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RAMSEY CARDINALS AND CONSTRUCTIBILITY

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Large cardinal properties divide rather strikingly into two groups. "Small" large cardinal properties such as weak compactness always relativize to L, while in contrast "large" large cardinal properties such as Ramsey are incompatible with L. These properties seem to be similar otherwise and this sense of similarity is reinforced by the fact that many of the large cardinals do exist in the L-like model $L(\mu)$. This paper will show that the division is caused by an artificially restrictive class of "constructible" sets rather than an essential difference in the properties themselves. Specifically, we consider the class K of sets constructible from mice as defined by Dodd and Jensen [3] and prove

THEOREM 1. *If ρ is Ramsey then ρ is Ramsey in K.*

A modification of the proof will show

THEOREM 2. *If ρ is Jonson, then ρ is Ramsey in K.*

Since Ramsey implies Jonson this shows that the notions are equiconsistent. Theorem 2 was proved by Kunen [5] under the assumption that $V = L(\mu)$.

The proof of Theorems 1 and 2 depends heavily on results of Dodd and Jensen [3] about K and mice. These results are stated without proof in §2. §2 also contains elementary (to a reader familiar with the theory of iterated ultrapowers) proofs of special cases of some of these lemmas sufficient to give a self contained proof of the following corollary of Theorem 1:

COROLLARY. *If $\rho \leq \kappa$, ρ is Ramsey and $L(\mu) \models \mu$ is a measure on κ then $L(\mu) \models \rho$ is Ramsey.*

§2 is also intended to give an intuitive introduction to the subject of mice and give some foundation to the claim that the sets in K are as constructible (or at least almost as constructible) as the sets in L. For a full introduction, including proofs, see [3]. Also, [6] is projected to contain a discussion of an extension of the theory of mice.

Theorems 1 and 2 are proved in §§3 and 4, respectively. §1 gives a characterization of Ramsey cardinals in terms of ultrafilters.

§1. Ramsey cardinals and iterable ultrafilters. For any ordinal ρ, we write $Q(\rho)$ for $\bigcup\{P(\rho^n): n < \omega\}$. If $F \subset Q(\rho)$ then an iterable ultrafilter on F is a set $U \subset Q(\rho)$ such that

(1) $U \cap P(\rho^n)$ is a uniform filter on $P(\rho^n)$ for $n \in \omega$.

(2) If $n \in \omega$ and $x \in F \cap P(\rho^n)$, then either $x \in U$ or $\rho^n \backslash x \in U$.

(3) If $n = m + m'$ and $x \subset \rho^n$ then $x \in U$ iff $\{\alpha \in \rho^m: \{\alpha' \in \rho^{m'}: \alpha\alpha' \in x\} \in U\} \in U$, where $\alpha\alpha'$ is the concatenation of the sequences α and α'.

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Condition (3) means that, if $U_n = U \cap P(\rho^n)$, then $U_n = U_m \times U_{m'}$. Together with the uniformity of U_n, (3) implies that $\{(\alpha_0, \ldots, \alpha_{n-1})\colon \alpha_0 < \cdots < \alpha_{n-1}\} \in U_n$ so that we can identify ρ^n and $[\rho]^n$ modulo U_n.

We say U is λ complete if for each $n \in \omega$ (or, equivalently, for $n = 1$), each $\lambda' < \lambda$ and each sequence $(x_\nu\colon \nu < \lambda')$ of sets in U_n, $\bigcap(x_\nu\colon \nu < \lambda') \neq \emptyset$. After proving Theorem 3 we found that it had already been discovered by J. Henle and E. Kleinberg [4].

THEOREM 3. *ρ is Ramsey iff every $F \subset Q(\rho)$ of cardinality ρ has a ρ-complete iterable ultrafilter.*

PROOF. The proof by Erdös and Hajnal that every measurable cardinal is Ramsey [1, Lemma 9a] does not use the normality of the measure. It shows that if the required iterable ultrafilters exist then ρ is Ramsey. Now suppose that ρ is Ramsey and $F \subset Q(\rho)$ has cardinality ρ.

