ON FINITE GROUPS WITH AN ABELIAN SYLOW GROUP

RICHARD BRAUER AND HENRY S. LEONARD, JR.

1. Introduction. We shall consider finite groups $\mathfrak{G}$ of order of g which satisfy the following condition:

(*) *There exists a prime p dividing g such that if $P \neq 1$ is an element of a p-Sylow group $\mathfrak{P}$ of $\mathfrak{G}$ then the centralizer $\mathfrak{C}(P)$ of P in $\mathfrak{G}$ coincides with the centralizer $\mathfrak{C}(\mathfrak{P})$ of $\mathfrak{P}$ in $\mathfrak{G}$.*

This assumption is satisfied for a number of important classes of groups. It also plays a role in discussing finite collineation groups in a given number of dimensions.

Of course (*) implies that $\mathfrak{P}$ is abelian. It is possible to obtain rather detailed information about the irreducible characters of groups $\mathfrak{G}$ in this class (§ 4). The method used here is that of considering the exceptional characters of $\mathfrak{G}$ with regard to the normalizer $\mathfrak{N}(\mathfrak{P})$ of $\mathfrak{P}$ (§ 3) and comparing the results with those obtained by studying the p-blocks of characters of $\mathfrak{G}$. It may be mentioned that the same method can be applied under wider assumptions than (*), but the results are less definite.

If g is divisible by p but not by p^2, it is clear that (*) will always be satisfied. The corresponding class of groups has been studied by one of us previously **(3; 4)**. Our results contain many but not all of the previous results. On the other hand, our new method is more elementary.

As a consequence of our results we show in § 5 that if a group $\mathfrak{G}$ satisfying condition (*) has a faithful representation of degree less than $(p^n - 1)^{\frac{1}{2}}$, where p^n is the order of the p-Sylow group $\mathfrak{P}$, then $\mathfrak{P}$ is normal in $\mathfrak{G}$. This is a generalization of a theorem of Blichfeldt **(1)**.

In order to make the paper more self-contained, we start with a specially elementary treatment of the theory of exceptional characters in the form needed here.† The method used can be applied under wider assumptions.

2. Exceptional characters. Let $\mathfrak{G}$ be a group of finite order g and let $\mathfrak{H}$ be a subgroup of order h. We assume that there exists a non-empty set $\mathscr{L}$

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†Exceptional characters were first used by the first author in his original version of the work published in the joint paper **(9)**. The proof was based on **(5,** Theorem 1**)**. M. Suzuki **(10)** then gave a much more elementary proof. For refinements cf. **(11)**.

of conjugate classes $\mathfrak{L}$ of $\mathfrak{H}$ such that if $L \in \mathfrak{L}$, $G \in \mathfrak{G}$, and $G^{-1}LG \in \mathfrak{H}$, then $G \in \mathfrak{H}$, that is, that only the elements of $\mathfrak{H}$ transform an element of $\mathfrak{L}$ into an element of $\mathfrak{H}$. Clearly this condition on $\mathfrak{L}$ is equivalent to the following two conditions: (I) If $L \in \mathfrak{L}$ then the centralizer $\mathfrak{C}(L)$ of L in $\mathfrak{G}$ lies in $\mathfrak{H}$. (II) If $\mathfrak{L}$ is included in the conjugate class $\mathfrak{K}$ of $\mathfrak{G}$ then $\mathfrak{K} \cap \mathfrak{H} = \mathfrak{L}$. It is not required that $\mathscr{L}$ consist of all classes of $\mathfrak{H}$ with these properties.

Let $\psi_1, \psi_2, \ldots, \psi_l$ denote the irreducible characters of $\mathfrak{H}$. We arrange them in "families" $F_1, F_2, \ldots$, such that ψ_i and ψ_j belong to the same family if and only if $\psi_i(H) = \psi_j(H)$ for all $H \in \mathfrak{H}$ which do not belong to a class $\mathfrak{L} \in \mathscr{L}$. We shall say that a family F is *proper* if it has more than one member.

In the following, $\mathfrak{G}$, $\mathfrak{H}$, and $\mathscr{L}$ will be fixed. The notion of exceptional characters of $\mathfrak{G}$ will depend on $\mathfrak{H}$ and $\mathscr{L}$. For convenience we shall call the classes $\mathfrak{L} \in \mathscr{L}$ *special classes* and their elements *special elements*.

(2A) *Let $F = \{\psi_1, \psi_2, \ldots, \psi_m\}$ be a proper family. There exist m irreducible characters $\chi_1, \chi_2, \ldots, \chi_m$ of $\mathfrak{G}$ such that*

$$\chi_i|\mathfrak{H} = \epsilon\psi_i + S, \qquad (i = 1, 2, \ldots, m), \tag{2.1}$$

where ϵ is a sign ± 1 independent of i, and S is a character of $\mathfrak{H}$ independent of i in which $\psi_1, \psi_2, \ldots, \psi_m$ appear with the same multiplicity. If χ is an irreducible character of $\mathfrak{G}$ other than $\chi_1, \chi_2, \ldots, \chi_m$ then $\psi_1, \psi_2, \ldots, \psi_m$ appear with the same multiplicity in $\chi|\mathfrak{H}$.

Proof. If $\chi_1, \chi_2, \ldots, \chi_k$ are the irreducible characters of $\mathfrak{G}$, we can set

$$\chi_i|\mathfrak{H} = \sum_{j=1}^{l} a_{ij}\psi_j, \qquad (i = 1, 2, \ldots, k), \tag{2.2}$$

where the a_{ij} are non-negative integers. Then

$$ha_{ij} = \sum_{H \in \mathfrak{H}} \chi_i(H)\overline{\psi}_j(H). \tag{2.3}$$

We shall view the k numbers a_{ij} with fixed j as the coefficients of a column A_j with i serving as row index. Likewise, if $G \in \mathfrak{G}$, we arrange the k numbers $\chi_i(G)$ in a column $X(G)$. Then (2.3) becomes

$$hA_j = \sum_{H \in \mathfrak{H}} X(H)\overline{\psi}_j(H), \qquad (j = 1, 2, \ldots, l). \tag{2.3*}$$

The inner product of two columns is defined in the usual manner. If $G \in \mathfrak{G}$, we have

$$h(X(G), A_j - A_1) = \sum_{H \in \mathfrak{H}} (X(G), X(H))(\psi_j(H) - \psi_1(H)).$$

Assume here that $1 \leqslant j \leqslant m$. It will suffice to let H range over the special elements of $\mathfrak{H}$ since otherwise $\psi_j(H) = \psi_1(H)$. By the orthogonality relations $(X(G), X(H)) = 0$ if G and H are not conjugate in $\mathfrak{G}$. If G and H are conjugate, $(X(G), X(H))$ is the order $c(H)$ of the centralizer $\mathfrak{C}(H)$. For a special

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element H, $\mathfrak{C}(H)$ is also the centralizer of H in $\mathfrak{H}$ and the class $\mathfrak{K}$ of H in $\mathfrak{H}$ consists of $h/c(H)$ elements. Thus, for $j = 1, 2, \ldots, m$,

$$(2.4) \qquad (X(G), A_j - A_1) = \begin{cases} \psi_j(H) - \psi_1(H) \\ 0, \end{cases}$$

where the first case applies when G is conjugate in $\mathfrak{G}$ to special elements $H \in \mathfrak{H}$ and the second case when G is not conjugate to such elements.

