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Global Units Modulo Circular Units: Descent Without Iwasawa's Main Conjecture

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Abstract. Iwasawa's classical asymptotical formula relates the orders of the p-parts X_n of the ideal class groups along a $\mathbb{Z}_p$-extension F_∞/F of a number field F to Iwasawa structural invariants λ and μ attached to the inverse limit $X_\infty = \varprojlim X_n$. It relies on "good" descent properties satisfied by X_n. If F is abelian and F_∞ is cyclotomic, it is known that the p-parts of the orders of the global units modulo circular units U_n/C_n are asymptotically equivalent to the p-parts of the ideal class numbers. This suggests that these quotients U_n/C_n, so to speak unit class groups, also satisfy good descent properties. We show this directly, *i.e.,* without using Iwasawa's Main Conjecture.

Introduction

Let K be a number field and p an odd prime ($p \neq 2$) and let K_∞/K be a $\mathbb{Z}_p$-extension (quite soon K_∞/K will be the cyclotomic $\mathbb{Z}_p$-extension). Recall the usual notations: $\Gamma = \mathrm{Gal}(K_\infty/K)$ is the Galois group of K_∞/K, K_n is the n-th layer of K_∞ (so that $[K_n : K] = p^n$), $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$, and $G_n = \mathrm{Gal}(K_n/K) \cong \Gamma/\Gamma_n$. Let us consider a sequence $(M_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}_p[G_n]$-modules equipped with norm maps $M_n \longrightarrow M_{n-1}$ and the inverse limit of this sequence $M_\infty = \varprojlim M_n$ seen as a $\Lambda = \mathbb{Z}_p[[\Gamma]]$-module. The general philosophy of Iwasawa theory is to study the simpler Λ-structure of M_∞, then to try and recollect information about the M_n's themselves from that structure. For instance, if M_∞ is Λ-torsion, one can attach two invariants λ and μ to M_∞. If we assume further that

(i) the Γ_n coinvariants $(M_\infty)_{\Gamma_n}$ are finite,
(ii) the sequence $(M_n)_{n\in\mathbb{N}}$ behaves "nicely" viz descent,

then one can prove that the orders of the M_n are asymptotically equivalent to $p^{\lambda n+\mu p^n}$. These two assumptions are expected to occur whenever one chooses for (M_n) significant modules (from the number theoretic point of view). However proving them may require some effort. Actually, (i) for the canonical Iwasawa module $\mathrm{tor}_\Lambda(\mathfrak{X}_\infty)$ is (one of the many equivalent formulations of) Leopoldt's conjecture. The historical example occurs when we specialize $M_n = X_n$, the p-part of ideal class group of K_n. Then the asymptotic formulas are a theorem of Iwasawa. The proof of this theorem uses an auxiliary module Y_∞ which is pseudo-isomorphic to X_∞ but with better descent properties.

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In this paper we are interested in similar statements for unit class groups, that is, for $M_n = \overline{U}_n/\overline{C}_n$, the p-part of the quotient of units modulo the subgroup of circular units of K_n. In order for the latter merely to exist we need to assume that all K_n are abelian over $\mathbb{Q}$; hence K is abelian and K_∞/K is cyclotomic. Let us assume further that K is totally real, which is not a loss of generality as long as we are only interested in $p \neq 2$. Now by the Sinnott index formulas we know that the orders of $\overline{U}_n/\overline{C}_n$ are asymptotically equivalent to the orders of X_n. And as a consequence of Iwasawa's Main Conjecture, the structural invariants λ and μ of both inverse limits $\overline{U}_\infty/\overline{C}_\infty = \varprojlim \overline{U}_n/\overline{C}_n$ and of $X_\infty = \varprojlim X_n$ are equal. Using these two theorems we have immediately a somewhat indirect proof of an analogue of Iwasawa's theorem for unit class groups. Clearly a direct proof of this fact must exist and the first goal of this paper is to write it down. Theorem 5.3 shows that the orders of the unit class groups $\overline{U}_n/\overline{C}_n$ along the finite steps of the $\mathbb{Z}_p$-extension are those prescribed by the structural Iwasawa invariants of the inverse limit $\overline{U}_\infty/\overline{C}_\infty$. It makes no use of any precise link between ideal and unit class groups, *e.g.,* it does not use the Main Conjecture or even Sinnott's index formula. On the other hand if one does use some classical results together with Theorem 5.3, one gets easier proofs of beautiful and well known theorems (see §6). For instance Sinnott's index formulas together with Theorem 5.3 imply the equality of the λ and μ invariants of the two class groups without using Iwasawa Main Conjecture. The equality between these two λ invariants is actually an important step (sometimes called "class number trick") in the proof of Iwasawa's Main Conjecture. If we use further Ferrero and Washington's theorem, we prove that all the μ-invariants involved here are trivial. Maybe some ideas in the present approach could be used in a different framework where the equality of the two characteristic ideals (that is Iwasawa's Main Conjecture) is still open. In [N2, §5], T. Nguyen Quang Do has also proved that unit classes have asymptotically good descent properties by using Iwasawa's Main Conjecture in its strongest form, that is, following the path outlined just above.

We conclude this introduction by recalling the (now) traditional notations of cyclotomic Iwasawa theory. For any number field F we put $S = S(F)$ for the set of places of F dividing p. We will adopt the following notations:

- For any abelian group A, we'll denote by $\overline{A}$ the p-completion of A, *i.e.,* the inverse limit $\overline{A} = \varprojlim A/A^{p^m}$. If A is finitely generated over $\mathbb{Z}$, then $\bar{A} \cong A \otimes \mathbb{Z}_p$.
- U_F is the group of units of F.
- U'_F is the group of S-units of F, that is, elements of $F^\times$ whose valuations are trivial for all finite places v of F such that $v \notin S$.
- X_F is the p-part of $\mathrm{cl}(F)$ which in turn is the ideal class group of F.
- X'_F is the p-part of $\mathrm{cl}'(F)$ which in turn is the quotient of $cl(F)$ modulo the subgroup generated by classes of primes in S.
- $\mathfrak{X}_F$ is the Galois group over F of the maximal S-ramified (*i.e.,* unramified outside S) abelian pro-p-extension of F.
- $\mathcal{N}_F$ is the multiplicative group of the semi-local numbers. As a mere $\mathbb{Z}_p$-module $\mathcal{N}_F = \prod_{v \in S} \overline{F_v^\times}$, where F_v is the completion of F at the place $v \in S$.

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- $\mathcal{U}_F$ is the group of semi-local units of F. As a mere $\mathbb{Z}_p$-module $\mathcal{U}_F = \prod_{v\in S} U_v^1$, where U_v^1 is the set of principal units of F_v, that is, units $\equiv 1$ modulo the maximal ideal of F_v.

So for instance the $\overline{U}_n$'s above could have been understood as $U_{K_n} \otimes \mathbb{Z}_p$ and $X_F = \overline{\mathrm{cl}(F)}$.

1 Consequences of Leopoldt's Conjecture

If F is Galois over $\mathbb{Q}$, the groups $\mathcal{U}_F$ and $\mathcal{N}_F$ come equipped with the induced action of $\mathrm{Gal}(F/\mathbb{Q})$. In other words if we fix one place v in S and if we put $G_v = \mathrm{Gal}(F_v/\mathbb{Q}_p)$, then we have (as $\mathbb{Z}_p$-modules):

$$\mathcal{U}_F \cong U_v^1 \otimes_{\mathbb{Z}_p[G_v]} \mathbb{Z}_p[\mathrm{Gal}(F/\mathbb{Q})] \text{ and } \mathcal{N}_F \cong \overline{F_v^\times} \otimes_{\mathbb{Z}_p[G_v]} \mathbb{Z}_p[\mathrm{Gal}(F/\mathbb{Q})].$$