By a theorem of Erdös and Rado [2], if ρ is Ramsey then for every function f with domain $[\rho]^{<\omega}$ there is a sequence $(I_n\colon n \in \omega)$ and an $X \subset \rho$ such that $I_n \subset n$, $\bar{\bar{X}} = \rho$ and for every $(\alpha_0, \ldots, \alpha_{n-1})$ and $(\alpha'_0, \ldots, \alpha'_{n-1}) \in [X]^n$,

$$(*) \qquad f(\alpha_0, \ldots, \alpha_{n-1}) = f(\alpha'_0, \ldots, \alpha'_{n-1}) \quad \text{iff } \forall i \in I_n\ \alpha_i = \alpha'_i.$$

Let $(F_\alpha\colon \alpha < \rho)$ be an enumeration of F and define

$$f(\alpha_0, \ldots, \alpha_n) = \{\alpha < \alpha_0\colon F_\alpha \subset \rho^n \text{ and } (\alpha_1, \ldots, \alpha_n) \in F_\alpha\}.$$

If $(I_n\colon n \in \omega)$ and X are such that $(*)$ holds then it is easy to check that $I_n = \{0\}$ for all n and if $\alpha < \gamma$ and $\gamma \in X$ then $X \backslash (\gamma + 1)$ is homogeneous for F_α. Define

$$U_1 = \{x \subset \rho\colon X \backslash x \text{ is bounded}\}$$

and for $x \subset \rho^{n+1}$, $x \in U_{n+1}$ iff

$$\{\alpha_0 \in \rho\colon \{(\alpha_1, \ldots, \alpha_n) : (\alpha_0, \ldots, \alpha_n) \in x\} \in U_n\} \in U_1.$$

Then $U = \bigcup\{U_n\colon n \in \omega\}$ is the desired iterable ultrafilter on F. QED Theorem 3

T. Jech pointed out to the author that Theorem 3 allows the use of backward Easton extensions, as is done in [7] for weakly compact cardinals, to prove that con(ZFC + $\exists\rho\ \rho$ is Ramsey) implies con(ZFC + $\exists\rho(\rho$ is Ramsey and $2^\rho > \rho^+)$).

Z^* is a fragment of ZF which we will not specify except to mention that any mouse satisfies Z^*. If N is a standard model of Z^* then an iterable N-ultrafilter is an iterable ultrafilter on $Q(\rho) \cap N$ such that

(4) if $f\colon \rho^{1+n} \to \rho$ is a function in N then

$$\{\beta\beta' \in \rho^{2n}\colon \forall\alpha < \beta_0(f(\alpha, \beta) < \beta_0 \Rightarrow f(\alpha, \beta) = f(\alpha, \beta'))\} \in U,$$

where $\beta = (\beta_0, \ldots, \beta_{n-1})$ and $\beta\beta' = (\beta_0, \ldots, \beta_{n-1}, \beta'_0, \ldots, \beta'_{n-1})$.

The special case of $f\colon \rho^n \to \lambda$, $\lambda < \rho$, shows that U is ρ complete in N and the case $f\colon \rho^n \to 2$ shows that U is an ultrafilter on $Q(\rho) \cap N$. The property of being an iterable N-ultrafilter is weaker than that of being a normal N-ultrafilter in the sense of Kunen [5], but our definition is exactly what is needed to apply Kunen's definition of the iterated ultrafilter $\mathrm{ult}_\alpha(N, U)$. We say that U is wellfounded if $\mathrm{Ult}_{\omega_1}(N, U)$, and hence every $\mathrm{ult}_\alpha(N, U)$, is wellfounded. By the argument of [5, Lemma 3.6] every countably complete iterable ultrafilter (and hence every ρ-complete iterable ultrafilter for $\rho > \omega$) is wellfounded.