Combining (2.3*) and (2.4), we have

$$h(A_r, A_j - A_1) = \sum_{H \in \mathfrak{H}} \bar{\psi}_r(H)(\psi_j(H) - \psi_1(H)), \qquad (1 \leqslant r \leqslant l; 1 \leqslant j \leqslant m).$$

Hence, if δ_{ij} has the usual significance,

$$(2.5) \qquad (A_r, A_j - A_1) = \delta_{jr} - \delta_{1r}, \qquad (1 \leqslant r \leqslant l; 1 \leqslant j \leqslant m).$$

This implies that $(A_j - A_1, A_j - A_1) = 2$ for $2 \leqslant j \leqslant m$. Hence $A_j - A_1$ contains two coefficients ± 1 while all other coefficients vanish. If $\mathfrak{H} \neq \mathfrak{G}$, the unit element 1 is not special and (2.4) shows that $(X(1), A_j - A_1) = 0$. Since $X(1)$ has positive coefficients, the two non-vanishing coefficients in $A_j - A_1$ have opposite signs. If $\mathfrak{H} = \mathfrak{G}$, this is still true, since here the matrix $[a_{ij}]$ in (2.2) is the unit matrix. If $1 < i < j \leqslant m$ then $(A_i - A_1, A_j - A_1) = 1$ by (2.5). It now follows easily that the characters $\chi_1, \chi_2, \ldots, \chi_k$ can be taken in such an order that the first m rows in the $m - 1$ columns $A_2 - A_1, A_3 - A_1, \ldots, A_m - A_1$ are given by

$$\begin{matrix} -\epsilon & -\epsilon & \ldots & -\epsilon \\ \epsilon & 0 & \ldots & 0 \\ 0 & \epsilon & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & \ldots & \epsilon \end{matrix} \qquad \text{with } \epsilon = \pm 1$$

while the remaining rows vanish. In other words

$$(2.6) \qquad a_{ij} - a_{i1} = \begin{cases} -\epsilon \text{ for } i = 1, 2 \leqslant j \leqslant m \\ \epsilon \text{ for } i = j, 2 \leqslant j \leqslant m \\ 0 \text{ otherwise.} \end{cases}$$

(For $j = 1$, the left side of (2.6) vanishes trivially.)

It now follows from (2.5) that

$$(2.7) \qquad -a_{1r}\epsilon + a_{jr}\epsilon = \delta_{jr} - \delta_{1r}, \qquad (1 \leqslant r \leqslant l, 1 \leqslant j \leqslant m).$$

In particular

$$(2.7^*) \qquad a_{jr} = a_{1r}, \qquad (1 \leqslant j \leqslant m, r > m).$$

We find from (2.7) and (2.6) that

$$(2.8) \qquad a_{jr} - \epsilon\delta_{jr} = a_{1r} - \epsilon\delta_{1r} = a_{11} - \epsilon, \quad (1 \leqslant j \leqslant m, 1 \leqslant r \leqslant m).$$

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Here $a_{11} - \epsilon = a_{12} \geqslant 0$. On combining (2.8) and (2.7*) with (2.2) we see that

$$\chi_j|\mathfrak{H} - \epsilon\psi_j = (a_{11} - \epsilon) \sum_{r=1}^{m} \psi_r + \sum_{r>m} a_{1r}\psi_r, \qquad (1 \leqslant j \leqslant m).$$

Thus this expression is a character S of $\mathfrak{H}$ which is independent of j and which contains $\psi_1, \psi_2, \ldots, \psi_m$ with the same multiplicity. This yields (2.1). The last statement of (2A) is immediate from (2.2) and (2.6) with $i > m$.

We call χ_j the *exceptional character* of $\mathfrak{G}$ corresponding to ψ_j, $(1 \leqslant j \leqslant m)$. It is clear from (2A) that χ_j is uniquely determined by ψ_j.

On combining (2.4) and (2.6) we have

(2B) *If G is an element of $\mathfrak{G}$ which is not conjugate to a special element then the exceptional characters χ_j of $\mathfrak{G}$ corresponding to the characters of $\mathfrak{H}$ in a proper family F all take the same value for G. In particular, if $\mathfrak{G} \neq \mathfrak{H}$ then these characters χ_j all have the same degree.*

The last statement is obtained by taking $G = 1$ which is permitted for $\mathfrak{G} \neq \mathfrak{H}$.

Let F^* be a proper family of characters of $\mathfrak{H}$ different from the family F in (2A). Then $\psi_r \in F^*$ appears with the same multiplicity in $\chi_1|\mathfrak{H}, \chi_2|\mathfrak{H}, \ldots, \chi_m|\mathfrak{H}$. This shows that $\chi_1, \chi_2, \ldots, \chi_m$ are different from the exceptional characters of $\mathfrak{G}$ associated with the characters in F^*. We shall say that an irreducible character χ of $\mathfrak{G}$ is *exceptional*, if χ is the exceptional character of $\mathfrak{G}$ corresponding to a character of H in some *proper* family. By definition, exceptional characters of $\mathfrak{G}$ are irreducible. Combining the preceding remark with (2A), we have

(2C) *If χ is an exceptional character of $\mathfrak{G}$ and ψ the irreducible character of $\mathfrak{H}$ with which it is associated, then the correspondence $\chi \rightarrow \psi$ is a $(1-1)$ correspondence between the exceptional characters of $\mathfrak{G}$ and the irreducible characters of $\mathfrak{H}$ belonging to proper families. If ψ belongs to the proper family F_0 then we have formulae*

$$\chi|\mathfrak{H} = \epsilon(F_0)\psi + \sum_{F} a(F_0, F) \sum_{\psi \in F} \psi \tag{2.9}$$

where F ranges over all families of characters of $\mathfrak{H}$, proper or improper. Here $\epsilon(F_0)$ is a sign depending only on F_0, and $a(F_0, F)$ is a non-negative rational integer depending only on F_0 and F.

Likewise, if χ_μ is an irreducible character of $\mathfrak{G}$ which is non-exceptional then we have a formula

$$\chi_\mu|\mathfrak{H} = \sum_{F} a(\mu, F) \sum_{\psi \in F} \psi \tag{2.10}$$

where the $a(\mu, F)$ are non-negative rational integers.

3. Application of the exceptional characters. Let p be a prime number. We consider groups $\mathfrak{G}$ of finite order g divisible by p which satisfy the following assumption:

(*) *If $\mathfrak{P}$ is a p-Sylow subgroup of $\mathfrak{G}$ and $P \neq 1$ is an element of $\mathfrak{P}$, then the centralizer $\mathfrak{C}(P)$ of P in $\mathfrak{G}$ coincides with the centralizer $\mathfrak{C}(\mathfrak{P})$ of $\mathfrak{P}$ in $\mathfrak{G}$.*

It follows from (*) that $\mathfrak{P}$ is abelian. The order of $\mathfrak{P}$ will be denoted by p^n. Set $\mathfrak{C} = \mathfrak{C}(\mathfrak{P})$. As is well known,

$$\mathfrak{C} = \mathfrak{P} \times \mathfrak{V} \tag{3.1}$$

where $\mathfrak{V}$ is a group of order v prime to p. We choose $\mathfrak{H}$ as the normalizer $\mathfrak{N}(\mathfrak{P})$ of $\mathfrak{P}$ in $\mathfrak{G}$. Both $\mathfrak{P}$ and $\mathfrak{V}$ are normal in $\mathfrak{H}$. Since $\mathfrak{P}$ is the only p-Sylow subgroup of $\mathfrak{H}$, every p-singular class $\mathfrak{K}$ of $\mathfrak{H}$ consists of elements of the form PV with $P \in \mathfrak{P}, P \neq 1$, $V \in \mathfrak{V}$. It follows from (*) that $\mathfrak{P} \subseteq \mathfrak{C}(PV) \subseteq \mathfrak{C} \subseteq \mathfrak{H}$. Now Sylow's theorem shows that if $G^{-1}PVG \in \mathfrak{H}$ for some $G \in \mathfrak{G}$, then $G \in \mathfrak{H}$. Hence we may choose for $\mathscr{L}$ in § 2 the set of all p-singular classes $\mathfrak{K}$ of $\mathfrak{H}$.