These isomorphisms define the action of $\mathrm{Gal}(F/\mathbb{Q})$ on $\mathcal{U}_F$ and on $\mathcal{N}_F$. Let us now consider the cyclotomic $\mathbb{Z}_p$-extension K_∞/K of our totally real abelian over $\mathbb{Q}$ number field K and the n-th finite layers K_n/K, *i.e.*, K_n is the unique subfield of K_∞ with $[K_n : K] = p^n$. We will denote by C_n (and of course we will be more interested in $\overline{C}_n$) the group of circular units of K_n as defined by Sinnott [Si]. We will consistently indicate arithmetical objects related to K_n by the subscript $_n$. So $\overline{U}_n$, $\mathcal{U}_n$ and so on make sense. For any extension L/F of global fields, the formula

$$N_{L/F}((x_w)_{w|p}) = \Big(\prod_{w|v} N_{L_w/K_v}(x_w)\Big)_{v|p}$$

defines a Galois equivariant morphism $N_{L/F}\colon \mathcal{N}_L \longrightarrow \mathcal{N}_F$ which is compatible with the usual norms on global units for instance. As ever $\overline{U}_\infty$, $\overline{C}_\infty$ $\mathcal{N}_\infty$, $\mathcal{U}_\infty$, $\overline{U}'_\infty$ and so on denotes the inverse limit (related to norm maps) of $\overline{U}_n$, $\overline{C}_n$, $\mathcal{N}_n$, $\mathcal{U}_n$, $\overline{U}'_n$ and so on. Before considering more precise properties of descent kernels (and cokernels) of unit classes, we need to first establish their finiteness. This can surely be extracted from various and older references as a consequence of Leopoldt's conjecture (which is true here since all K_n are abelian over $\mathbb{Q}$). However a more precise and general result on quotients of semi-local units modulo circular units can be found in [T]. The point for our present approach is that the proofs of [T] do not make any use of the Main Conjecture and only need Coleman morphisms.

We will need and freely use the following consequence of [T, Theorem 3.1].

Theorem 1.1 *Let $G = \mathrm{Gal}(K/\mathbb{Q})$. Recall that $\Gamma = \mathrm{Gal}(K_\infty/K)$ and that $\Lambda = \mathbb{Z}_p[[\Gamma]]$. Fix a generator γ of Γ. For all non-trivial Dirichlet characters of the first kind ψ of $\widehat{G}$, following* [T], *let us denote by $g_\psi(T)$ the Iwasawa power series associated to the Kubota–Leopoldt p-adic L-function $L_p(\psi, s)$ (with our fixed choice of γ).*

(i) *Assume p is tamely ramified in K so that every characters of G are of the first kind. Then up to a power of p, the characteristic ideal of the Λ-torsion module $\mathcal{U}_\infty/\overline{C}_\infty$ is generated by the product $\prod_{\psi\in\widehat{G},\psi\neq 1} g_\psi(T)$.*

(ii) *For all $n \in \mathbb{N}$, the Γ_n-coinvariants $(\mathcal{U}_\infty/\overline{C}_\infty)_{\Gamma_n}$ and $(\overline{U}_\infty/\overline{C}_\infty)_{\Gamma_n}$ are finite.*

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Proof Let us consider ψ-parts M^ψ as defined in [T] for any $\mathbb{Z}_p[G]$-module M. These ψ parts are naturally $\mathbb{Z}_p[\psi]$-modules and $\mathbb{Z}_p[G]$-submodules of M. For technicalities about ψ-components which are often left to readers, see Beliaeva's thesis [Bt]. From [T, Theorem 3.1] we use that $\mathcal{U}_\infty^\psi/\overline{C}_\infty^\psi$ is $\mathbb{Z}_p$-torsion free and that its characteristic series over $\Lambda[\psi]$ is $g_\psi(T)$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_p[\psi]/\mathbb{Q}_p)$; then $(\mathcal{U}_\infty/\overline{C}_\infty)^{\psi^\sigma}$ is isomorphic to $(\mathcal{U}_\infty/\overline{C}_\infty)^\psi$ and $g_{\psi^\sigma}(T) = \sigma(g_\psi(T))$. For R equal to Λ or to $\Lambda[\psi]$, let us abbreviate by $\mathrm{Char}_R(M)$ the characteristic ideal of the R-module M. It is an easy linear algebra exercise to check that if a power series f generates $\mathrm{Char}_{\Lambda[\psi]}(M)$, then $N_{\mathbb{Q}_p[\psi]/\mathbb{Q}_p}(f)$ generates $\mathrm{Char}_\Lambda(M)$. Hence we get

$$\mathrm{Char}_\Lambda((\mathcal{U}_\infty^\psi/\overline{C}_\infty^\psi)) = (N_{\mathbb{Q}_p[\psi]/\mathbb{Q}_p}(g_\psi(T))) = \Big(\prod_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p[\psi]/\mathbb{Q}_p)} g_{\psi^\sigma}(T)\Big).$$

Let us fix a set Ψ of representatives of $\widehat{G}$ up to $\mathbb{Q}_p$-conjugation classes. If a $\mathbb{Z}_p[G]$-module M is $\mathbb{Z}_p$-torsion free then the submodules of M, $(M^\psi)_{\psi\in\Psi}$, are mutually direct summands and the quotient $M/\oplus_{\psi\in\Psi} M^\psi$ is annihilated by $\#G$. This applies to $\mathcal{U}_\infty$ and $\overline{C}_\infty$, so that we may use the inclusions $\bigoplus_{\psi\in\Psi}\mathcal{U}_\infty^\psi \subset \mathcal{U}_\infty$ and $\bigoplus_{\psi\in\Psi}\overline{C}_\infty^\psi \subset \overline{C}_\infty$ to form the following snake diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{\psi\in\Psi}\overline{C}_\infty^\psi & \longrightarrow & \bigoplus_{\psi\in\Psi}\mathcal{U}_\infty^\psi & \longrightarrow & \bigoplus_{\psi\in\Psi}\mathcal{U}_\infty^\psi/\overline{C}_\infty^\psi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \alpha & & \\
0 & \longrightarrow & \overline{C}_\infty & \longrightarrow & \mathcal{U}_\infty & \longrightarrow & \mathcal{U}_\infty/\overline{C}_\infty & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z}_p - \text{torsion} & \longrightarrow & \mathbb{Z}_p - \text{torsion} & \longrightarrow & \text{Coker } \alpha & \longrightarrow & 0
\end{array}$$

Now the kernel of α must be $\mathbb{Z}_p$-torsion as a submodule of the cokernel of the first inclusion, and hence is trivial because all $\mathcal{U}_\infty^\psi/\overline{C}_\infty^\psi$ are $\mathbb{Z}_p$-free. This proves that, up to a power of p, the characteristic ideal $\mathrm{Char}_\Lambda(\mathcal{U}_\infty/\overline{C}_\infty)$ is equal to $\mathrm{Char}_\Lambda(\bigoplus_{\psi\in\Psi}\mathcal{U}_\infty/\overline{C}_\infty)$, which is what we wanted.

The second claim is equivalent to the fact that both characteristic ideals are prime to all $(T+1)^{p^n} - 1$ for all $n \in \mathbb{N}$. From the inclusion $\overline{U}_\infty/\overline{C}_\infty \subset \mathcal{U}_\infty/\overline{C}_\infty$ we see that the first characteristic ideal divides the second. Let ζ be a p-power order root of unity, and let ρ be the unique character of the second kind such that $\rho(\gamma) = \zeta^{-1}$. Then we have $g_\psi(\zeta - 1) = L_p(1, \psi\rho)$. By Leopoldt's conjecture, $L_p(1, \psi\rho) \neq 0$, (see [W, Corollary 5.30]). Hence in the special case where p is tame in K, (ii) follows from (i). We now want to remove this hypothesis. Let $I_p \subset \mathrm{Gal}(K_\infty/\mathbb{Q})$ be the inertia subgroup of p, and let $\overline{I}_p \subset I_p$ be its pro-p-part. Then the field $L = K_\infty^{\overline{I}_p}$ is a finite abelian over $\mathbb{Q}$ number field and p is at most tamely ramified in L. Since all modules M_∞ depend on K_∞ and not on K itself, to prove that the hypothesis is almost no loss of generality, we just have to show the following lemma.

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Lemma 1.2 $L_\infty = K_\infty$.