LEMMA 1. *If ρ is Ramsey then for every standard model N of Z^* with* card $(P(\rho) \cap N) = \rho$ *there is a wellfounded iterable N ultrafilter on ρ.*

PROOF. We first argue that the parameter α in (4) can be eliminated: Suppose D is an iterable ultrafilter on $Q(\rho) \cap N$ such that

(5) for all $g \in N$, $g: \rho^n \to \rho$, if $\{x \in \rho^n: g(x) < x_0\} \in D$ then $\exists \beta_0$ such that $\{x \in \rho^n: g(x) = \beta_0\} \in D$.

Then (4) holds. Suppose that to the contrary there is $f \in N$ with $f: \rho^{1+n} \to \rho$ such that $C = \{xx': \exists \alpha < x\, f(\alpha x) < x_0$ and $f(\alpha x) \neq f(\alpha x')\} \in D$. Let $h(xx') =$ the least α such that $\alpha < x_0$, $f(\alpha x) < x_0$, and $f(\alpha x) \neq f(\alpha x')$ if such α exists and be undefined otherwise. Then $h \in N$ and $\{xx': h(xx') < x_0\} \in D$ so for some α_0, $A = \{xx': h(xx') = \alpha_0\} \in D$. Then $\{x: f(\alpha_0 x) < x_0\} \in D$ so for some β_0, $B = \{x: f(\alpha_0 x) = \beta_0\} \in D$. So $C \cap A \cap B \times B = \varnothing$ contradicting our assumption that C, A, and B are in D.

Now let U be an arbitrary ρ-complete iterable ultrafilter on $Q(\rho) \cap N$; we will construct an iterable ultrafilter D satisfying (5). If $f, g \in N, f, g: \rho^n \to \rho$ then we say $f < g$ if $\{x \in \rho^n: f(x) < g(x)\} \in D$. This definition can be extended to f and g with domains ρ^n and ρ^m with $n \neq m$ by introducing dummy variables. Since U is countably complete, $<$ is a wellordering. Let f be the $<$-least function such that $\{x, x' \in \rho^{2n}: fx \neq fx'\} \in U$. Then, again by the countable completeness of U, $\{xx': fx < fx'\} \in U$. Let $D = \{X \subset \rho^s: s \in \omega$ and $\{x_1 \cdots x_s \in (\rho^n)^s: (f(x_1), \ldots, f(x_s)) \in X\} \in U\}$. Then D is a ρ-complete iterable ultrafilter on $Q(\rho) \cap N$. If is g such that $\{y \in \rho^s: g(y) < y\} \in D$ then $\{x_1 \cdots x_s \in (\rho^n)^s: g(f(x_1), \ldots, f(x_s)) < f(x_1)\} \in U$ and if $g^*(x_1 \cdots x_s) = g(f(x_1), \ldots, f(x_s))$ then $g^* < f$. Then $\{zz': g^*(z) = g^*(z')\} \in U$ so for some β_0, $\{z: g^*(z) = \beta_0\} \in U$,

$$\{x_1 \cdots x_s: g(f(x_1), \ldots, f(x_s)) = \beta_0\} \in U,$$

so $\{y: g(y) = \beta_0\} \in D$. QED Lemma 1

COROLLARY. *If ρ is Ramsey then for every $x \subset \rho$, $x^\#$ exists.*

PROOF. Take $N = L_\alpha(x) \prec L_{\rho^+}(x)$ and let U be a wellfounded iterable N ultrafilter. There is a closed unbounded set of indiscernibles in $\mathrm{ult}_{\omega_1}(N, U) = L_{\alpha'}(i(x))$. But $i_{\omega_1}(x) \cap \rho = x$ so this is a set of indiscernibles for $L_{\alpha'}(x)$ and hence for $L(x)$. Thus $x^\#$ exists.