We next discuss the irreducible characters of $\mathfrak{H}$. If $\vartheta_1, \vartheta_2, \ldots, \vartheta_a$ are the irreducible characters of $\mathfrak{V}$ and $\lambda_1, \lambda_2, \ldots, \lambda_b$ those of $\mathfrak{P}$, and if we define $\lambda_\beta\vartheta_\alpha$ by

$$(\lambda_\beta\vartheta_\alpha)(PV) = \lambda_\beta(P)\vartheta_\alpha(V), \qquad (P \in \mathfrak{P},\ V \in \mathfrak{V}),$$

then the ab products $\lambda_\beta\vartheta_\alpha$ are the ab distinct irreducible characters of the direct product $\mathfrak{C}$ in (3.1). Every irreducible character ψ of $\mathfrak{H}$ appears as a constituent in a character $(\lambda_\beta\vartheta_\alpha)^*$ of $\mathfrak{H}$ induced by an irreducible character $\lambda_\beta\vartheta_\alpha$ of $\mathfrak{C}$. Here

$$(\lambda_\beta\vartheta_\alpha)^*(H) = \begin{cases} 0, & (H \notin \mathfrak{C}), \\ \sum_N (\lambda_\beta^N\vartheta_\alpha^N)(H), & (H \in \mathfrak{C}), \end{cases} \tag{3.2}$$

where N ranges over a complete residue system R of $\mathfrak{H}$ (mod $\mathfrak{C}$). It follows that we have $(\lambda_\beta\vartheta_\alpha)^* = (\lambda_{\beta'}\vartheta_{\alpha'})^*$ if and only if there exists an $N \in \mathfrak{H}$ such that $\lambda_{\beta'} = \lambda_\beta{}^N$ and $\vartheta_{\alpha'} = \vartheta_\alpha{}^N$. In particular, to obtain all characters $(\lambda_\beta\vartheta_\alpha)^*$ it will suffice to let ϑ_α range over a set $\mathfrak{S}$ of irreducible characters of $\mathfrak{V}$ such that each irreducible character of $\mathfrak{V}$ is an associate in $\mathfrak{H}$ of exactly one element of $\mathfrak{S}$.

If $N \in \mathfrak{H} - \mathfrak{C}$ then N transforms the abelian group $\mathfrak{P}$ into itself leaving only the unit element fixed. Hence every $P \neq 1$ in $\mathfrak{P}$ has $(\mathfrak{H} : \mathfrak{C})$ conjugates in $\mathfrak{H}$. Thus if w conjugate classes of $\mathfrak{H}$ contain such elements P and if $(\mathfrak{P} : 1) = p^n$, then

$$p^n - 1 = w(\mathfrak{H} : \mathfrak{C}). \tag{3.3}$$

Moreover for $N \in \mathfrak{H} - \mathfrak{C}$, the mapping $\lambda \to \lambda^N$ permutes the irreducible characters λ of $\mathfrak{P}$ leaving only the principal character 1 fixed.† Hence if $\lambda_\beta \neq 1$ in (3.2) then the characters $\lambda_\beta{}^N$ appearing in (3.2) are distinct. If ψ is an irreducible constituent of $(\lambda_\beta\vartheta_\alpha)^*$, then $\psi|\mathfrak{C}$ is a sum of terms $\lambda_\beta{}^N\vartheta_\alpha{}^N$ such that with each term its distinct associates appear. Since we have just seen that all $(\mathfrak{H} : \mathfrak{C})$ associates are distinct, we have $(\lambda_\beta\vartheta_\alpha)^* = \psi$; that is,

†Cf., e.g., the "permutation lemma" in **(2)**.

$(\lambda_\beta\vartheta_\alpha)^*$ is irreducible if $\lambda_\beta \neq 1$. We shall have $(\lambda_\beta\vartheta_\alpha)^* = (\lambda_{\beta'}\vartheta_\alpha)^*$ if and only if there exists an $N \in \mathfrak{H}$ such that $\lambda_{\beta'} = \lambda_\beta{}^N$ and $\vartheta_\alpha = \vartheta_\alpha{}^N$. The latter condition means that N must belong to the inertial group $\mathfrak{F}(\vartheta_\alpha) \supseteq \mathfrak{C}$ of ϑ_α in $\mathfrak{H}$.

If we set

$$(3.4) \qquad (\mathfrak{H} : \mathfrak{F}(\vartheta_\alpha)) = r_\alpha, \qquad (\mathfrak{F}(\vartheta_\alpha) : \mathfrak{C}) = s_\alpha,$$

it follows from our remarks that we have $(p^n - 1)/s_\alpha$ irreducible characters ψ of $\mathfrak{H}$ of the form $(\lambda_\beta\vartheta_\alpha)^*$ with fixed α and $\lambda_\beta \neq 1$. We shall denote this set of characters by $F(\vartheta_\alpha)$. Then, by (3.3) and (3.4), $F(\vartheta_\alpha)$ consists of wr_α distinct irreducible characters ψ of $\mathfrak{H}$. As shown by (3.2), each $\psi \in F(\vartheta_\alpha)$ vanishes for the elements $H \in \mathfrak{H} - \mathfrak{C}$. If V is a p-regular element of $\mathfrak{C}$ then $V \in \mathfrak{V}$ and we have

$$(3.5) \qquad \psi(V) = \sum_{N \in R} \vartheta_\alpha^N(V), \qquad (V \in \mathfrak{V}).$$

Hence all characters in $F(\vartheta_\alpha)$ have the same value for elements of $\mathfrak{H}$ which are not special in the sense of § 2. Thus the family $F(\vartheta_\alpha)$ is included in one of the families F in § 2. We note that (3.5) can be written

$$(3.5^*) \qquad \psi(V) = s_\alpha\zeta_\alpha(V) \text{ with } \zeta_\alpha(V) = \sum_M \vartheta_\alpha^M(V) \qquad (V \in \mathfrak{V}),$$

where M ranges over a complete residue system of $\mathfrak{H}(\text{mod } \mathfrak{F}(\vartheta_\alpha))$.

It follows from (3.2) that the kernel of a character $\psi \in F(\vartheta)$ cannot include $\mathfrak{P}$. If we let ϑ range over the set $\mathfrak{S}$, the families $F(\vartheta)$ obtained will be disjoint. The remarks above show that all irreducible characters ψ of $\mathfrak{H}$ will belong to such a family $F(\vartheta)$ except those which appear as a constituent of some $(1\vartheta_\rho)^*$ with $\vartheta_\rho \in \mathfrak{S}$. Let us denote the set of distinct irreducible constituents ψ of $(1\vartheta_\rho)^*$ by $U(\vartheta_\rho)$. It follows from (3.2) with $\lambda_\beta = 1$ that all these ψ have kernels which include $\mathfrak{P}$. Consequently no $\psi \in U(\vartheta_\rho)$ can belong to an $F(\vartheta)$.

We summarize some of the results in the following proposition.