Proof Let $\mathbb{B}_\infty/\mathbb{Q}$ be the $\mathbb{Z}_p$-extension of $\mathbb{Q}$. By construction we have $L \subset K_\infty$ and therefore $L_\infty \subset K_\infty$. By definition of L, the group $\mathrm{Gal}(K_\infty/L_\infty)$ is then a subgroup of $\overline{I}_p$, hence K_∞/L_∞ is a p-extension totally ramified at p. On the other hand, $\mathbb{B}_\infty \subset L_\infty$ and K_∞ is abelian over $\mathbb{Q}$. Let $\mathbb{Q}^{\mathrm{ab}}$ be the maximal abelian extension of $\mathbb{Q}$. Then $\mathbb{Q}^{\mathrm{ab}}/\mathbb{B}_\infty$ is tamely ramified at (the only) prime of $\mathbb{B}_\infty$ above p. This shows $L_\infty = K_\infty$. ■

Now with that lemma we have already proved the finiteness assertion when we take coinvariants along $\widetilde{\Gamma}_n = \mathrm{Gal}(K_\infty/L_n)$. We conclude the proof of the second part of Theorem 1.1 by pointing out that even if the first few Γ_n differ from $\tilde{\Gamma}_n$, we get $\Gamma_n = \tilde{\Gamma}_n$ as soon as K_∞/K_n is totally ramified at p. Due to canonical surjections $(M_\infty)_{\Gamma_n} \twoheadrightarrow (M_\infty)_{\Gamma_{n-1}}$, this does not change the finiteness of $(\mathcal{U}_\infty/\overline{C}_\infty)_{\Gamma_n}$. ■

Remark As long as one is only concerned with characteristic ideals up to a power of p and finiteness of coinvariants, Theorem 1.1 is well known as part of the folklore, and can be traced back to Iwasawa ([I1]; see also [G]) in some cases.

Now that the finiteness of $(\mathcal{U}_\infty/\overline{C}_\infty)_{\Gamma_n}$ (resp. $(\overline{U}_\infty/\overline{C}_\infty)_{\Gamma_n}$) has been proved, we want to relate these modules to their counterparts $\mathcal{U}_n/\overline{C}_n$ (resp. $\overline{U}_n/\overline{C}_n$) at finite levels. On the way we will study descent for various other multiplicative Galois modules.

2 Background: Easy Part of Descent

Let $M_\infty = \varprojlim(M_n)$ be a Λ-module. Of course the projections $M_\infty \longrightarrow M_n$ factor through $(M_\infty)_{\Gamma_n} \longrightarrow M_n$, but in general these maps have non-trivial kernels and cokernels. Let us denote $\mathrm{Ker}_n(M_\infty)$ for the kernel of $(M_\infty)_{\Gamma_n} \longrightarrow M_n$, $\widetilde{M}_n \subset M_n$ for its image, and $\mathrm{Coker}_n(M_\infty) = M_n/\widetilde{M}_n$ for its cokernel. Let l be a prime number ($l = p$ is allowed) and E be a finite extension of $\mathbb{Q}_l$. Put E_∞/E for a $\mathbb{Z}_p$-extension of E, and E_n for the n-th finite layers. Let us abbreviate $L_E = \overline{E^\times}$, consistently $L_n = \overline{(E_n)^\times}$, and $L_\infty = \varprojlim L_n$. For all $n \in \mathbb{N} \cup \infty$, local class field theory identifies L_n with $\mathrm{Gal}(M_n/E_n)$, where M_n is the maximal abelian p-extension of E_n.

Lemma 2.1 *The natural map $(L_\infty)_{\Gamma_n} \longrightarrow L_n$ fits into an exact sequence*

$$0 \longrightarrow (L_\infty)_{\Gamma_n} \longrightarrow L_n \xrightarrow{\text{Artin}} \mathrm{Gal}(E_\infty/E_n) \longrightarrow).$$

Proof This is well known to experts. We follow the cohomological short cut of [N1]. The group $\Gamma_n \simeq \mathbb{Z}_p$ is pro-p-free, hence we have $H^2(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Consider the inflation-restriction sequence associated with the extension of groups

$$1 \longrightarrow \mathcal{H} \longrightarrow \mathcal{G}_n \longrightarrow \Gamma_n \longrightarrow 1,$$

where $\mathcal{G}_n$ is the absolute Galois group of E_n:

$$0 \longrightarrow H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1(\mathcal{G}_n, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n} \longrightarrow 0$$

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Applying Pontryagin duality and class field theory we get

$$0 \longrightarrow (L_\infty)_{\Gamma_n} \longrightarrow L_n \xrightarrow{\text{Artin}} \mathrm{Gal}(E_\infty/E_n) \longrightarrow 0. \qquad \blacksquare$$

Let us come back to our global field K and its cyclotomic $\mathbb{Z}_p$-extension.

Proposition 2.2 *Let S_n be the set of places of K_n dividing p. For all n and $v \in S_n$ put $K_{n,v}$ for the completion at v of K_n and $(K_{n,v})^{\mathrm{cyc}}_\infty/K_{n,v}$ for its cyclotomic $\mathbb{Z}_p$-extension. For all n, the natural map $(\mathcal{N}_\infty)_{\Gamma_n} \longrightarrow \mathcal{N}_n$ fits into an exact sequence*

$$0 \longrightarrow (\mathcal{N}_\infty)_{\Gamma_n} \longrightarrow \mathcal{N}_n \xrightarrow{\oplus_{v\in S_n}\text{Artin at } v} \bigoplus_{v\in S_n} \mathrm{Gal}((K_{n,v})^{\mathrm{cyc}}_\infty/K_{n,v}) \longrightarrow 0.$$

Proof Let us examine the places above p along K_∞/K. These primes have nontrivial conjugated (hence equal) decomposition subgroups. There exists a $d \in \mathbb{N}$ such that

(i) no primes above p splits anymore in K_∞/K_d and
(ii) all primes above p are totally split in K_d/K.

For all $n \geq d$ we then have $S_n \cong S_d$, $\mathcal{N}_n = \bigoplus_{v\in S_d} L_{K_{n,v}}$ and $\mathrm{Gal}(K_\infty/K_n)$ identifies with the local Galois groups $\mathrm{Gal}((K_{n,v})^{\mathrm{cyc}}_\infty/K_{n,v})$ (at every $v \in S_n$). So in the case $n \geq d$, the proposition follows from Lemma 2.1. Next suppose that $0 \leq n < d$, and let G_n^m be the Galois group $\mathrm{Gal}(K_m/K_n)$. Of course we have $(\mathcal{N}_\infty)_{\Gamma_n} \cong ((\mathcal{N}_\infty)_{\Gamma_d})_{G_n^d}$. Using the previous case we have an exact sequence

$$(\dagger) \qquad 0 \longrightarrow (\mathcal{N}_\infty)_{\Gamma_d} \longrightarrow \mathcal{N}_d \longrightarrow \bigoplus_{v\in S_d} \mathrm{Gal}((K_{d,v})^{\mathrm{cyc}}_\infty/K_{d,v}) \longrightarrow 0.$$

As $(K_{d,v})^{\mathrm{cyc}}_\infty/K_{d,v}$ is cyclotomic, the action of the local Galois group $\mathrm{Gal}(K_{d,v}/\mathbb{Q}_p)$ on the group $\mathrm{Gal}((K_{d,v})^{\mathrm{cyc}}_\infty/K_{d,v})$ is trivial. Hence we have an isomorphism of Galois modules

$$\bigoplus_{v\in S_d} \mathrm{Gal}((K_{d,v})^{\mathrm{cyc}}_\infty/K_{d,v}) \cong \mathbb{Z}_p[S_d].$$

Since the places above p split totally in K_d/K, over G_0^d these two modules are cohomologically trivial, and the same is true for $\mathcal{N}_d \cong \mathcal{N}_0 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_0^d]$ (as G_0^d-modules). Therefore in the sequence $(\dagger)$, two (hence all) terms are G_0^d-cohomologically trivial. The triviality of $\widehat{H}^1(G_n^d, (\mathcal{N}_\infty)_{\Gamma_d})$ proves the injectivity of $(\mathcal{N}_\infty)_{\Gamma_n} \longrightarrow \mathcal{N}_n$. It then suffices to apply N_{K_d/K_n} to the sequence $(\dagger)$ and use the triviality of the three $\widehat{H}^0(G_n^d, \cdot\,)$ to get the full sequence

$$0 \longrightarrow (\mathcal{N}_\infty)_{\Gamma_n} \longrightarrow \mathcal{N}_n \xrightarrow{\oplus_{v\in S_n}\text{Artin at } v} \bigoplus_{v\in S_n} \mathrm{Gal}((K_{n,v})^{\mathrm{cyc}}_\infty/K_{n,v}) \longrightarrow 0. \qquad \blacksquare$$

Let d be as before. Let D_n be the decomposition subfield for the place p in K_n (*i.e*, p is totally split in D_n and no place of D_n above p splits anymore in K_n/K). Then for all $n \geq d$, we have $D_n = D_d$. Let us put $D = D_d$ for the decomposition subfield for the place p in K_∞.