§**2. About mice.** This section has two purposes. The first is to list results of Dodd and Jensen that are used, but not proved, in this paper. For this purpose the reader can skip everything (including starred proofs) in this section except the statements of the lemmas. For proofs, see [3] or [6].

The second purpose is to state and prove the special cases which are needed for the promised self-contained proof of the corollary to Theorem 1. For this purpose, we define a pet mouse to be a structure $m = (L_\alpha(\nu), \in, \nu)$ such that

(1) $L_\alpha(\nu) \models \mathrm{ZF}^- + \nu$ is a κ_m-complete normal measure on κ_m,

(2) all iterated ultrapowers of $L_\alpha(\nu)$ by ν are wellfounded (this follows from (1) if $\omega_1 \in \alpha$), and

(3) there is a finite set $p_m \subset \alpha$ and a $\gamma_m < \kappa_m$ such that every element of $L_\alpha(\nu)$ is definable in $L_\alpha(\nu)$ from parameters in $\gamma_m \cup P_m \cup \{\nu\}$.

Roughly speaking, this definition of a pet mouse is the same as that of a mouse except that instead of (1) a mouse only needs to satisfy that ν is an iterable $L(\nu)$-

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ultrafilter. If there is an inner model with a measurable cardinal then every set constructible from a mouse is constructible from a pet mouse. Our self contained proof of the corollary to Theorem 1 uses this fact. For each of Lemmas 2—5 we give a *proof. This is a proof of the special case of the lemma obtained by adding "$L(\mu)$ exists" to the hypothesis and substituting "pet mouse" for "mouse." The proof in §3 of Theorem 1 goes through with these special cases under the assumption that $L(\mu)$ exists. The corollary follows by Lemma 3.

LEMMA 2. *There is a wellordering $<$ of the class of mice such that if $m < m'$ then $m \in L(m')$.*

*PROOF. If $m = L_\alpha(\nu)$ is a pet mouse and θ is an ordinal then the θ-extension of m is $\mathrm{Ult}_\theta(L_\alpha(\nu), \nu)$. The θ-extension of m is clearly constructible from m. On the other hand, (3) implies that m is isomorphic to the set of elements definable in its θ-extension from parameters from γ_m, $i_\theta^\nu(p_m)$ and $\{i_\theta^\nu(\nu)\}$, so m is constructible from its θ-extension. If m and m' are two pet mice, pick a regular cardinal larger than either. Then if C is the closed unbounded filter on θ, the θ-extensions of m and m' are $L_\alpha(C)$ and $L_{\alpha'}(C)$ for some ordinals α and α'. We put $m < m'$ iff $\alpha < \alpha'$ and check that this defines a wellordering. QED Lemma 2

We sometimes say m is longer than m' if $m > m'$. This terminology is, of course, suggested by the above proof. K is defined to be the class of sets constructible from mice. Lemma 2 shows that $K \models \mathrm{ZF} + \mathrm{AC}$. Also, K is absolute in the sense that any model of $V = K$ is in fact the class of sets constructible from an initial segment of the class of mice. Recall that $L(\mu) \models \mu$ is a measure on $\kappa > \rho$.

LEMMA 3. *If $L(\mu)$ exists then $K \subset L(\mu)$ and $K \cap P(\kappa) = L(\mu) \cap P(\kappa)$.*

*PROOF. Since $P(\kappa) \cap L(\mu) = P(\kappa) \cap \mathrm{ult}_1(L(\mu), \mu)$ it is enough to show $K \cap P(\kappa) = L(\mu) \cap P(\kappa)$ where μ is a measure on some $\lambda > \kappa$. If $x \in P(\kappa) \cap L(\mu)$ then x is definable in $L_{\lambda^+}(\mu)$ by a formula with a finite set $p \subset \lambda^+$ of parameters. Let $X \prec L_{\lambda^+}(\mu)$ be the smallest elementary substructure containing $\kappa \cup \{\kappa, \mu\} \cup p$ and let $L_\alpha(\mu') \cong X$ be transitive. Then $L_\alpha(\mu')$ is a pet mouse so $x \in K$.