(3A) *Let G be a group satisfying condition* (*), *$\mathfrak{P}$ a p-Sylow group, $\mathfrak{C} = \mathfrak{C}(\mathfrak{P}) = \mathfrak{P} \times \mathfrak{V}$, $\mathfrak{N}(\mathfrak{P}) = \mathfrak{H}$. Let $\mathfrak{S}$ be a system of irreducible characters ϑ_α of $\mathfrak{V}$ such that every irreducible character of $\mathfrak{V}$ is an associate in $\mathfrak{H}$ of exactly one $\vartheta_\alpha \in \mathfrak{S}$. The irreducible characters ψ of $\mathfrak{H}$ whose kernel does not include $\mathfrak{P}$ are distributed in the disjoint families $F(\vartheta_\alpha)$, $\vartheta_\alpha \in \mathfrak{S}$. Here $F(\vartheta_\alpha)$ consists of the characters of $\mathfrak{H}$ induced by characters $\lambda\vartheta_\alpha$ of $\mathfrak{C}$ where λ is a non-principal irreducible character of $\mathfrak{P}$. The number of members of $F(\vartheta_\alpha)$ is $(p^n - 1)/s_\alpha = wr_\alpha$ where $s_\alpha = (\mathfrak{F}(\vartheta_\alpha) : \mathfrak{C})$, $r_\alpha = (\mathfrak{H} : \mathfrak{F}(\vartheta_\alpha))$, $p^n - 1 = w(\mathfrak{H} : \mathfrak{C})$, and where $\mathfrak{F}(\vartheta_\alpha)$ is the inertial group of ϑ_α in $\mathfrak{H}$.*

We now study the characters in the set $U(\vartheta_\rho)$. It follows from (3.2) that for $\psi \in U(\vartheta_\rho)$, $P \in \mathfrak{P}$, $V \in \mathfrak{V}$,

$$(3.6) \qquad \psi(PV) = e(\psi) \sum_M \vartheta_\rho^M(V) = e(\psi)\zeta_\rho(V),$$

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where M ranges over a residue system of $\mathfrak{H}$ (mod $\mathfrak{F}(\vartheta_\rho)$) and where $e(\psi)$ is a natural integer depending only on ψ. In particular this shows that, for two distinct characters ϑ_ρ and $\vartheta_{\rho'}$ in $\mathfrak{S}$, the sets $U(\vartheta_\rho)$ and $U(\vartheta_{\rho'})$ are disjoint.

(3B) *Let $U(\vartheta_\rho)$ denote the set of irreducible constituents of the character of $\mathfrak{H}$ induced by the character $1\vartheta_\rho$ of $\mathfrak{C}$. Each irreducible character of $\mathfrak{H}$ whose kernel includes $\mathfrak{P}$ belongs to exactly one set $U(\vartheta_\rho)$ with $\vartheta_\rho \in \mathfrak{S}$.*

In applying the lemma of § 2 each ψ in a set $U(\vartheta_\rho)$ will be considered as forming a family of one element; the remaining families will be the sets $F(\vartheta_\alpha)$. We claim that if we sum over all ψ in $F(\vartheta_\alpha)$, we have

$$\sum_{\psi} \psi(H) = \begin{cases} 0, & (H \notin \mathfrak{C}), \\ -\zeta_\alpha(V), & (H \in \mathfrak{C} - \mathfrak{V}), \\ (p^n - 1)\zeta_\alpha(V), & (H = V \in \mathfrak{V}), \end{cases} \tag{3.7}$$

where $H = PV$ with $P \in \mathfrak{P}$, $V \in \mathfrak{V}$, for $H \in \mathfrak{C}$. This is immediate from (3.2) for $H \notin \mathfrak{C}$ and from (3.5*) for $H \in \mathfrak{V}$. Assume that $H \in \mathfrak{C} - \mathfrak{V}$. If we let λ range over the $p^n - 1$ non-principal irreducible characters of $\mathfrak{P}$, it follows from (3.4) and the remarks preceding it that each of the characters ψ in $F(\vartheta_\alpha)$ is obtained s_α times in the form $(\lambda\vartheta_\alpha)^*$, and we have

$$s_\alpha \sum_{\psi} \psi|\mathfrak{C} = \sum_{\lambda\neq 1} \sum_{N\in R} \lambda^N \vartheta_\alpha^N .$$

For $H = PV$ with $P \neq 1$, we have $\sum_{\lambda\neq 1}\lambda^N(P) = -1$. Thus

$$\sum_{\psi} \psi(H) = -(1/s_\alpha) \sum_{N\in R} \vartheta_\alpha^N(V) = -\zeta_\alpha(V)$$

by (3.5*). This completes the proof of (3.7).

The family $F(\vartheta_\alpha)$ is proper except when $w = 1$ and $r_\alpha = 1$, that is, when $(\mathfrak{H} : \mathfrak{C}) = p^n - 1$ and ϑ_α is associated only with itself. If $F(\vartheta_\alpha)$ is a proper family, there exist exactly wr_α exceptional characters χ corresponding to the characters $\psi \in F(\vartheta_\alpha)$. If $\chi_\beta{}^{(\alpha)}$ is the exceptional character of $\mathfrak{G}$ corresponding to $(\lambda_\beta\vartheta_\alpha)^* \in F(\vartheta_\alpha)$, then by (2.9)

$$\chi_\beta^{(\alpha)}|\mathfrak{H} = \epsilon(F(\vartheta_\alpha))(\lambda_\beta\vartheta_\alpha)^* + \sum_F a(F(\vartheta_\alpha),F) \sum_{\psi\in F} \psi. \tag{3.8}$$

For $P \in \mathfrak{P}$, $P \neq 1$, $V \in \mathfrak{V}$, using (3.2), (3.6), (3.7), we have

$$\chi_\beta^{(\alpha)}(PV) = \epsilon(F(\vartheta_\alpha)) \sum_{N\in R} \lambda_\beta^N(P)\vartheta_\alpha^N(V) + \sum_\rho b_\rho^{(\alpha)}\zeta_\rho(V), \tag{3.9}$$

where ρ ranges over the values for which $\vartheta_\rho \in \mathfrak{S}$, and where the $b_\rho{}^{(\alpha)}$ are rational integers independent of P. Applying the same method for the element $V \in \mathfrak{V}$, we see that

$$\chi_\beta^{(\alpha)}(V) - \epsilon(F(\vartheta_\alpha))(\lambda_\beta\vartheta_\alpha)^*(V) \equiv \chi_\beta^{(\alpha)}(PV) - \epsilon(F(\vartheta_\alpha))(\lambda_\beta\vartheta_\alpha)^*(PV) \tag{3.10}$$

mod p^n in the ring of all algebraic integers.

If χ_μ is a non-exceptional character of $\mathfrak{G}$, we apply (2.10). It follows that we have formulae

$$\chi_\mu(PV) = \sum_\rho b_{\mu\rho}\zeta_\rho(V) \tag{3.11}$$

for $P \neq 1$ in $\mathfrak{P}$ and V in $\mathfrak{V}$, where ρ has the same range as in (3.9) and where the $b_{\mu\rho}$ are rational integers. Also

$$\chi_\mu(PV) \equiv \chi_\mu(V) \pmod{p^n}. \tag{3.12}$$

4. The decomposition numbers. We now study the p-blocks B of groups $\mathfrak{G}$ which satisfy the assumption (*). If $\mathfrak{D}$ is the defect group of B, we may assume $\mathfrak{D} \subseteq \mathfrak{P}$. There exist p-regular elements $G \in \mathfrak{G}$ such that $\mathfrak{D}$ is a p-Sylow subgroup of $\mathfrak{C}(G)$ **(6**, § 8**)**. It follows from (*) that either $\mathfrak{D} = \mathfrak{P}$ or $\mathfrak{D} = \{1\}$. Hence

(4A) *If $\mathfrak{G}$ satisfies assumption* (*) *for a prime p then every p-block B of $\mathfrak{G}$ has either full defect n or defect* 0.

