S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ or $e^{Q}\otimes S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ respectively, then $X_m(\alpha+n\delta)(e^{\sigma+p\delta}\otimes 1)$ is in $e^{\sigma+Z\delta}\otimes S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ or $e^{Q}\otimes S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ respectively.*

**Proof** This follows from (78), (77), and Lemma 2.2. ∎

Now assume that $f$ in (76) is not a constant. The fact that $(\alpha+n\delta\mid\mu)=n$ and $(\alpha+n\delta\mid\delta)=0$ will be used when applying (56). Since $T_+(\alpha+n\delta,z)(e^{\sigma+p\delta}\otimes f)=-\sum_{r>0}\frac{1}{r}(\alpha+n\delta)(r)z^{-r}(e^{\sigma+p\delta}\otimes f)$ we see from (16), (3), and (76) that only $r\in\{q_1,\ldots,q_s\}$ can contribute a non-zero term. And so

$$\begin{aligned}&-\sum_{r>0}\frac{1}{r}(\alpha+n\delta)(r)z^{-r}(e^{\sigma+p\delta}\otimes f)\\ &\quad=-n\sum_{i=1}^{s}\big(e^{\sigma+p\delta}\otimes a_i\mu(-q_1)^{a_1}\cdots\mu(-q_i)^{a_i-1}\cdots\mu(-q_s)^{a_s}\delta(k)\big)z^{-q_i}.\end{aligned} \tag{79}$$

We want to rewrite (79) in a more complicated way that generalises for $T_+^l$, $l$ a positive integer. First replace $-\sum_{i=1}^{s}$ by $(-)^1\sum$ and $n$ by $n^1$. Let $w=(w_1,\ldots,w_s)\in\mathbf{Z}_{\geq 0}^s$. Then $(-)^1\sum$ in (79) ranges over all possible $s$-tuples in $\mathbf{Z}_{\geq 0}^s$ with $\sum_{i=1}^{s}w_i=1$. Each such $s$-tuple $w$ gives a term $f_w z^{-\sum_{i=1}^{s}w_iq_i}$, where the coefficient, $f_w$, of $z^{-\sum_{i=1}^{s}w_iq_i}$ has $\mu$-length $\leq$ ($\mu$-length of $f$)$-1$. The polynomial $f_w$ is a scalar multiple of the polynomial obtained by replacing each $\mu(-q_i)^{a_i}$ in (76) by its $w_i$-th derivative with respect to $\mu(-q_i)$. By replacing 1 by $l$ in this version of (79) we get a Leibniz-rule-type formula for $T_+^l(\alpha+n\delta)(e^{\sigma+p\delta}\otimes f)$. A consequence of this visualised formula for $T_+^l$ is that if $l>a=\sum_{i=1}^{s}a_i$, the $\mu$-length of $f$ then $T_+^l(\alpha+n\delta)(e^{\sigma+p\delta}\otimes f)=0$.

Let $\mathcal{W}=\{(w_1,\ldots,w_s)\in\mathbf{Z}_{\geq 0}^s : 0\leq\sum_{i=1}^{s}w_i\leq a\}$. Let $F=\exp T_+(\alpha+n\delta,z)$. Then $F(e^{\sigma+p\delta}\otimes f)=\sum_{w\in\mathcal{W}}e^{\sigma+p\delta}\otimes c_wf_wz^{-\sum_{i=1}^{s}w_iq_i}$ for some scalars $c_w$, where $c_{(0,0,\ldots,0)}=1$ and $f_{(0,0,\ldots,0)}=f$.

We now recall (56) to get

$$\sum_{m\in\mathbf{Z}}X_m(\alpha+n\delta)z^{-m}(e^{\sigma+p\delta}\otimes f)=z^{(\alpha|\alpha)/2}\exp T_-(\alpha+n\delta,z)e^{\alpha+n\delta}z^{(\alpha+n\delta)(0)}F(e^{\sigma+p\delta}\otimes f)$$

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where $\exp T_-(\alpha+n\delta,z) = \sum_{r=0}^{\infty} S_r(\alpha+n\delta)z^r$. So $X(\alpha+n\delta,z)(e^{\sigma+p\delta}\otimes f) = \epsilon\sum_{r=0}^{\infty}\big(e^{\sigma+\alpha+(p+n)\delta}\otimes S_r(\alpha+n\delta)z^{r+\frac{1}{2}(\alpha|\alpha)+(\alpha|\sigma)}\big)F(e^{\sigma+p\delta}\otimes f)$. Since the constant $\epsilon=\pm 1$ can be absorbed by $F$ we shall suppress it. We get that $\sum_{m\in\mathbf{Z}} X_m(\alpha+n\delta)z^{-m}(e^{\sigma+p\delta}\otimes f) = \sum_{w\in\mathcal{W}}\sum_{r=0}^{\infty}\big(e^{\sigma+\alpha+(p+n)\delta}\otimes S_r(\alpha+n\delta)c_w f_w z^{r+\frac{1}{2}(\alpha|\alpha)+(\alpha|\sigma)}z^{-\sum_{i=1}^{s}w_iq_i}\big)$.