If m is a pet mouse on some κ_m, take $\theta = \sup(\kappa_m^+, \kappa^{++})$. Then the θ-expansion of m is in $L(\mu)$ since the closed unbounded filter on θ is strong in $L(\mu)$. But then $m \in L(\mu)$ and since m was arbitrary, $K \subset L(\mu)$, QED Lemma 3

LEMMA 4. *If α is an ordinal and $x \in P(\alpha) \cap K$ then there is a mouse m on some $\kappa_m > \alpha$ with $x \in m$.*

*PROOF. Apply Lemma 3 with $\mathrm{Ult}_\alpha(L(\mu), \mu)$ for $L(\mu)$. QED Lemma 4

COROLLARY. $K \models \mathrm{GCH}$.

PROOF. If $x \in P(\rho) \cap K$, then by Lemma 4, $x \in m$ for some mouse m. Clearly we can take $\gamma_m = \rho$, so $\bar{m} = \rho$. An examination of the proof of Lemma 2 shows that if $m' < m$ and $x \in m' \cap P(\rho)$ then x is in the θ-extension of m and hence in m. Thus the ordering of mice induces a wellordering of $P(\rho) \cap K$ of order type at most $\rho^{+(K)}$, so $K \models 2^\rho = \rho^+$.

LEMMA 5. *Suppose U is a wellfounded iterable K-ultrafilter. Then $L(U) \models U$ is a ρ-complete normal ultrafilter.*

*PROOF. We can assume $\rho < \kappa$. Then by Lemma 3, U is an iterable $L(\mu)$-ultrafilter and the lemma follows by a result of Kunen [5, Theorem 6.9].

The next two lemmas will only be needed for the proof of Theorem 2 in §4.

LEMMA 6 [3, §4, Lemma 15]. *If ν is the first ordinal moved by an elementary embedding $i: K \to K$ and $i(\nu)$ is regular, then $i(\nu)$ is measurable in an inner model.*

We actually use the proof of Lemma 6 rather than Lemma 6 itself. Lemma 7 is a corollary of the covering lemma [3, §5].

LEMMA 7. *If κ is a singular cardinal then either κ is singular in K or κ is measurable in an inner model.*

§3. Ramsey cardinals.

LEMMA 8. *If ρ is Ramsey and $\rho^{+(K)} < \rho^{+}$ then there is an inner model with ρ measurable.*

PROOF. Let U be a wellfounded iterable K-ultrafilter on ρ, which exists since $2^{\rho(K)} = \rho^{+(K)} < \rho^{+}$. By Lemma 5, $L(U) \models U$ is a ρ-complete ultrafilter on ρ. QED Lemma 8

We now complete the proof of Theorem 1.

By Lemma 8 we can assume $\rho^{+(K)} = \rho^{+}$. Let $F \subset Q(\rho)$ have cardinality ρ in K. By Lemma 4, there is a mouse m such that $\bar{\bar{m}} = \rho < \kappa_m$ and $F \subset m$. We can suppose $F = Q(\rho) \cap m$. Let D be a ρ-complete iterable m-ultrafilter on ρ. We can assume that D is generated from $D \cap m$ by ρ-completeness and iterability. Then $m_0 = \mathrm{ult}_{\rho^+}(m, D)$ is also a mouse and we can let m_1 be the ρ^{++} mouse extension of m_0. Let $i_0: m \to m_0$ and $i_1: m_0 \to m_1$, so $i_1 i_0(\kappa_m) = \rho^{++}$ and $i_0(\rho) = i_1 i_0(\rho) = \rho^{+}$. The extended mouse m_1 is in K. The map $i_1 i_0 : m \to m_1$ is also in K, since it is the transitive collapse of the set of elements of m_1 definable in m_1 from parameters in $\rho \cup p_{m_1} \cup \{\mu_{m_1}\}$. If D' is the closed, unbounded filter on ρ^+ then $D \cap m = \{x \in Q(\rho) \cap m: i_0(x) = i_1 i_0(x) \in D'\}$. We will show that $D' \cap m_1 \in K$ and hence conclude our proof that $D \in K$.