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Proposition 2.3 *Recall that for a Λ-module $M_\infty = \varprojlim(M_n)$, we denote* $\mathrm{Ker}_n(M_\infty)$ *for the kernel,* $\widetilde{M}_n$ *for the image and* $\mathrm{Coker}_n(M_\infty)$ *for the cokernel of the natural map* $(M_\infty)_{\Gamma_n} \longrightarrow M_n$.

(i) *For all $n \geq 0$,* $\mathrm{Ker}_n(\mathcal{U}_\infty)$ *is isomorphic to $\mathbb{Z}_p[S_n]$, hence its $\mathbb{Z}_p$-rank is $\#S_n$.*

(ii) *For all $n \geq 0$,* $\mathrm{Coker}_n(\mathcal{U}_\infty) \cong N_{K_n/D_n}(\mathcal{U}_n)$*, hence its $\mathbb{Z}_p$-rank is $\#S_n$. In other words we have an exact sequence*

$$0 \longrightarrow \widetilde{\mathcal{U}}_n \longrightarrow \mathcal{U}_n \xrightarrow{N_{K_n/D_n}} \mathcal{U}_{D_n}.$$

Proof For any local field E, the (normalized) valuation v of E gives an exact sequence $0 \to U_E^1 \to \overline{E^\times} \xrightarrow{v} \mathbb{Z}_p \to 0$, where U_E^1 is the set of principal units of E. Taking the semi-local version of this sequence and projective limits, we obtain

$$0 \longrightarrow \mathcal{U}_\infty \longrightarrow \mathcal{N}_\infty \longrightarrow \mathbb{Z}_p[S_d] \longrightarrow 0.$$

But for all n, $\mathcal{N}_n^\Gamma = \mathcal{N}_0$ contains no infinitely p-divisible element. Therefore $\mathcal{N}_\infty^\Gamma = \{0\}$ and taking Γ_n-cohomology on this sequence we get:

$$0 \longrightarrow \mathbb{Z}_p[S_d]^{\Gamma_n} \longrightarrow (\mathcal{U}_\infty)_{\Gamma_n} \longrightarrow (\mathcal{N}_\infty)_{\Gamma_n} \longrightarrow (\mathbb{Z}_p[S_d])_{\Gamma_n} \longrightarrow 0.$$

Now since $(\mathcal{N}_\infty)_{\Gamma_n} \longrightarrow \mathcal{N}_n$ is a monomorphism, $\mathrm{Ker}_n(\mathcal{U}_\infty)$ is identified with $\mathbb{Z}_p[S_d]^{\Gamma_n} \cong \mathbb{Z}_p[S_n]$. This proves (i).

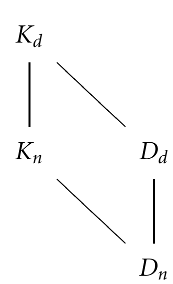
For (ii), we use the notations d, D_n, and D of the proof of Proposition 2.2. We first prove the case $n \geq d$. By compactness we have $\widetilde{\mathcal{U}}_n = \bigcap_{m \geq n} N_{K_m/K_n}(\mathcal{U}_m)$. Because $n \geq d$, the global norm $N_{K_m/K_n} \colon \mathcal{U}_m \longrightarrow \mathcal{U}_n$ is nothing but the direct sum place by place of the local norms of the extensions $K_{m,w}/K_{n,v}$ (at each unique $w \in S_m$ above a fixed $v \in S_n$). Fix a place $v \in S_n$ and for all $m \geq n$ still call v the unique place of S_m above v and also v the unique place of D_n under v. Note that $D_{n,v} = \mathbb{Q}_p$. Let $u \in U^1_{K_{n,v}}$, then using local class field theory we see that $u \in \bigcap_{m \geq n} N_{K_{m,v}/K_{n,v}}(U^1_{K_{m,v}})$ if and only if $N_{K_{n,v}/\mathbb{Q}_p}(u) \in \bigcap_{n \in \mathbb{N}} N_{(\mathbb{Q}_p)_n^{\mathrm{cyc}}/\mathbb{Q}_p}(U^1_{(\mathbb{Q}_p)_n^{\mathrm{cyc}}})$, where $(\mathbb{Q}_p)_n^{\mathrm{cyc}}$ denotes the n-th step of the cyclotomic (hence totally ramified) $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$. By local class field theory $(\,\cdot\,, (\mathbb{Q}_p)_\infty^{\mathrm{cyc}}/\mathbb{Q}_p)$ is an isomorphism from $U^1_{\mathbb{Q}_p}$ to $\mathrm{Gal}((\mathbb{Q}_p)_\infty^{\mathrm{cyc}}/\mathbb{Q}_p)$. The equivalence

$$u \in \bigcap_{m \geq n} N_{K_{m,v}/K_{n,v}}(U^1_{K_{m,v}}) \iff N_{K_{n,v}/D_{n,v}}(u) = 1$$

follows. As no place above p splits in K_n/D_n, this gives

$$\widetilde{\mathcal{U}}_n = \mathrm{Ker}(N_{K_n/D_n} \colon \mathcal{U}_n \longrightarrow \mathcal{U}_{D_n}).$$

Next pick an n such that $0 \leq n < d$. Consider the diagram of fields



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There every prime above p is totally split in the extensions K_d/K_n, and D_d/D_n and no prime above p splits at all in K_n/D_n nor in K_d/D_d. It follows that N_{K_d/K_n} is surjective onto $\mathcal{U}_n$ and that

$$\widetilde{\mathcal{U}}_n = N_{K_d/K_n}(\widetilde{\mathcal{U}}_d) \subset \operatorname{Ker}(N_{K_n/D_n} : \mathcal{U}_n \longrightarrow \mathcal{U}_{D_n}).$$

Conversely pick $u = (u_v)_{v\in S_n} \in \operatorname{Ker}(N_{K_n/D_n} : \mathcal{U}_n \longrightarrow \mathcal{U}_{D_n})$. At each $v \in S_n$ choose a single $w(v) \in S_d$ above v and define $t = (t_w)_{w\in S_d} \in \mathcal{U}_d$ by putting $t_w = 1$ if there does not exist v such that $w = w(v)$ and $t_{w(v)} = u_v$ for all v in S_n. Then we have $N_{K_d/K_n}(t) = u$ and $t \in \widetilde{\mathcal{U}}_d$ which shows that $u \in \widetilde{\mathcal{U}}_n$. ■

Similar but not so precise statements about the sequence of global units could be deduced from Proposition 2.3 and from the following Proposition 2.4. As they are not needed we don't state them. To end this section we recall the analogous proposition for the sequence $\overline{U}'_n$ of (p)-units, which is a result of Kuz′min [K, Theorems 7.2 and 7.3]. Recall that K is totally real, so that $r_1 = [K : \mathbb{Q}]$, $r_2 = 0$, and all $\overline{U}'_n$ are $\mathbb{Z}_p$-torsion free. To avoid ugly notations $\widetilde{\overline{M}}$ we will let $\widetilde{U}'_n$ denote the image of $\overline{U}'_\infty$ in $\overline{U}'_n$.

Proposition 2.4 (i) *The Λ-module $\overline{U}'_\infty$ is free of rank $[K:\mathbb{Q}]$.*
(ii) *For all n, the natural map $(\overline{U}'_\infty)_{\Gamma_n} \longrightarrow \overline{U}'_n$ is injective.*
(iii) *$\widetilde{U}'_n \cong (\overline{U}'_\infty)_{\Gamma_n}$ is a free $\mathbb{Z}_p[G_n]$-module of rank $[K:\mathbb{Q}]$ (hence is $\mathbb{Z}_p$ free of rank $p^n[K:\mathbb{Q}] = [K_n:\mathbb{Q}]$).*

Proof (i) is Theorem 7.2 and (ii) is Theorem 7.3 of [K]. There the number field K is arbitrary and a considerable amount of effort is made to avoid using Leopoldt's conjecture. Another proof (also without using Leopoldt's conjecture) is in [KNF]. On the other hand, our abelian number field K does satisfy Leopoldt's conjecture, so we may give, for the convenience of the reader, the following shorter proof. We may suppose $n = 0$ (else replace K by K_n). Let $\mathfrak{X}_\infty$ be the standard Iwasawa module $\mathfrak{X}_\infty = \varprojlim(\mathfrak{X}_n)$, where $\mathfrak{X}_n$ is the Galois group over K_n of its maximal abelian S-ramified p-extension. (Note that of course this definition works also for $n = \infty$.) Recall from the end of the introduction the notation X'_n for the p-part of the (p)-class group of K_n. From class field theory we have the exact sequence (sometimes called decomposition)

$$0 \longrightarrow \overline{U}'_\infty \longrightarrow \mathcal{N}_\infty \longrightarrow \mathfrak{X}_\infty \longrightarrow X'_\infty \longrightarrow 0.$$

Put $\mathcal{D}_\infty = \operatorname{Im}(\mathcal{N}_\infty \longrightarrow \mathfrak{X}_\infty)$. By Leopoldt's conjecture for K, we have $\mathfrak{X}^\Gamma_\infty = 0$ therefore $\mathcal{D}^\Gamma_\infty = 0$. This implies that the induced map $(\overline{U}'_\infty)_\Gamma \longrightarrow (\mathcal{N}_\infty)_\Gamma$ is a monomorphism. Now (ii) follows from Proposition 2.2.