Matching powers of $z$, we let $r_w=\sum_{i=1}^{s}w_iq_i - m - \frac{1}{2}(\alpha\mid\alpha)-(\alpha\mid\sigma)$. Then equating coefficients of $z^{-m}$, we get

$$X_m(\alpha+n\delta)(e^{\sigma+p\delta}\otimes f) = e^{\sigma+\alpha+(p+n)\delta}\otimes\sum_{w\in\mathcal{W}} S_{r_w}(\alpha+n\delta)c_w f_w. \tag{80}$$

The relevant thing about (80) for what follows is that $f_w$ is obtained from $f$ in (76) by lowering the $\mu-$length of $f$.

***Lemma 5.3*** *If $e^{\sigma+p\delta}\otimes f$ is in $e^{\sigma+\mathbf{Z}\delta}\otimes S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ or $e^{Q}\otimes S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ respectively. Then $X_m(\alpha+n\delta)(e^{\sigma+p\delta}\otimes f)$ is in $e^{\sigma+\alpha+\mathbf{Z}\delta}\otimes S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ or $e^{Q}\otimes S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ respectively.*

**Proof** We saw in the proof of Theorem 2.3 that $\mathcal{S}_l$ is invariant under reduction of $\mu$-length. So Lemma 5.3 follows from (80), (77), and Lemma 2.2. ■

Let $H(\lambda)_l=\mathbf{C}[\lambda+\mathbf{Z}\delta]\otimes_{\mathbf{C}} S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ and $K(0)_l=\mathbf{C}[Q]\otimes_{\mathbf{C}} S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$.

***Remark 5.4*** In the proof of Propositions 5.5 and 5.6 we shall use the fact that the invariance of $H(\lambda)_l$ and $K(0)_l$ under Vir depends only on the invariance of $S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ under the oscillator operators. This follows from (21) to (23). Since $\mathcal{S}_l\subseteq\mathcal{S}_{l+1}$ we have the inclusions stated in Propositions 5.5 and 5.6.

***Proposition 5.5*** *Let $l$ be any integer. Then for every $\lambda\in Q$, $H(\lambda)_l$ is an $\tilde{\mathcal{H}}(\dot{Q},2)$-submodule of $H(\lambda)$ and $H(\lambda)_l$ is an $\tilde{\mathcal{H}}(\dot{Q},2)$-submodule of $H(\lambda)_{l+1}$.*

**Proof** Just before (60) we saw that $\tilde{\mathcal{H}}(\dot{Q},2)$ is generated inside $\tilde{\mathcal{T}}_{[2]}$ by the Virasoro operators $L_k$ on $V(\Gamma)$ and $\mathcal{H}(\dot{Q},2)$. As in Lemma 2.2 $S\big(\mathcal{A}(\dot{Q})_-\big)\mathcal{S}_l$ is closed under $L_k$ and $\delta(m)$. So $H(\lambda)_l$ is closed under Vir and $\delta(m)$. Using Proposition 3.7 we need only check invariance of $H(\lambda)_l$ under $X_m(n\delta)$. This follows from Lemma 5.3 with $\alpha=0$. ■

***Proposition 5.6*** *Let $l$ be any integer. Then $K(0)_l$ is a $\tilde{\mathcal{T}}_{[2]}$-submodule of $K(0)$ and $K(0)_l$ is a $\tilde{\mathcal{T}}_{[2]}$-submodule of $K(0)_{l+1}$.*

**Proof** Invariance of $K(0)_l$ under Vir follows from Remark 5.4 and Proposition 5.5 while invariance of $K(0)_l$ under $X_m(\alpha+n\delta)$ and $\delta(m)$ follows from Lemma 5.3 and Lemma 2.2 respectively. ■

***Proposition 5.7*** *Let $\lambda\in\Gamma$. Then every non-zero submodule of $V_\Gamma(\lambda)$ is an indecomposable $\tilde{A}(Q)$-module.*

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**Proof** Let $M$ be a non-zero submodule of $V_\Gamma(\lambda)$. Since $\mathcal{A}(Q) \subset \tilde{\mathcal{A}}(Q)$ it is enough to show that $M$ is an indecomposable $\mathcal{A}(Q)$-module. We show that if $M = A + B$ with both $A$ and $B$ non-zero then $A \cap B \neq 0$. Let $x = e^\lambda \otimes f \in A$ and $y = e^\lambda \otimes g \in B$. By acting on them with appropriate $\delta(n)$ as specified in (16) we may assume that neither $x$ nor $y$ has a $\mu$-term, *i.e.* $f$ and $g$ are in $S(\mathcal{A}(Q)_-)$. We then get from (18) that $0 \neq e^\lambda \otimes fg \in A \cap B$. ■

Since $\mathcal{A}(Q)$ is a subalgebra of both $\tilde{\mathcal{H}}(\dot{Q}, 2)$ and $\tilde{\mathcal{T}}_{[2]}$, the proof of Proposition 5.7 gives analogous results for $H(\lambda)$ and $K(0)$ over their respective algebras.