m_0 may contain subsets of ρ not in m. Every such set, however, is of the form $i_0(f)(d_0, \ldots, d_k)$ where $f \in m$, $d_0 < \cdots < d_k$ and $d_i \in C = \{i^D_\alpha(\rho): \alpha < \rho^+\}$ for $i = 0, \ldots, k$. Since $i_0(f)(d_0, \ldots, d_k)$ is independent of the choice of $d_0, \ldots, d_k$, there are only ρ such subsets. Since $\kappa^{+(K)} = \rho^{+}$, there is a mouse n with $\kappa_n < \rho^{+}$ and an $x \subset \rho$ such that $x \in n \backslash m_0$. Then the ρ^{++} extension n' of n is longer than m_1 since $x \in n' \backslash m_1$. But n' has a closed unbounded set $C' \subset \rho^{+}$ of indiscernibles and since $m_1 \in n'$, $C' \backslash \delta$ is a set of indiscernibles for m_1 for some $\delta < \rho^{+}$. QED Theorem 1

If ρ is an ordinal and there is a standard model M of ZFC + "ρ is Ramsey" with $ON \subset M$ then Theorem 1 and Lemma 2 imply that the intersection of all such models is such a model of the form $L(a)$ where a codes up $\{m: \bar{\bar{m}} = \rho$ and $m < m'\}$ for some mouse m'. If $M(\rho)$ is this minimal model then how does $M(\rho)$ depend on ρ? The analogy with $L(\mu)$ might suggest that $M(\rho) \subset M(\rho')$ if $\rho > \rho'$ but this need not be the case, and in fact the wellordering of mice implies that the relation $M(\rho) \subset M(\rho')$ is a wellorder. If $L(\mu)$ exists, with μ a measure on ρ, then $M(i^{\mu}(\rho)) \supset M(\rho)$ and in fact $M(\rho) = L(P(\rho)) \cap M(i^{\mu}(\rho))$. In general we have the

PROPOSITION. *If $\rho < \rho'$ and ρ is Ramsey in $M(\rho')$ then $P(\rho) \cap M(\rho') \subset M(\rho) \subset M(\rho')$.*

PROOF. Since ρ is Ramsey in $M(\rho')$, $M(\rho) \subset M(\rho')$. If $P(\rho) \cap M(\rho') \not\subset M(\rho)$ then we would have $\mathrm{card}(M(\rho) \cap P(\rho)) = \rho$ in $M(\rho')$. Then $M(\rho) = L(A)$ for an $A \subset \nu < \rho^{+}$, $A \in M(\rho')$, and $\mathrm{card}(L(A^{\sharp}) \cap P(\rho)) = \rho$ in $M(\rho')$. Then since ρ is Ramsey in $M(\rho')$, there is a wellfounded iterable $L(A^{\sharp})$ ultrafilter D in $M(\rho')$. But then if $i: L(A^{\sharp}) \to \mathrm{ult}_{\rho'}(L(A^{\sharp}), D)$ then $\mathrm{ult}_{\rho'}(L(A^{\sharp}), D) \models$ "$L(i(A)) = M(i(\rho))$ and $i(A^{\sharp})$ is its sharp". But $i(\rho) = \rho'$ so $M(\rho')$ contains its own sharp, which is absurd. QED Proposition

It is easy to see that if $\rho' > \rho$ then ρ' is not Ramsey in $M(\rho)$. The only other fact we know is that if $\rho < \rho'$ and ρ' is a cardinal in $M(\rho)$ then $M(\rho) \subset M(\rho')$.