K is totally real and $p \neq 2$, so $(\overline{U}'_\infty)_\Gamma$ is $\mathbb{Z}_p$-free as a submodule of $\overline{U}'_0$. As for $\mathcal{N}_\infty$, we have $(\overline{U}'_\infty)^\Gamma = 0$ (same argument applies). These two facts suffice to show the Λ-freeness of $\overline{U}'_\infty$. To compute the rank we consider the sequence

$$0 \longrightarrow (\overline{U}'_\infty)_\Gamma \longrightarrow (\mathcal{N}_\infty)_\Gamma \longrightarrow (\mathcal{D}_\infty)_\Gamma \longrightarrow 0.$$

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As $\mathcal{D}_\infty$ is a torsion-Λ-module (as a submodule of $\mathfrak{X}_\infty$) with trivial Γ-invariants, its Γ-coinvariants are finite. Hence $(\overline{U}'_\infty)_\Gamma$ has the same $\mathbb{Z}_p$-rank as $(\mathcal{N}_\infty)_\Gamma$. By Proposition 2.2 this rank is $\operatorname{rank}_{\mathbb{Z}_p}(\mathcal{N}_0) - \#S_0 = [K:\mathbb{Q}]$. By Nakayama's lemma the Λ-rank of $\overline{U}'_\infty$ is also $[K:\mathbb{Q}]$. This concludes the proof of (i).

(iii) is an immediate corollary of (i) and (ii). ∎

3 From Semi-Local to Global and Vice-Versa

We now state and proceed to prove our main result in this paper. We want to show that descent works asymptotically well for $M_\infty = \overline{U}_\infty/\overline{C}_\infty$ or for $M_\infty = \mathcal{U}^{(0)}_\infty/\overline{C}_\infty$ (which will be proven to be equivalent). (See explanations and notations below for the symbol $^{(0)}$.) This means that in both cases above, $\operatorname{Ker}_n(M_\infty)$ and $\operatorname{Coker}_n(M_\infty)$ are finite of bounded orders. Our strategy of proof is the following. First we will show that bounding the kernels and cokernels associated to both modules is equivalent. Then we will use the injectivity of descent on $\overline{U}'_\infty$ (Proposition 2.4) to bound the kernels of descent for $\overline{U}_\infty/\overline{C}_\infty$. Then we use local class field theory (Proposition 2.3) to bound the cokernels of descent for $\mathcal{U}^{(0)}_\infty/\overline{C}_\infty$.

But before this we have to change the sequence $\mathcal{U}_n/\overline{C}_n$ slightly. Indeed, for all n, the module $\mathcal{U}_n/\overline{C}_n$ is (by Leopoldt's conjecture) of $\mathbb{Z}_p$-rank 1 while $(\mathcal{U}_\infty/\overline{C}_\infty)_{\Gamma_n}$ is torsion. This rank 1 comes from $N_{K_n/\mathbb{Q}}(\overline{U}_n) = \{0\}$, or if we adopt the class field theory point of view, it represents the rank of the $\mathbb{Z}_p$-extension K_∞/K. Let us put the

Notation. Let F be a number field. In the sequel, $\mathcal{U}^{(0)}_F$ will denote the kernel of

$$N_{F/\mathbb{Q}} \colon \mathcal{U}_F \longrightarrow \mathcal{U}_\mathbb{Q}.$$

Consistently $\mathcal{U}^{(0)}_n$ will denote the kernel of $N_{K_n/\mathbb{Q}}$ and $\mathcal{U}^{(0)}_\infty = \varprojlim \mathcal{U}^{(0)}_n$.

By Proposition 2.3 we have $\widetilde{\mathcal{U}}_n \subset \mathcal{U}^{(0)}_n$ and therefore $\mathcal{U}_\infty = \varprojlim(\mathcal{U}^{(0)}_n)$. Moreover $\mathcal{U}^{(0)}_n/\overline{C}_n$ is torsion and since $N_{K_n/\mathbb{Q}}(\mathcal{U}_n)$ is $\mathbb{Z}_p$-torsion free, we have $\mathcal{U}^{(0)}_n/\overline{C}_n = \operatorname{Tor}_{\mathbb{Z}_p}(\mathcal{U}_n/\overline{C}_n)$, and accordingly $\mathcal{U}^{(0)}_n/\overline{U}_n = \operatorname{Tor}_{\mathbb{Z}_p}(\mathcal{U}_n/\overline{U}_n)$. For all these reasons, it is clearly more convenient to use the sequence $(\mathcal{U}^{(0)}_n)_{n\in\mathbb{N}}$ instead of $(\mathcal{U}_n)_{n\in\mathbb{N}}$. This convention gives sense to the notations $\widetilde{\mathcal{U}}^{(0)}_n$, $\operatorname{Ker}_n(\mathcal{U}^{(0)}_\infty)$, $\operatorname{Coker}_n(\mathcal{U}^{(0)}_\infty)$ and consistently to the same notations associated to sequences $\mathcal{U}^{(0)}_n/\overline{C}_n$, $\mathcal{U}^{(0)}_n/\overline{U}_n$ and so on. Of course, due to Proposition 2.3, only the various Coker_n will actually change when we replace $\mathcal{U}_n$ by $\mathcal{U}^{(0)}_n$.

Lemma 2.1 has a global analogue which is the following.

Lemma 3.1 *The natural map $(\mathfrak{X}_\infty)_{\Gamma_n} \longrightarrow \mathfrak{X}_n$ fits into the exact sequence:*

$$0 \longrightarrow (\mathfrak{X}_\infty)_{\Gamma_n} \longrightarrow \mathfrak{X}_n \xrightarrow{\text{res}} \operatorname{Gal}(K_\infty/K_n) \longrightarrow 0.$$

In other words, descent provides an isomorphism $(\mathfrak{X}_\infty)_{\Gamma_n} \cong \operatorname{Tor}_{\mathbb{Z}_p} \mathfrak{X}_n$.

Proof The proof of $(\mathfrak{X}_\infty)_{\Gamma_n} \hookrightarrow \mathfrak{X}_n$ is exactly the same as for Lemma 2.1; we only need to replace the $\mathcal{G}_n$ there by the group $G_S(K_n)$ which is the Galois group of the maximal S-ramified extension of K_n. The remaining part of the exact sequence comes from maximality properties defining $(\mathfrak{X}_\infty)_{\Gamma_n}$ and $\mathfrak{X}_n$. Let M_n be the maximal abelian S ramified p-extension of K_n. Then, by Leopoldt's conjecture, one has $\mathrm{Tor}_{\mathbb{Z}_p}(\mathfrak{X}_n) = \mathrm{Gal}(M_n/K_\infty)$, which gives the isomorphism. ■

Proposition 3.2 *Recall that X_n stands for the p-part of the class group of K_n, and that $\mathrm{Ker}_n(X_\infty)$ is the kernel of the natural map $(X_\infty)_{\Gamma_n} \longrightarrow X_n$.*

(i) *The map $(\mathcal{U}_\infty^{(0)}/\overline{U}_\infty)_{\Gamma_n} \longrightarrow \mathcal{U}_n^{(0)}/\overline{U}_n$ fits into an exact sequence*

$$0 \longrightarrow (X_\infty)^{\Gamma_n} \longrightarrow (\mathcal{U}_\infty^{(0)}/\overline{U}_\infty)_{\Gamma_n} \longrightarrow \mathcal{U}_n^{(0)}/\overline{U}_n \longrightarrow \mathrm{Ker}_n(X_\infty) \longrightarrow 0.$$