***Proposition 5.8*** *Let $\lambda \in Q$. Then the modules $H(\lambda)$, $V_\Gamma(\lambda)$, and $K(0)$ do not contain irreducible submodules over the respective Lie algebras, $\tilde{\mathcal{H}}(\dot{Q}, 2)$, $\tilde{\mathcal{A}}(Q)$, and $\tilde{\mathcal{T}}_{[2]}$.*

**Proof** Fix $X \in \{V_\Gamma(\lambda), H(\lambda), K(0)\}$, and let $X_l$ be $V_\Gamma(\lambda)_l$, $H(\lambda)_l$, or $K(0)_l\}$ as the case may be. Now, for $m > 0$, we have that $0 \neq \delta(-m)X_l \subseteq X_{l-1}$. Let $M$ be a non-zero submodule of $X$. Then we must have $0 \neq M \cap X_l \neq M$ for some integer $l$ because $X = \bigcup_{l \in \mathbf{Z}} X_l$ and $\bigcap_{l \in \mathbf{Z}} X_l = \{0\}$. ■

We have been able to do computations in $\tilde{\mathcal{T}}_{[n]}$ and $\tilde{\mathcal{H}}(L, n)$ for $n \leq 2$ and restricted choices of $n$. As can be seen by comparing [EM] and [MEY], the jump from $\mathcal{T}_{[2]}$ to $\mathcal{T}_{[n]}$, $n$ arbitrary is fraught with difficulties.

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## References


[BC] S. Berman and A. Cox, *Enveloping algebras and representations of toroidal Lie algebras*. Pacific J. Math. **165**(1994), 239–267.

[BO] R. Borcherds, *Vertex algebras, Kac-Moody algebras and the Monster*. Proc. Nat. Acad. Sci. U.S.A. **83**(1986), 3068–3071.

[COL] A. J. Coleman, *Groups and Physics*. Notices Amer. Math. Soc. **44**(1997), 8–17.

[E1] S. Eswara Rao, *Iterated loop modules and a filtration for vertex representations of toroidal Lie algebras*. Pacific J. Math. **171**(1995), 511–528.

[E2] ———, *Classification of loop modules with finite dimensional weight spaces*. Math. Ann. **305**(1996), 651–663.

[EM] S. Eswara Rao and R. V. Moody, *Vertex representations for the universal central extensions for the n-toroidal Lie algebras and a generalization of the Virasoro algebra*. Comm. Math. Phys. **159**(1994), 239–266.

[F] M. A. Fabbri, *Virasoro-toroidal algebras and vertex representations*. C. R. Math. Rep. Acad. Sci. Canada **14**(1992), 77–82.

[M] M. A. Fabbri and R. V. Moody, *Irreducible representations of Virasoro-toroidal Lie algebras*. Comm. Math. Phys. **159**(1994), 1–13.

[FK] I. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models* Invent. Math. **63**(1980), 23–66.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex operator algebras and the monster*. Academic Press, 1988.

[GO] P. Goddard and D. Olive, *Algebras, lattices, and strings*. In: Vertex operators in Mathematics and Physics (eds. J. Lepowsky, S. Mandelstram and I. M. Singer), MSRI Berkeley Publication **3**, Berkeley, California, 1984, 51–96.

[KR] V. G. Kac and A. K. Raina, *Bombay Lectures on Highest Weight Representations of Infinite-Dimensional Lie Algebras*. Singapore World Scientific, 1987.

[KS] C. Kassel, *Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra*. J. Pure Appl. Algebra **34**(1985), 265–275.

[MEY] R. V. Moody, S. Eswara Rao and T. Yokonuma, *Toroidal Lie algebras and vertex representations*. Geom. Dedicata **35**(1990), 283–307.

[MP1] R. V. Moody and A. Pianzola, *Infinite-dimensional Lie algebras (a unifying overview)*. Algebras Groups Geom. **4**(1987), 211–230.

[MP2] ______, *Lie Algebras with triangular decomposition*. J. Wiley, 1995.

[Z] H. Zhang, *A class of representations over the Virasoro algebra*. J. Algebra **190**(1997), 1–10.



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