§4. **Jonson cardinals.** A cardinal ρ is Jonson if for every structure $\mathfrak{A} = (A, R_0, \ldots, R_n, \ldots)_{n \in \omega}$ with $\bar{\bar{A}} = \rho$ there is $A' \subset A$ such that $\bar{\bar{A}}' = \rho$, $A' \neq A$ and $(A', R_0', \ldots, R_n', \ldots) \prec \mathfrak{A}$ where R_n' is the restriction of R_n to A'.

In this section we will modify the proof of Theorem 1 to prove

THEOREM 2. *If ρ is Jonson then ρ is Ramsey in K.*

PROOF. If there is an inner model with some $\kappa \leq \rho$ measurable then Theorem 2 is immediate, so we can assume there is no such inner model. As before the proof divides into two cases accroding to whether $\rho^{+(K)} = \rho^+$ or not. The proof of Lemmas 9.1 and 9.2 on which the cases depend will be deferred to the end of this section.

Case 1. $\rho^{+(K)} = \rho^+$.

The proof of Theorem 1 will go through if we can show that ρ is regular in K and for every $f\colon [\rho]^{<\omega} \to 2$ in K there is a homogeneous set for f (which need not be in K). We will adapt Kunen's proof in [5] that if $V = L(\mu)$ then every Jonson cardinal is Ramsey.

If κ is an ordinal, then $K_\kappa = \{m\colon m$ is a mouse and $ON \cap m < \kappa\}$.

LEMMA 9.1. *Let $f\colon [\rho]^{<\omega} \to 2$ be in K. Then there is $\mathfrak{A} = (A, \in, f')$ and $i\colon \mathfrak{A} \cong \prec (K_{\rho^+}, \in, f)$ such that A is transitive*, card$(i''A \cap \rho) = \rho$, and $K_\rho \not\subset A$.

Let such an $\mathfrak{A}$ be given and suppose m is a mouse in $K_\rho \backslash A$. Then the expansion m' of m to ρ is also not in A. Since ρ is regular in m' and m' is longer than any mouse in A, ρ is regular in A and hence in K. Since $A \models f' \in K$, $f' \in m''$ for a mouse $m'' \in A$. We must have $m'' < m'$ so $f' \in m'$. But m' has a set I of indiscernibles of cardinality ρ. For some $\lambda < \rho$, $I \backslash \lambda$ is homogeneous for f' and so $i''(I \backslash \lambda)$ is homogeneous for f.

Case 2. $\rho^{+(K)} < \rho^+$.

To use the proof from §3, we have to find a wellfounded iterable K ultrafilter on ρ. The basic lemma is

LEMMA 9.2. *Suppose $\rho^{+(K)} = \lambda < \rho^+$. Then there is an $\mathfrak{A} = (A, \in)$ and $i\colon \mathfrak{A} \cong \prec (K_\lambda, \in)$ such that A is transitive, $i''A$ is cofinal in λ, and $K_\rho \not\subset A$.*

Take such an A and pick $m \in K_\rho$ and let m' be its expansion to ρ. As in case 1, ρ is regular in K and by Lemma 7 it follows that ρ is regular in the real world.

Let $I \subset \rho$ be the closed, unbounded set of indiscernibles for m' and define

$$U = \{x \subset \rho^n : n \in \omega \text{ and } \exists \lambda < \rho\ [i''(I \backslash \lambda)]^n \subset x\}.$$

Since ρ is regular, U is ρ complete and hence wellfounded. We will show U is an iterable K ultrafilter by checking condition (5) from the proof of Lemma 1 in §1. Suppose $g \in K$, $g\colon \rho^n \to \rho$ and $\{x \in \rho^n\colon g(x) < x_0\} \in U$. We have to show that for some β, $\{x \in \rho^n\colon g(x) = \beta\} \in U$.