(ii) $\mathrm{Ker}_n(\mathcal{U}_\infty^{(0)}/\overline{U}_\infty)$ *and* $\mathrm{Coker}_n(\mathcal{U}_\infty^{(0)}/\overline{U}_\infty)$ *are finite and of bounded orders.*

Proof By global class field theory we have an exact sequence (sometimes called inertia exact sequence):

$$\text{(R)} \qquad 0 \longrightarrow \mathcal{U}_n/\overline{U}_n \longrightarrow \mathfrak{X}_n \longrightarrow X_n \longrightarrow 0.$$

Taking the $\mathbb{Z}_p$-torsion counterpart of the sequence (R) we have:

$$0 \longrightarrow \mathcal{U}_n^{(0)}/\overline{U}_n \longrightarrow \mathrm{Tor}_{\mathbb{Z}_p}(\mathfrak{X}_n) \longrightarrow X_n.$$

On the other hand, if we take limits up to K_∞ on (R), then apply Γ_n-cohomology, we obtain (by Leopoldt, $\mathfrak{X}_\infty^{\Gamma_n} = 0$ and therefore $X_\infty^{\Gamma_n}$ is finite):

$$0 \longrightarrow X_\infty^{\Gamma_n} \longrightarrow (\mathcal{U}_\infty^{(0)}/\overline{U}_\infty)_{\Gamma_n} \longrightarrow (\mathfrak{X}_\infty)_{\Gamma_n} \longrightarrow (X_\infty)_{\Gamma_n} \longrightarrow 0.$$

Then putting together the last three sequences we obtain the following diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
(X_\infty)^{\Gamma_n} & \hookrightarrow & (\mathcal{U}_\infty^{(0)}/\overline{U}_\infty)_{\Gamma_n} & \longrightarrow & (\mathfrak{X}_\infty)_{\Gamma_n} & \longrightarrow & (X_\infty)_{\Gamma_n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{U}_n^{(0)}/\overline{U}_n & \longrightarrow & \mathrm{Tor}_{\mathbb{Z}_p}(\mathfrak{X}_n) & \longrightarrow & X_n & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Proposition 3.2(i) then follows from the snake lemma.

By Leopoldt's conjecture the $(X_\infty)^{\Gamma_n}$'s are finite. As X_∞ is a noetherian Λ-module the ascending union $\bigcup_{n\in\mathbb{N}}(X_\infty)^{\Gamma_n}$ stabilizes. This shows that the orders of $\mathrm{Ker}_n(\mathcal{U}_\infty^{(0)}/\overline{U}_\infty)$ stabilize. As for $\mathrm{Coker}_n(\mathcal{U}_\infty^{(0)}/\overline{U}_\infty) \cong \mathrm{Ker}_n(X_\infty)$, the maps $X_{n+1} \longrightarrow X_n$ (and consequently $X_\infty \longrightarrow X_n$) are surjective as soon as K_{n+1}/K_n

does ramify. By the classical Iwasawa theorems (see [W, §13]), the orders of (X_n) is asymptotically equivalent to $p^{\lambda_X n+\mu_X p^n}$, where λ_X and μ_X are the structural invariants of X_∞. Since the order of $(X_\infty)^\Gamma$ is finite the orders of $(X_\infty)_{\Gamma_n}$ are also asymptotically equivalent to $p^{\lambda_X n+\mu_X p^n}$. This proves that the orders of $\mathrm{Ker}_n(X_\infty)$, hence those of $\mathrm{Coker}_n(\mathcal{U}_\infty^{(0)}/\overline{U}_\infty)$, are bounded and concludes the proof of (ii). ■

By Leopoldt's Conjecture we have $(\mathcal{U}_\infty^{(0)}/\overline{U}_\infty)^{\Gamma_n} \subset \mathfrak{X}_\infty^{\Gamma_n} = 0$ for all n, and therefore an exact sequence

$$(\overline{U}_\infty/\overline{C}_\infty)_{\Gamma_n} \hookrightarrow (\mathcal{U}_\infty^{(0)}/\overline{C}_\infty)_{\Gamma_n} \twoheadrightarrow (\mathcal{U}_\infty^{(0)}/\overline{U}_\infty)_{\Gamma_n}.$$

By the snake lemma with the analogous exact sequence at finite level we obtain the sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Ker}_n(\overline{U}_\infty/\overline{C}_\infty) \longrightarrow \mathrm{Ker}_n(\mathcal{U}_\infty^{(0)}/\overline{C}_\infty) \longrightarrow \mathrm{Ker}_n(\mathcal{U}_\infty^{(0)}/\overline{U}_\infty) \longrightarrow \\ \longrightarrow \mathrm{Coker}_n(\overline{U}_\infty/\overline{C}_\infty) \longrightarrow \mathrm{Coker}_n(\mathcal{U}_\infty^{(0)}/\overline{C}_\infty) \longrightarrow \mathrm{Coker}_n(\mathcal{U}_\infty^{(0)}/\overline{U}_\infty) \longrightarrow 0. \end{aligned}$$

Now using this sequence and the Proposition 3.2, we have proved our first key lemma:

Lemma 3.3 (i) *$\mathrm{Ker}_n(\overline{U}_\infty/\overline{C}_\infty)$ and $\mathrm{Ker}_n(\mathcal{U}_\infty^{(0)}/\overline{C}_\infty)$ are finite. Their orders are simultaneously bounded or not.*

(ii) *$\mathrm{Coker}_n(\overline{U}_\infty/\overline{C}_\infty)$ and $\mathrm{Coker}_n(\mathcal{U}_\infty^{(0)}/\overline{C}_\infty)$ are finite. Their orders are simultaneously bounded or not.*

Remark We have proved that, even if not bounded, the sequences of orders would have been asymptotically equivalent. We will not use this, because we will now proceed to bound these orders!

4 Descent Kernels

The second key lemma is the following.

Lemma 4.1

(i) *The orders of the kernels of the natural maps $(\overline{U}_\infty/\overline{C}_\infty)_{\Gamma_n} \longrightarrow \overline{U}_n/\overline{C}_n$ are bounded.*

(ii) *The orders of the kernels of the natural maps $(\mathcal{U}_\infty^{(0)}/\overline{C}_\infty)_{\Gamma_n} \longrightarrow \mathcal{U}_n^{(0)}/\overline{C}_n$ are bounded.*

Proof From the commutative diagram:

$$\begin{array}{ccccccccc} & & (\overline{C}_\infty)_{\Gamma_n} & \longrightarrow & (\overline{U}_\infty)_{\Gamma_n} & \longrightarrow & (\overline{U}_\infty/\overline{C}_\infty)_{\Gamma_n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{C}_n & \longrightarrow & \overline{U}_n & \longrightarrow & \overline{U}_n/\overline{C}_n & \longrightarrow & 0 \end{array}$$

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we deduce the exact sequence:

$$0 \longrightarrow \frac{\mathrm{Ker}_n(\overline{U}_\infty)}{\mathrm{Im}(\mathrm{Ker}_n(\overline{C}_\infty))} \longrightarrow \mathrm{Ker}_n(\overline{U}_\infty/\overline{C}_\infty) \longrightarrow \overline{C}_n/\widetilde{C}_n.$$

By Theorem 1.1(ii), $\mathrm{Ker}_n(\overline{U}_\infty/\overline{C}_\infty)$ is finite. To control $\mathrm{Coker}_n(\overline{C}_\infty)$ we use the following lemma.

Lemma 4.2 *There exists an N such that for all $n \geq N$ we have $\overline{C}_n/\widetilde{C}_n \cong \overline{C}_N/\widetilde{C}_N$.*

Proof Let I be the inertia subfield of p for $K_\infty/\mathbb{Q}$. By Lemma 2.5 of [Bjr], for n large enough (n such that $I \subset K_n$ is large enough), we have $\overline{C}_n = \widetilde{C}_n C_I$. It follows that $\mathrm{Coker}_n(\overline{C}_\infty) \cong C_I/(\widetilde{C}_n \bigcap C_I)$. Now the increasing sequence $\widetilde{C}_n \bigcap \overline{C}_I$ has to stabilize because $\overline{C}_I$ is of finite $\mathbb{Z}_p$-rank. This shows Lemma 4.2. ■

To prove Lemma 4.1 it then suffices to bound the orders of

$$\mathrm{Ker}_n(\overline{U}_\infty)/\,\mathrm{Im}(\mathrm{Ker}_n(\overline{C}_\infty)).$$

For that, we prove that this sequence of quotients stabilizes.