We investigate the blocks B of full defect further. Here $\mathfrak{D} = \mathfrak{P}$, $\mathfrak{D}\mathfrak{C}(\mathfrak{D}) = \mathfrak{C} = \mathfrak{P} \times \mathfrak{V}$ by (3.1), and $\mathfrak{D}\mathfrak{C}(\mathfrak{D})/\mathfrak{D} \cong \mathfrak{V}$ is of order prime to p. The number s of blocks B of full defect is equal to the number of classes $\{\vartheta\}$ of irreducible characters of $\mathfrak{V}$ associated in $\mathfrak{N}(\mathfrak{D})/\mathfrak{D} = \mathfrak{H}/\mathfrak{P}$ **(6**, § 12**)**, that is, classes $\{\vartheta\}$ of irreducible characters associated in $\mathfrak{H}$. Thus s is the number of $\vartheta_\alpha \in \mathfrak{S}$.

On the other hand we use the main result of **(7)**. As is easily seen, each block $\mathfrak{b}$ of $\mathfrak{C}$ consists of the p^n irreducible characters $\lambda_\beta\vartheta_j$ for some fixed j. We shall denote this block by $\mathfrak{b}(\vartheta_j)$. There is only one modular irreducible character in $\mathfrak{b}(\vartheta_j)$, and it may be identified with ϑ_j. If χ_i is an irreducible character of $\mathfrak{G}$ belonging to a block B, then

$$\chi_i(PV) = \sum_j d_{ij}^P\vartheta_j(V), \qquad (P \in \mathfrak{P}, P \neq 1, V \in \mathfrak{V}), \tag{4.1}$$

where the $d_{ij}{}^P$ are the generalized decomposition numbers and where $d_{ij}{}^P \neq 0$ only for values of j for which

$$\mathfrak{b}(\vartheta_j)^{\mathfrak{G}} = B. \tag{4.2}$$

For each block B of defect n, there exist ϑ_j for which (4.2) holds. Indeed, if this were not so, then by (4.1) $\chi_i(P) = 0$ for all $P \neq 1$. If χ_i is non-exceptional then by (3.12) $\chi_i(1) \equiv 0 \pmod{p^n}$, that is, χ_i would be of defect 0, a contradiction. If χ_i is exceptional then taking $V = 1$ and summing in (3.10) over the classes relative to $\mathfrak{H}$ of elements $P \neq 1$ in $\mathfrak{P}$, again we would have $\chi_i(1) \equiv 0 \pmod{p^n}$.

Let $F(\vartheta_\alpha)$ be a proper family, take χ_i as an exceptional character $\chi_\beta{}^{(\alpha)}$ and let B be the block to which $\chi_\beta{}^{(\alpha)}$ belongs. Comparison of (4.1) with (3.9) yields

$$\epsilon(F(\vartheta_\alpha)) \sum_{N\in R} \lambda_\beta^N(P)\vartheta_\alpha^N(V) + \sum_\rho b_\rho^{(\alpha)}\zeta_\rho(V) = \sum_j d_{ij}^P\vartheta_j(V), \tag{4.3}$$

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where ζ_ρ is given by (3.5*). Since this holds for all $V \in \mathfrak{V}$, each irreducible character ϑ of $\mathfrak{V}$ appears with the same coefficient on both sides. The coefficient of $\vartheta_\alpha{}^N$ is

$$\epsilon(F(\vartheta_\alpha)) \sum_T \lambda_\beta^{TN}(P) + b_\alpha^{(\alpha)},$$

where T ranges over a residue system of $\mathfrak{F}(\vartheta_\alpha)$ (mod $\mathfrak{C}$). This expression would vanish for all $P \neq 1$ in $\mathfrak{P}$ only if all irreducible characters $\lambda \neq 1$ appeared among the $\lambda_\beta{}^{TN}$. This would mean that $s_\alpha = p^n - 1$. But then (3.3) and (3.4) would imply that $wr_\alpha = 1$, a contradiction since $F(\vartheta_\alpha)$ consists of wr_α elements and would not be a proper family. Hence each $\vartheta_\alpha{}^N \in \{\vartheta_\alpha\}$ must appear among the ϑ_j on the right in (4.3), that is, (4.2) holds for all $\vartheta_j \in \{\vartheta_\alpha\}$.

If $F(\vartheta_\alpha)$ is not a proper family, we have $r_\alpha = 1$ (in addition to $w = 1$). Then the class $\{\vartheta_\alpha\}$ consists only of ϑ_α. We may therefore say for all classes $\{\vartheta_\alpha\}$ that if ϑ_j in (4.2) ranges over $\{\vartheta_\alpha\}$, the block B on the right is the same for all j. Thus (4.2) establishes a mapping of the set of classes $\{\vartheta_\alpha\}$ into the set of blocks B of full defect. We have already noted that the mapping is onto and that the number of classes $\{\vartheta_\alpha\}$ and of blocks B of full defect are equal. Hence we have a $(1 - 1)$ correspondence. We shall now denote the block B corresponding to the class $\{\vartheta_\alpha\}$ by $B(\vartheta_\alpha)$. Then we have shown:

(4B) *Every block B of $\mathfrak{G}$ of full defect is obtained exactly once in the form $B(\vartheta_\alpha)$ when ϑ_α ranges over $\mathfrak{S}$. If $wr_\alpha > 1$, the wr_α exceptional characters $\chi_\beta{}^{(\alpha)}$ belong to $B(\vartheta_\alpha)$.*

(4C) *For $\chi_i \in B(\vartheta_\alpha)$, we have formulae*

$$\chi_i(PV) = \sum_j d_{ij}^P \vartheta_j(V), \qquad (P \in \mathfrak{P}, P \neq 1, V \in \mathfrak{V}),$$

where ϑ_j ranges over the class $\{\vartheta_\alpha\}$.

It follows from (4C) that in (3.9) and (4.3) $b_\rho{}^{(\alpha)} = 0$ for $\rho \neq \alpha$. Thus

(4D) *If χ_i is an exceptional character $\chi_\beta{}^{(\alpha)}$ then for $\vartheta_j = \vartheta_\alpha{}^M$ we have*

$$d_{ij}^P = \epsilon_\alpha \sum_T \lambda_\beta^{TM}(P) + b_\alpha^{(\alpha)}, \qquad (P \in \mathfrak{P}, P \neq 1). \tag{4.4}$$

Here T ranges over a residue system of $\mathfrak{F}(\vartheta_\alpha)$ (mod $\mathfrak{C}$), $b_\alpha{}^{(\alpha)}$ is a rational integer which does not depend on P, and $\epsilon_\alpha = \epsilon(F(\vartheta_\alpha))$ is a sign ± 1 depending only on α. Hence

$$\chi_\beta^{(\alpha)}(PV) = \sum_M \left(\epsilon_\alpha \sum_T \lambda_\beta^{TM}(P) + b_\alpha^{(\alpha)}\right) \vartheta_\alpha^M(V)$$

for $P \in \mathfrak{P}$, $P \neq 1$, $V \in \mathfrak{V}$, with M ranging over a residue system of $\mathfrak{H}$ (mod $\mathfrak{F}(\vartheta_\alpha)$).

We may also take for χ_i a non-exceptional irreducible character of $\mathfrak{G}$ of positive defect and apply (4C) to (3.11). It follows that if $\chi_i \in B(\vartheta_\alpha)$ then $b_{i\rho} = 0$ for $\rho \neq \alpha$. Hence

(4E) *If χ_i is a non-exceptional irreducible character of $\mathfrak{G}$ belonging to $B(\vartheta_\alpha)$ and $\vartheta_j = \vartheta_\alpha{}^M$, then $d_{ij}{}^P$ is a rational integer $b_i = b_{i\alpha}$ which depends neither on P for $P \neq 1$ nor on M. Hence*

$$\chi_i(PV) = b_i \sum_M \vartheta_\alpha^M(V), \qquad (P \in \mathfrak{P}, P \neq 1, V \in \mathfrak{V}),$$

where M ranges over a residue system of $\mathfrak{H}$ (mod $\mathfrak{F}(\vartheta_\alpha)$, that is, ϑ_α^M ranges over the distinct associates of ϑ_α in $\mathfrak{H}$.