Since $i''A$ is confinal in $\rho^{+(K)}$ there is function σ in A such that dom $\sigma = \rho$, $\sigma(\alpha)\colon \rho^n \to \rho$ for all $\alpha < \rho$, and $g = i(\sigma)(\delta)$ for some δ. Since $g \in m'$, g is definable in m' using only members of I less than λ for some $\lambda < \rho$. We can pick λ so that $i(\lambda) > \delta$. If x, x' are any two members of $[I \backslash \lambda]^n$, then $\mathfrak{A}$ satisfies

$$\forall \nu < \lambda(\sigma(\nu)(x) < x_0 \Rightarrow \sigma(\nu)(x) = \sigma(\nu)(x'))$$

where $x = (x_0, \ldots, x_{n-1})$. Then since i is elementary and $\delta < i(\lambda)$

$$g(i(x)) < i(x_0) \Rightarrow g(i(x)) = g(i(x')).$$

But $\{x: g(x) < x_0\} \in U$ so in particular $g(x) < x_0$ for some $x \in [i''I\backslash\lambda]^n$. But then $g(x') = g(x)$ for all $x' \in [i''I\backslash\lambda]^n$ and $\{x': g(x') = g(x)\} \in U$. This completes the proof for case 2.

The proof of Lemmas 9.1 and 9.2 is based on the following modification of Lemma 6:

LEMMA 6′. *Suppose $\alpha \in ON$, $(X, \in) \prec (K_\alpha, \in)$, and the least ordinal $\gamma \in X$ such that $\gamma \not\subseteq X$ is regular. Then if K_{γ^+} is contained in the transitive closure of X' for every X' such that $(X, \in) \prec (X', \in) \prec (K_\alpha, \in)$ then γ is measurable in an inner model.*

Lemma 6 is obtained from Lemma 6′ by taking $\alpha = ON$ and using the fact that if $j: K \to M$ is an elementary embedding then $M \cong K$. Lemma 6′ is proved by the same proof as Lemma 6 [3, §4, Lemma 15] with minor changes.

PROOF OF LEMMAS 9.1 AND 9.2. For 9.1 let $\alpha < \rho^+$ be such that $(K_\alpha, \in) \prec (K_{\rho^+}, \in)$ and $f \in K_\alpha$. For 9.2, let $\alpha = \rho^{+(K)}$. Let $F: \rho \to K_\alpha$ onto and let $G: \rho \to \alpha$ be strictly increasing and cofinal in α. Let $H: \rho^2 \to \rho$ be such that $\beta = \bigcup \{H(\gamma, \beta): \gamma < \operatorname{cf} \beta\}$ for all singular $\beta < \rho$. Since ρ is Jonson there is $(X, \in, F'G', H') \prec (K_\alpha, \in, F, G, H)$ such that $\bar{\bar{X}} = \rho$, $X \neq K_\alpha$ and, for Lemma 9.1, $f \in X$. Because of F, $\rho \not\subset X$ and because of G, X is cofinal in α.

Using F we see that $\rho \not\subset X$ and $|X \cap \rho| = \rho$. Thus the least ordinal γ such that $\gamma \notin X$ is in ρ, so $\gamma^+ \leq \rho$. Using H we see that γ is regular. Since there is no inner model with a measurable cardinal, Lemma 6′ implies that there is an X' with $(X, \in) \prec (X', \in) \prec (K_\alpha, \in)$ such that K_{γ^+}, and hence K_ρ, is not contained in the transitive closure of X'. But $|X' \cap \rho| \geq |X \cap \rho| = \rho$ because of F and $X' \supset X$ is cofinal in α because of G. In case 9.1, $f \in X \subseteq X'$ so $f \in X'$. Thus, in either case the transitive collapse $(A, \in)$ of $(X', \in)$ is the desired model.

QED Lemma 9, Theorem 2

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