Proposition 4.3 *For any noetherian Λ-module M_∞, let* $\mathrm{Minv}(M_\infty)$ *denote the submodule of M_∞ defined as follows:*

$$\mathrm{Minv}(M_\infty) = \bigcup_{n \in \mathbb{N}} (M_\infty)^{\Gamma_n}.$$

There exists N such that for all $n \geq N$ we have

$$\mathrm{Ker}_n(\overline{U}_\infty)/\,\mathrm{Im}(\mathrm{Ker}_n(\overline{C}_\infty)) \cong \frac{\mathrm{Minv}(\overline{U}'_\infty/\overline{U}_\infty)}{\mathrm{Im}(\mathrm{Minv}(\overline{U}'_\infty/\overline{C}_\infty))}.$$

Proof Starting from the sequence

$$0 \longrightarrow \overline{C}_\infty \longrightarrow \overline{U}'_\infty \longrightarrow \overline{U}'_\infty/\overline{C}_\infty \longrightarrow 0$$

one gets the following diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & (\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_n} & \longrightarrow & (\overline{C}_\infty)_{\Gamma_n} & \longrightarrow & (\overline{U}'_\infty)_{\Gamma_n} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & \overline{C}_n & \longrightarrow & \overline{U}'_n
\end{array}$$

By Proposition 2.4, the kernels $\mathrm{Ker}_n(\overline{U}'_\infty)$ are trivial. Hence we have an isomorphism $\mathrm{Ker}_n(\overline{C}_\infty) \simeq (\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_n}$. Since $\overline{U}'_\infty$ is a noetherian module, so is $\overline{U}'_\infty/\overline{C}_\infty$ and therefore the increasing sequence $(\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_n}$ stabilizes and for larege n we have $\mathrm{Ker}_n(\overline{C}_\infty) \simeq \mathrm{Minv}(\overline{U}'_\infty/\overline{C}_\infty)$. The same arguments substitution $\overline{U}_\infty$ for $\overline{C}_\infty$ proves that (provided n is greater than some N) we have $\mathrm{Ker}_n(\overline{U}_\infty) \cong \mathrm{Minv}(\overline{U}'_\infty/\overline{U}_\infty)$. This shows the proposition and also (putting everything together) the first part of Lemma 4.1. The second part of Lemma 4.1 follows then from Lemma 3.3. ■

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5 Descent Cokernels

The third and final key lemma is

Lemma 5.1 (i) *The orders of the cokernels of the natural maps* $(\overline{U}_\infty/\overline{C}_\infty)_{\Gamma_n} \longrightarrow \overline{U}_n/\overline{C}_n$ *are bounded.*

(ii) *The orders of the cokernels of the natural maps* $(\mathcal{U}^{(0)}_\infty/\overline{C}_\infty)_{\Gamma_n} \longrightarrow \mathcal{U}^{(0)}_n/\overline{C}_n$ *are bounded.*

Proof Starting with the snake diagram

$$\begin{array}{ccccccccc} & & (\overline{C}_\infty)_{\Gamma_n} & \longrightarrow & (\mathcal{U}_\infty)_{\Gamma_n} & \longrightarrow & (\mathcal{U}_\infty/\overline{C}_\infty)_{\Gamma_n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{C}_n & \longrightarrow & \mathcal{U}^{(0)}_n & \longrightarrow & \mathcal{U}^{(0)}_n/\overline{C}_n & \longrightarrow & 0 \end{array}$$

one gets the sequence

$$\mathrm{Ker}_n(\mathcal{U}^{(0)}_\infty/\overline{C}_\infty) \longrightarrow \overline{C}_n/\widetilde{C}_n \longrightarrow \mathcal{U}^{(0)}_n/\widetilde{\mathcal{U}}_n \longrightarrow \mathrm{Coker}_n(\mathcal{U}^{(0)}_\infty/\overline{C}_\infty) \longrightarrow 0.$$

Recall that D is the maximal subfield of K_∞ such that p is totally split in D. We may assume without loss of generality that $D \subset K_n$ (else enlarge n). By Lemma 3.3, $\mathrm{Ker}_n(\mathcal{U}^{(0)}_\infty/\overline{C}_\infty)$ is bounded. By Proposition 2.3 we have $\widetilde{\mathcal{U}}_n = \mathrm{Ker}(N_{K_n/D}\colon \mathcal{U}^{(0)}_n \longrightarrow \mathcal{U}_D)$. One then gets an isomorphism $\mathcal{U}^{(0)}_n/\widetilde{\mathcal{U}}_n \cong N_{K_n/D}(\mathcal{U}^{(0)}_n)$, and using this isomorphism, the preceding sequence reads:

$$0 \longrightarrow N_{K_n/D}(\overline{C}_n) \longrightarrow N_{K_n/D}(\mathcal{U}^{(0)}_n) \longrightarrow \mathrm{Coker}_n(\mathcal{U}^{(0)}_\infty/\overline{C}_\infty) \longrightarrow 0$$

Now, Lemma 5.1(ii) follows from the next lemma. ■

Lemma 5.2 *Let* c_n *denote* $[K_n\colon D]$ *(asymptotically* c_n *is equivalent to* p^n*).*

(i) $(\overline{C}_D)^{c_n}$ *is a submodule of bounded finite index in* $N_{K_n/D}(\overline{C}_n)$

(ii) $(\mathcal{U}^{(0)}_D)^{c_n}$ *is a submodule of bounded finite index in* $N_{K_n/D}(\mathcal{U}^{(0)}_n)$

(iii) $(\mathcal{U}^{(0)}_D)^{c_n}/\overline{C}_D^{c_n}$ *is asymptotically equivalent to* $\mathcal{U}^{(0)}_D/\overline{C}_D$.

Proof Assertion (iii) is immediate. The finite constant group $\mathcal{U}^{(0)}_D/\overline{C}_D$ maps onto $(\mathcal{U}^{(0)}_D)^{c_n}/(\overline{C}_D)^{c_n}$. Since the norm $N_{K_n/D}$ acts as c_n on $\overline{C}_D$ and $\mathcal{U}^{(0)}_D$ themselves, the inclusions and finiteness of indices in (i) and (ii) are clear. We have to show that these finite indices are bounded.

For assertion (i) we use [Bjr, Lemma 2.5] again, that is, $\overline{C}_n = \widetilde{C}_n\overline{C}_I$. Moreover, as $\widetilde{C}_n \subset \widetilde{\mathcal{U}}^{(0)}_n$, we have $N_{K_n/D}(\widetilde{C}_n) = 0$ by Proposition 2.3. (Without using semi-local units, $N_{K_n/D}(\widetilde{C}_n) = 0$ can be checked directly using distribution relations on a generating system of $\widetilde{C}_n$.) Hence we get $N_{K_n/D}(\overline{C}_n) = N_{K_n/D}(\widetilde{C}_n\overline{C}_I) = N_{K_n/D}(\overline{C}_I) = N_{I/D}(\overline{C}_I)^{[K_n:I]}$. This gives 1 because, as c_n itself, $[K_n : I]$ is asymptotically equivalent to p^n and $N_{I/D}(\overline{C}_I)$ is of (constant) finite index in $\overline{C}_D$.

Assertion (ii) is an easy exercise using local class field theory. Indeed, recall that $\mathcal{U}_n = \bigoplus_{v|p} \overline{U}^1_{K_{n,v}}$. Then the global norm $N_{K_n/D}$ acts on each summand as the local norm $N_{K_{n,v}/\mathbb{Q}_p}$. By local class field theory, the quotient $\overline{U}^1_{\mathbb{Q}_p}/N_{K_{n,v}/\mathbb{Q}_p}(\overline{U}^1_{K_{n,v}})$ is isomorphic to the p-part of the ramification subgroup of $\mathrm{Gal}(K_{n,v}/\mathbb{Q}_p)$. These wild ramification subgroups are cyclic with orders asymptotically equivalent to p^n. Summing up, it follows that $N_{K_n/D}(\mathcal{U}_n)$ contains $\mathcal{U}_D^{c_n}$ with bounded finite index. A fortiori $N_{K_n/D}(\mathcal{U}_n^{(0)})$ contains $(\mathcal{U}_D^{(0)})^{c_n}$ with bounded finite index. This concludes the proof of Lemma 5.2 and therefore of the second claim in Lemma 5.1. The first claim in 5.1 then follows from Lemma 3.3. ■

With Lemmas 4.1 and 5.1 we have fullfilled our goal. We have directly proved that the natural descent homomorphisms $(\overline{U}_\infty/\overline{C}_\infty)_{\Gamma_n} \longrightarrow \overline{U}_n/\overline{C}_n$ have bounded kernels and cokernels. As a consequence, without using Iwasawa's Main Conjecture or Sinnott's index formula, we get an analogue for unit classes of Iwasawa's theorem. Recall that any torsion Λ-module M_∞ has an invariant λ which is the Weierstrass degree of (any) generators of its characteristic ideal and an invariant μ which is the maximal power of p dividing (any) generators of its characteristic ideal. By purely abstract algebra it is classical and easy to prove that *if they are finite* the orders of $(M_\infty)_{\Gamma_n}$ are asymptotically equivalent to $p^{\lambda n+\mu p^n}$.