If $w = 1$, that is, if $(\mathfrak{N}(\mathfrak{P}) : \mathfrak{C}(\mathfrak{P}))$ has the maximal value $p^n - 1$ possible when assumption (*) is satisfied, we may have blocks $B(\vartheta_\alpha)$ which do not contain any exceptional character of $\mathfrak{G}$. Indeed, this will be so if $r_\alpha = 1$ since, by (4B), the exceptional characters belong to the blocks $B(\vartheta_\alpha)$ with $wr_\alpha > 1$. If $w = 1$ and $r_\alpha = 1$ we shall now pick an irreducible character arbitrarily from $B(\vartheta_\alpha)$, denote it by $\chi^{(\alpha)}$, and from now on count it as the exceptional character of $B(\vartheta_\alpha)$. We show that (4D) remains valid. Since $r_\alpha = 1$, we have $\mathfrak{F}(\vartheta_\alpha) = \mathfrak{H}$. Then T in (4.4) ranges over a complete residue system of $\mathfrak{H}$ (mod $\mathfrak{C}$), $\lambda_\beta{}^{TM}$ ranges over all irreducible characters $\neq 1$ of $\mathfrak{P}$, and $\sum_T \lambda_\beta{}^{TM}(P) = -1$ for every $P \neq 1$ in $\mathfrak{P}$. By (4E) applied to $\chi_i = \chi^{(\alpha)}$, $d_{ij}{}^P$ is a rational integer which for $\vartheta_j = \vartheta_\alpha{}^M$ is independent of P ($P \neq 1$) and M. Hence (4.4) is true, if we take $\epsilon_\alpha = 1$ and

$$b_\alpha^{(\alpha)} = d_{ij}^P + 1.$$

(4F) *If $B(\vartheta_\alpha)$ is a block of $\mathfrak{G}$ of full defect and if $B(\vartheta_\alpha)$ contains l_α modular irreducible characters of $\mathfrak{G}$, then $B(\vartheta_\alpha)$ contains $wr_\alpha + l_\alpha$ ordinary irreducible characters: wr_α exceptional ones and l_α non-exceptional ones.*

Proof. As shown in (7, (7D)), the total number of irreducible characters in $B(\vartheta_\alpha)$ is obtained by letting P range over a system of representatives of the conjugate classes of order a power of p, determining for each the number $l_\alpha(P)$ of modular irreducible characters of $\mathfrak{C}(P)$ corresponding to $B(\vartheta_\alpha)$ and taking $\sum_P l_\alpha(P)$. We know that for $P \neq 1$ the modular irreducible characters of $\mathfrak{C}(P)$ corresponding to $B(\vartheta_\alpha)$ are the r_α characters $\vartheta_\alpha{}^N \in \{\vartheta_\alpha\}$. By (3.3) there are w such representatives P. For $P = 1$ we have $l_\alpha = l_\alpha(1)$ by definition. Thus $\sum_P l_\alpha(P) = wr_\alpha + l_\alpha$. This yields the first statement. According to (4B), wr_α of these characters are exceptional; in the case $wr_a = 1$ this was a matter of definition. The remaining l_α characters then must be non-exceptional.

In particular, (4F) shows that every block $B(\vartheta_\alpha)$ contains non-exceptional irreducible characters.

As a corollary of the results of §§ 2 and 3, we have

(4G) *All exceptional characters in a block $B(\vartheta_\alpha)$ take the same value for p-regular elements $G \in \mathfrak{G}$. In particular, they all have the same degree.*

For the degrees of the characters, we have congruences mod p^n:

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(4H) *If χ_i is a non-exceptional character in $B(\vartheta_\alpha)$ and b_i has the same significance as in* (4E), *then* $\mathrm{Dg}\,\chi_i \equiv b_i r_\alpha \,\mathrm{Dg}\,\vartheta_\alpha \pmod{p^n}$. *In particular* $b_i \neq 0$. *If the notation is as in* (4D), *then*

$$\mathrm{Dg}\,\chi_\beta^{(\alpha)} \equiv \epsilon_\alpha r_\alpha s_\alpha \,\mathrm{Dg}\,\vartheta_\alpha + b_\alpha^{(\alpha)} r_\alpha \,\mathrm{Dg}\,\vartheta_\alpha \qquad \pmod{p^n}.$$

Proof. For the non-exceptional character, χ_i, it follows from (4E) for $V = 1$, $P \in \mathfrak{P}$, $P \neq 1$ that $\chi_i(P) = b_i r_\alpha \,\mathrm{Dg}\,\vartheta_\alpha$. Now (3.12) can be applied. Since $\mathrm{Dg}\,\chi_i$ cannot be divisible by p^n, we must have $b_i \neq 0$.

In the case of $\chi_\beta^{(\alpha)}$ we apply (3.10) for $V = 1$. Since $\mathrm{Dg}\,(\lambda_\beta \vartheta_\alpha)^* = (\mathfrak{H} : \mathfrak{C})\,\mathrm{Dg}\,\vartheta_\alpha$, the left side becomes $\mathrm{Dg}\,\chi_\beta^{(\alpha)} - \epsilon_\alpha r_\alpha s_\alpha \,\mathrm{Dg}\,\vartheta_\alpha$. Using (3.9) and the fact that $b_\rho^{(\alpha)} = 0$ for $\rho \neq \alpha$, we can write the right side in the form $b_\alpha^{(\alpha)} \zeta_\alpha(1) = b_\alpha^{(\alpha)} r_\alpha \,\mathrm{Dg}\,\vartheta_\alpha$, and the last part of (4H) is obtained.

We may also use (7, (3A)). Since for $P \in \mathfrak{P}$, $P \neq 1$, the Cartan matrix of each block of $\mathfrak{C}(P) = \mathfrak{P} \times \mathfrak{B}$ is (p^n), we find for $\phi_j^P = \vartheta_\alpha^M$,

$$\sum_i |d_{ij}^P|^2 = p^n,$$

and because of the orthogonality of these d-columns, we find

$$\sum_i \left| \sum_P \sum_j d_{ij}^P \right|^2 = w r_\alpha p^n, \tag{4.5}$$

where P ranges over a system of representatives of the p-classes $\neq 1$ of $\mathfrak{G}$ and for each P, j ranges over the r_α values belonging to the r_α characters $\vartheta_\alpha^M \in \{\vartheta_\alpha\}$. If χ_i is non-exceptional then by (4E)

$$\sum_P \sum_j d_{ij}^P = w r_\alpha b_i. \tag{4.6}$$

For $\chi_i = \chi_\beta^{(\alpha)}$, by (4.4)

$$\sum_P \sum_j d_{ij}^P = \epsilon_\alpha \sum_P \sum_{N \epsilon R} \lambda_\beta^N(P) + w r_\alpha b_\alpha^{(\alpha)} \tag{4.7}$$

since T in (4.4) ranges over a residue system of $\mathfrak{F}(\vartheta_\alpha) \pmod{\mathfrak{C}}$ while M has to range over a residue system of $\mathfrak{H} \pmod{\mathfrak{F}(\vartheta_\alpha)}$. As N ranges over a residue system R of $\mathfrak{H} \pmod{\mathfrak{C}}$, NPN^{-1} ranges over the set of elements of $\mathfrak{P}$ conjugate to P in $\mathfrak{G}$, and the first term on the right in (4.7) can be written as $\epsilon_\alpha \sum_Q \lambda_\beta(Q)$, where Q ranges over $\mathfrak{P} - \{1\}$. Thus (4.7) becomes

$$\sum_P \sum_j d_{ij}^P = -\epsilon_\alpha + w r_\alpha b_\alpha^{(\alpha)}. \tag{4.8}$$