Theorem 5.3 (i) *Let λ_1 and μ_1 denote the structural invariants of the Iwasawa module $\overline{U}_\infty/\overline{C}_\infty$. Then the orders of $\overline{U}_n/\overline{C}_n$ are asymptotically equivalent to $p^{\lambda_1 n+\mu_1 p^n}$.*

(ii) *Let λ_2 and μ_2 denotes the structural invariants of the Iwasawa module $\mathcal{U}_\infty/\overline{C}_\infty$. Then the orders of $\mathcal{U}_n^{(0)}/\overline{C}_n$ are asymptotically equivalent to $p^{\lambda_2 n+\mu_2 p^n}$.*

Proof By Propositions 4.1 and 5.1 the orders of $\overline{U}_n/\overline{C}_n$ are asymptotically equivalent to the orders of $(\overline{U}_\infty/\overline{C}_\infty)_{\Gamma_n}$. As they are finite, the last orders are equivalent to what we need. This shows (i). The same argument proves (ii) as well. ■

6 Two Applications of Iwasawa's Theorem for Unit Classes

Our first application is a structural link between unit and ideal classes at infinity.

Theorem 6.1 (i) *The Λ-modules $\overline{U}_\infty/\overline{C}_\infty$ and X_∞ share the same structural invariants $\lambda_1 = \lambda_X$ and $\mu_1 = \mu_X$.*

(ii) *The Λ-modules $\mathcal{U}^{(0)}_\infty/\overline{C}_\infty$ and $\mathfrak{X}_\infty$ share the same structural invariants $\lambda_2 = \lambda_{\mathfrak{X}}$ and $\mu_2 = \mu_{\mathfrak{X}}$.*

Proof Consider the exact sequence of Λ-torsion modules:

$$\text{(DNA)} \qquad 0 \longrightarrow \overline{U}_\infty/\overline{C}_\infty \longrightarrow \mathcal{U}^{(0)}_\infty/\overline{C}_\infty \longrightarrow \mathfrak{X}_\infty \longrightarrow X_\infty \longrightarrow 0.$$

The invariants λ and μ are additive in exact sequences. Therefore, going through the above (DNA) sequence, we see that (i) is equivalent to (ii). Now, by Sinnott's index formula, the orders of X_n are asymptotically equivalent to the orders of $\overline{U}_n/\overline{C}_n$. Using Theorem 5.3 and Iwasawa's theorem we get that the orders of $(X_\infty)_{\Gamma_n}$ and $(\overline{U}_\infty/\overline{C}_\infty)_{\Gamma_n}$ are finite and asymptotically equivalent. Therefore the sequence $\lambda_1 n + \mu_1 p^n$ is equivalent to the sequence $\lambda_X n + \mu_X p^n$. Assertion (i) follows. ■

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As explained in the introduction, Theorem 6.1 is an immediate consequence of Iwasawa's Main Conjecture. However the point here is that we have achieved a direct proof by only making use of Iwasawa classical theorem, Sinnott's index formula, Coleman's morphism, and Leopoldt's conjecture (the last two via Theorem 1.1).

Conversely, Theorem 6.1 could be used to simplify the (now classical) proof of the Main Conjecture via Euler systems and the "class number trick". Let us recall the main lines. The p-adic L-functions are related via Coleman's theory to the characteristic series of $\mathcal{U}_\infty^{(0)}/\overline{C}_\infty$, and one version of the Main Conjecture asserts that $\mathcal{U}_\infty^{(0)}/\overline{C}_\infty$ and $\mathfrak{X}_\infty$ have the same characteristic series (up to power of p, and θ-componentwise for all Dirichlet characters θ of the first kind). This is done in two steps:

- Use the Euler System of cicular units to "bound class groups" and to show that the characteristic series of X_∞ divide that of $\overline{U}_\infty/\overline{C}_\infty$. Hence, following the sequence (DNA), to show that the characteristic series of $\mathfrak{X}_\infty$ divides that of $\mathcal{U}_\infty^{(0)}/\overline{C}_\infty$. For full details, see [G].
- To show the converse property, it suffices to prove the equality of the relevant λ-invariants. For this, one uses the "class number trick". Suppose that K is the maximal real subfield of $M := K(\zeta_p)$, which is not a loss of generality. By Kummer's duality, the Iwasawa invariants of $\mathfrak{X}_\infty(K)$ and of $X_\infty^-(M)$ are equals. Then Iwasawa's asymptotical formula and the (minus part) of the class number formula show what we want.

It is this "class number trick" which could be advantageously replaced by Theorem 6.1.

Up to now we have not used Kummer's duality or any knowledge about the minus part of class groups, or Ferrero and Washington's Theorem. We use them now to show the vanishing of the μ-invariant and thus remove the assertions "up to power of p" in all the discussions just above.

Theorem 6.2 (i) *The structural invariant μ of the module $\overline{U}_\infty/\overline{C}_\infty$ is trivial.*
(ii) *The structural invariant μ of the module $\mathcal{U}_\infty^{(0)}/\overline{C}_\infty$ is trivial.*

Proof We claim that all four modules in the above (DNA) sequence have trivial μ-invariants and we only need to prove it for three of them (actually two well-chosen modules would be enough). By Theorem 6.1(i) $\mu_1 = \mu_X$ and by [W, Theorem 7.15], $\mu_X = 0$. Let us draw the main lines of the proof written in [W] of the triviality of μ_X.

Step 1: The main ingredient is Ferrero and Washington's theorem [FW], which claims that the power series $g_\psi(T)$ of our first section is prime to p.

Step 2: Over $K(\zeta_p)$, using Step 1 and the analytic class number formula for the minus part, one deduces that the sequence of orders of X_n^- is equivalent to $p^{\lambda n}$, where λ is the Weierstrass degree of the product of relevant $g_\psi(T)$'s. By Iwasawa's theorem, this implies the triviality of the structural μ-invariant of X_∞^-.

Step 3: Using the classical mirror inequality $\mu^+ \leq \mu^-$ derived from Kummer duality, one recovers the triviality of the μ-invariants of the plus part X_∞^+ over $K(\zeta_p)$, which in turn implies the triviality of the μ-invariant of X_∞ for our base field K.

For the remaining module $\mathfrak{X}_\infty$, a possible proof follows the above first two steps by just replacing the analytic class number formula for the minus part by Leopoldt's formula for the order of the even part of the $\mathfrak{X}_n$ in terms of products of values at 1 of p-adic L-functions. Alternatively let us just examine the third step more carefully. Actually Kummer duality (we are only making use here of [NSW, Corollary 11.4.4], but the full original of Kummer duality is in [I2]) gives that $\mu(\mathfrak{X}^+_\infty) = \mu(X^-_\infty)$ over $K(\zeta_p)$, and the inequality in the third step follows from $\mathfrak{X}_\infty \twoheadrightarrow X_\infty$. Hence Step 2 gives $\mu(\mathfrak{X}^+_\infty) = 0$ over $K(\zeta_p)$ directly, which is all we need. The triviality of μ_2 follows and this concludes the proof of Theorem 6.2(ii). ■

Remark Assertion (i) of Theorem 6.2 was also proved by Greither in the appendix of [FG]. Greither's proof is slightly different because it makes no use of Leopoldt's conjecture, and for that reason needs to work with possibly infinite $|(\overline{U}_\infty/\overline{C}_\infty)_{\Gamma_n}|$. To deal with that difficulty, Greither introduces the notion of "tame" sequences of modules. (Roughly speaking these are sequences of modules whose inverse limits are without μ.)

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