Since we have $w r_\alpha$ characters $\chi_\beta^{(\alpha)}$ in $B(\vartheta_\alpha)$, (4.5) becomes

$$w^2 r_\alpha^2 \sum_i{}' \, b_i^2 + w r_\alpha (w r_\alpha b_\alpha^{(\alpha)} - \epsilon_\alpha)^2 = w r_\alpha p^n$$

where in $\sum'$ the index i ranges over the l_α values belonging to the non-exceptional characters in $B(\vartheta_\alpha)$. Using (3.3) and (3.4) we can write this in the form

$$\sum_i{}' \, b_i^2 + w r_\alpha b_\alpha^{(\alpha)2} - 2\epsilon_\alpha b_\alpha^{(\alpha)} = s_\alpha$$

or

$$(4.9) \qquad \sum_i{}' b_i^2 + (wr_\alpha - 1)b_\alpha^{(\alpha)2} + (b_\alpha^{(\alpha)} - \epsilon_\alpha)^2 = s_\alpha + 1.$$

At least one of the numbers $b_\alpha^{(\alpha)}$, $b_\alpha^{(\alpha)} - \epsilon_\alpha$ is not zero. Since by (4H) $b_i \neq 0$, it follows for $wr_\alpha \neq 1$ that $\sum_i{}' b_i^2$ has at most s_α terms. If $wr_\alpha = 1$ then $s_\alpha = p^n - 1 \equiv -1 \pmod{p^n}$, and since $\mathrm{Dg}\,\chi^{(\alpha)} \not\equiv 0 \pmod{p^n}$, (4H) shows that $b_\alpha^{(\alpha)} - \epsilon_\alpha \not\equiv 0 \pmod{p^n}$. Hence the first sum again contains at most s_α terms. Thus

(4I) *The number l_α of non-exceptional characters χ_i in $B(\vartheta_\alpha)$ is at most s_α.*

It follows from (4.9) that if $b_\alpha^{(\alpha)} \neq 0$ then

$$wr_\alpha - 1 \leqslant 1 - l_\alpha + (p^n - 1)/(wr_\alpha).$$

Finally, it follows from the orthogonality relations for decomposition numbers (**7**, (3A)) and (4.6), (4.8), and (4G) that we have

(4J) *If G is a p-regular element of $\mathfrak{G}$, then for every β,*

$$\sum{}' b_i\chi_i(G) + (wr_\alpha b_\alpha^{(\alpha)} - \epsilon_\alpha)\chi_\beta^{(\alpha)}(G) = 0$$

where χ_i ranges over the non-exceptional characters of $B(\vartheta_\alpha)$.

5. Estimates for the degrees of the representations of $\mathfrak{G}$. We use our results to give lower estimates for the degrees of the irreducible representations of a group $\mathfrak{G}$ which satisfies condition (*). The following statement is obvious from (4A).

(5A) *If the irreducible character χ of $\mathfrak{G}$ does not belong to a block of defect n, the degree* $\mathrm{Dg}\,\chi$ *is divisible by p^n.*

We next consider the irreducible characters χ_i in the block $B(\vartheta_\alpha)$ of full defect.

(5B) *Let χ_i be a non-exceptional character belonging to $B(\vartheta_\alpha)$. Then either* $\mathrm{Dg}\,\chi_i \geqslant p^n - 1$ *or $\mathfrak{P}$ is included in the kernel $\mathfrak{K}(\chi_i)$ of χ_i (that is, the kernel of the representation of $\mathfrak{G}$ with the character χ_i).*

Proof. Let b_i have the same significance as in (4E). By (4.9), $|b_i| < p^n$. It follows from (4E) that

$$\chi_i(P) = b_i r_\alpha \,\mathrm{Dg}\,\vartheta_\alpha, \qquad (P \in \mathfrak{P}, P \neq 1).$$

Now the orthogonality relations for group characters show that

$$(5.1) \qquad \chi_i(1) + (p^n - 1)b_i r_\alpha \,\mathrm{Dg}\,\vartheta_\alpha = ap^n$$

where a is the multiplicity of the principal character in $\chi_i|\mathfrak{P}$; $a \geqslant 0$. If $b_i < 0$, it follows from (5.1) that

$$\chi_i(1) \geqslant (p^n - 1)|b_i|r_\alpha \,\mathrm{Dg}\,\vartheta_\alpha \geqslant p^n - 1.$$

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If $b_i > 0$, we have $\chi_i(1) \geqslant \chi_i(P) > 0$ for all $P \in \mathfrak{P}$. According to (4H), $\chi_i(1) \equiv \chi_i(P) \pmod{p^n}$. If $\chi_i(1) > \chi_i(P)$ for some P, we have $\operatorname{Dg} \chi_i > p^n$. On the other hand, if $\chi_i(P) = \chi_i(1)$ for all $P \in \mathfrak{P}$, then, as is well known, $\mathfrak{P}$ is included in the kernel $\mathfrak{K}(\chi_i)$.

We now turn to a discussion of the exceptional characters $\chi_\beta^{(\alpha)}$. If $wr_\alpha = 1$, then $\chi_\beta^{(\alpha)}$ is not truly exceptional and (5B) still applies.

(5C) *Let* $\chi = \chi_\beta^{(\alpha)}$ *be an exceptional character of* $B(\vartheta_\alpha)$, *with* $wr_\alpha > 1$. *If* $\epsilon_\alpha = -1$, *then* $\operatorname{Dg} \chi \geqslant \frac{1}{2}(p^n - 1)$. *If* $\epsilon_\alpha = 1$ *and* $\operatorname{Dg} \chi < p^n$ *then* $b_\alpha^{(\alpha)} \geqslant 0$ *and*

$$\chi | \mathfrak{C} = \sum_{N \in R} \lambda_\beta^N \vartheta_\alpha^N + b_\alpha^{(\alpha)} \sum_M \vartheta_\alpha^M \tag{5.2}$$

where M *ranges over a residue system of* $\mathfrak{H}$ $(\operatorname{mod} \mathfrak{F}(\vartheta_\alpha))$. *In particular*

$$\operatorname{Dg} \chi = (s_\alpha + b_\alpha^{(\alpha)}) r_\alpha \operatorname{Dg} \vartheta_\alpha. \tag{5.3}$$

Proof. If $\epsilon_\alpha = -1$ then $a(F(\vartheta_\alpha), F(\vartheta_\alpha)) \geqslant 1$ in (3.8). Indeed, in (2.1) S must contain ψ_i as a constituent when $\epsilon = -1$. Since $(\lambda_\beta \vartheta_\alpha)^*$ has degree $r_\alpha s_\alpha \operatorname{Dg} \vartheta_\alpha$ and since $F(\vartheta_\alpha)$ consists of wr_α characters of this degree,

$$\operatorname{Dg} \chi \geqslant (wr_\alpha - 1) r_\alpha s_\alpha \operatorname{Dg} \vartheta_\alpha \geqslant \tfrac{1}{2} wr_\alpha r_\alpha s_\alpha \operatorname{Dg} \vartheta_\alpha = \tfrac{1}{2}(p^n - 1) r_\alpha \operatorname{Dg} \vartheta_\alpha$$

whence the first statement.

Assume now $\epsilon_\alpha = 1$. If in (3.8) $a(F(\vartheta_\alpha), F(\vartheta_{\alpha'})) \neq 0$ for some α', an argument similar to the one used for $\epsilon_\alpha = -1$ shows that $\operatorname{Dg} \chi \geqslant 1 + (p^n - 1) = p^n$. If this is not so, then in (3.8) $a(F(\vartheta_\alpha), F)$ vanishes except for families F consisting of one irreducible character belonging to a set $U(\vartheta_\rho)$. If $b_\rho^{(\alpha)}$ has the same significance as in (3.9), we have here

$$\chi | \mathfrak{C} = \sum_{N \in R} \lambda_\beta^N \vartheta_\alpha^N + \sum_\rho b_\rho^{(\alpha)} \zeta_\rho$$

and all $b_\rho^{(\alpha)}$ are non-negative. Since $b_\rho^{(\alpha)} = 0$ for $\rho \neq \alpha$, this yields (5.2) and (5.3).

(5D) *Let* $\mathfrak{G}$ *be a group which satisfies assumption* (*) *for a prime* p *and let* p^n *be the order of the* p*-Sylow group. If* $\mathfrak{G}$ *has an irreducible character* $\chi \neq 1$ *of degree* $x < (p^n - 1)^{\frac{1}{2}}$, *then either* $\mathfrak{G}$ *has a proper normal subgroup* $\mathfrak{K} \supseteq \mathfrak{P}$, *or* $\mathfrak{G}$ *has a normal subgroup* $\mathfrak{M}$ *of index* p^n *and* $x = 1$.

Proof. It follows from (5A), (5B), and (5C), that we may assume χ is of the form $\chi = \chi_\beta^{(\alpha)}$ and $wr_\alpha > 1$. Then by (4G) $\chi_1^{(\alpha)} - \chi_2^{(\alpha)}$ vanishes for all p-regular elements. Let ϑ_1 denote the principal character of $\mathfrak{B}$. It follows from (4C) that the principal character 1 of $\mathfrak{G}$ belongs to $B(\vartheta_1)$. We distinguish two cases:

Case I. $B(\vartheta_1)$ contains an irreducible non-exceptional character $\chi_i \neq 1$. If b_i has the same significance as in (4E), then $\chi_i - b_i$ vanishes for all p-singular elements of $\mathfrak{G}$. Hence

$$(\chi_1^{(\alpha)} - \chi_2^{(\alpha)})(\chi_i - b_i) = 0$$

identically; that is,

(5.4) $$\chi_1^{(\alpha)}\chi_i + b_i\chi_2^{(\alpha)} = \chi_2^{(\alpha)}\chi_i + b_i\chi_1^{(\alpha)}.$$

If $b_i > 0$ then $\chi_1^{(\alpha)}$ is a constituent of $\chi_1^{(\alpha)}\chi_i$. Using the orthogonality relations, we see that χ_i is a constituent of $\chi_1^{(\alpha)}\bar{\chi}_1^{(\alpha)}$. Hence

(5.5) $$\mathrm{Dg}\, \chi_i \leqslant (\mathrm{Dg}\, \chi)^2 = x^2 < p^n - 1.$$

Hence by (5B) the kernel $\mathfrak{K}(\chi_i)$ includes $\mathfrak{P}$, and, as $\chi_i \neq 1$, $\mathfrak{K}(\chi_i) \neq \mathfrak{G}$, we have the first alternative in (5D).

If $b_i < 0$, it follows from (5.4) that $\chi_1^{(\alpha)}$ is a constituent of $\chi_2^{(\alpha)}\chi_i$. Then χ_i is a constituent of $\chi_1^{(\alpha)}\bar{\chi}_2^{(\alpha)}$. This implies (5.5) and again (5D) holds. (Actually, the proof of (5B) shows that this case is impossible under our assumptions.)

Case II. The only irreducible non-exceptional character in $B(\vartheta_1)$ is the principal character of $\mathfrak{G}$. Then (4F) shows that $B(\vartheta_1)$ contains only one irreducible modular character. This implies that $\mathfrak{G}$ has a normal subgroup $\mathfrak{M}$ of index p^n **(8**, p. 587**)**. It follows at once that $\mathfrak{N}(\mathfrak{P}) = \mathfrak{C}(\mathfrak{P})$, that is, that $\mathfrak{H} = \mathfrak{C}$. In particular,

$$r_\rho = s_\rho = 1 \text{ for all } \rho, \qquad w = p^n - 1.$$

Since $s_\alpha = 1$, (4.9) shows that for $p^n > 3$, we must have $b_\alpha^{(\alpha)} = 0$. Moreover $B(\vartheta_\alpha)$ then contains only one non-exceptional character χ_i, and for this χ_i we have $b_i = \pm 1$. As shown by (4J) then $\mathrm{Dg}\, \chi_i = \mathrm{Dg}\, \chi_\beta^{(\alpha)} = x$. If $\vartheta_\alpha \neq \vartheta_1$, χ_i is not the principal character. Since $\mathrm{Dg}\, \chi_i < (p^n - 1)^{\frac{1}{2}}$, (5B) shows that $\mathfrak{G}$ has a proper normal subgroup $\mathfrak{K}(\chi_i) \supseteq \mathfrak{P}$. If $\vartheta_\alpha = \vartheta_1$ then $\chi_i = 1$ and we have $x = 1$.

It remains to discuss the cases $p^n \leqslant 3$. If $p^n = 2$ then $wr_\alpha = 1$, a contradiction. If $p^n = 3$ and $b_\alpha^{(\alpha)} = 0$, the argument above applies. If $b_\alpha^{(\alpha)} \neq 0$ then (4.9) shows that $b_\alpha^{(\alpha)} = \epsilon_\alpha$ and that $B(\vartheta_\alpha)$ still contains only one non-exceptional χ_i and that $b_i = \pm 1$. Then (4J) shows that $\mathrm{Dg}\, \chi_i = x$ and we can conclude the proof as in the case $p^n > 3$.

(5E) *Let $\mathfrak{G}$ be a finite group which satisfies the assumption* (*) *for some prime p. If the p-Sylow group $\mathfrak{P}$ has order p^n and if $\mathfrak{G}$ has a faithful representation $\mathfrak{X}$ of degree $x < (p^n - 1)^{\frac{1}{2}}$, then $\mathfrak{P}$ is normal in $\mathfrak{G}$.*

Proof. We may assume that (5E) has been proved for groups of smaller order than $\mathfrak{G}$. Since $\mathfrak{X}$ must have an irreducible constituent which is not the principal representation, (5D) applies. If $\mathfrak{G}$ has a proper normal subgroup $\mathfrak{K} \supseteq \mathfrak{P}$, application of (5E) to $\mathfrak{K}$ shows that $\mathfrak{P}$ is normal in $\mathfrak{K}$. Since $\mathfrak{P}$ then is characteristic in $\mathfrak{K}$, it is normal in $\mathfrak{G}$.

It remains to deal with the case that $\mathfrak{G}$ has a normal subgroup $\mathfrak{M}$ of index p^n and that all irreducible constituents of $\mathfrak{X}$ have degree 1. Since $\mathfrak{X}$ is a faithful representation, $\mathfrak{G}$ then is abelian, and, of course, $\mathfrak{P}$ is normal in $\mathfrak{G}$.

The case $n = 1$ has been studied elsewhere **(4)**. If $n \geqslant 2$, it follows from (5E) that if the order g of a finite group $\mathfrak{G}$ is divisible by the square p^2 of a prime p and if $\mathfrak{G}$ has a faithful representation $\mathfrak{X}$ of degree $x < p$, then either the p-Sylow group $\mathfrak{P}$ is normal in $\mathfrak{G}$, or there exist elements $P \neq 1$ in $\mathfrak{P}$ such that $\mathfrak{C}(P) \neq \mathfrak{C}(\mathfrak{P})$.

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Harvard University
and
Carnegie Institute of Technology