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THE n-INSERTIVE SUBGROUPS OF UNITS

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Let R be a finite ring. Let us denote its group of units by $G = G(R)$ and its Jacobson radical by $J = J(R)$. Let n be an arbitrary integer. We prove that R is an n-insertive ring if and only if G is an n-insertive group and show that every n-insertive finite ring is a direct sum of local rings. We prove that if n is a unit, then the local ring R is n-insertive if and only if its Jacobson group $1 + J$ is n-insertive and find an example to show that this is not true if n is a non-unit.

1. INTRODUCTION

Many properties of finite rings follow from the properties of their groups of units. For example, it was shown in [**1**] that a finite ring is commutative if and only if its group of units is commutative. The notion of commutativity can be generalised to the notion of n-insertiveness, as shown below. In this paper, we study the link between the n-insertiveness of a finite ring and the n-insertiveness of its group of units.

So, let R be a finite ring with identity $1 \neq 0$. Denote the group of units of R by $G = G(R)$ and the Jacobson radical of R by $J = J(R)$.

If n is an integer, we call R an n-insertive ring if, for $a, b \in R$ and $ab = n$, we have $arb = nr$ for every $r \in R$. Let H be a subgroup of G. We call H an n-insertive group if, for $a, b \in R$ and $ab = n$, we have $agb = ng$ for every $g \in H$.

LEMMA 1.1. *G is 1-insertive if and only if G is commutative.*

PROOF: Assume that G is 1-insertive. Choose $a \in G$ and denote $b = a^{-1}$. Since G is 1-insertive, we have $ab = 1$ and $agb = g$ for every $g \in G$. Therefore $ag = gb^{-1} = ga$ for every $g \in G$, so G is commutative.

On the other hand, if G is commutative, then R is commutative by a corollary of [**1**, Theorem 3.2]. This implies that R, and then of course also G, is 1-insertive. ∎

We know by [**3**, Lemma 1] that R is 1-insertive if and only if R is commutative. A corollary of [**1**, Theorem 3.2] tells us that R is commutative if and only if G is commutative. So, the above lemma implies that G is 1-insertive if and only if R is 1-insertive.

We prove that for every integer n the following holds: G is n-insertive if and only if R is n-insertive. We prove this by studying the structure of n-insertive rings, showing

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that every n-insertive ring (for an arbitrary integer n) is a direct sum of local rings. We also show that the converse of this statement is false. Namely, we find a local ring that is not n-insertive for any integer n.

The group $1 + J$ is a normal subgroup of G, called the Jacobson group. We study whether the n-insertiveness of $1 + J$ is equivalent to the n-insertiveness of R. Obviously, the answer is negative in general (consider for example the full matrix ring over some finite field). However, we prove that the answer is affirmative if R is a local ring and n is a unit. We also find an example of a non n-insertive local ring R with a n-insertive Jacobson group (for every integer non-unit n in R), thus proving that the above equivalence does not hold for an arbitrary n, even in the class of local rings.

2. The properties of n-insertive rings

Theorem 2.1. *Let n be an arbitrary integer. If G is n-insertive, then R is a direct sum of local rings.*

Proof: Assume that R is a directly indecomposable ring. Assume also that R is not local. Then there exists a non-trivial idempotent $e_1 \in R$. Denote $e_2 = 1 - e_1$. Since R is indecomposable, we either have $e_1Re_2 \neq 0$ or $e_2Re_1 \neq 0$, otherwise we would be able to decompose R as $R = e_1Re_1 \oplus e_2Re_2$. We can assume without any loss of generality that $e_1xe_2 \neq 0$ for some $x \in R$. Now, $(e_1 + ne_2)(ne_1 + e_2) = n$, so by our assumption $(e_1 + ne_2)g(ne_1 + e_2) = ng$ for every $g \in G$. Clearly, $1 + e_1xe_2 \in G$, since $(e_1xe_2)^2 = 0$. But $(e_1 + ne_2)(1 + e_1xe_2)(ne_1 + e_2) = n + e_1xe_2$, therefore $(n - 1)e_1xe_2 = 0$. We can therefore conclude that $n - 1 \notin G$. However, R is indecomposable, therefore it is a p-ring for some prime number p. Since $n - 1$ is a multiple of p, we can conclude that n has to be prime to p, and thus n must be a unit. Let us show that G is then 1-insertive. Choose $a, b \in R$ such that $ab = 1$ and choose $g \in G$. Then $a(bn) = n$ and therefore $agbn = gn$, so $agb = g$, because n is a unit. So, Lemma 1.1 implies that G is Abelian and therefore R is commutative by [**1**, Theorem 3.2]. This, together with the existence of e_1, is a contradiction with the indecomposability of R. Therefore, we can conclude that R is indeed a local ring. ∎

Example 2.2. The converse of the above statement is false. Let p be a prime number and let R be a ring of all 4×4 upper triangular matrices with entries from $GF(p^r)$, such that their entries on the (main) diagonal are constant. Obviously, G is a non-Abelian group. Therefore G is not 1-insertive and then G is also not n-insertive for any integer n, prime to p, by the proof of Theorem 2.1. If we take $p = 3$, we have

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0, \text{ but}$$

$$\begin{bmatrix}0&1&2&0\\0&0&0&2\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&1&1&0\\0&0&0&1\\0&0&0&1\\0&0&0&0\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\neq 0.$$

So, for $p = 3$, R is a local ring, but G is not n-insertive for some integers n (specifically $n = 0, 3, 6, \ldots$).

Corollary 2.3. *Let n be an arbitrary integer. Then R is n-insertive if and only if G is n-insertive.*

Proof: Since $G(R_1 \oplus R_2) = G(R_1) \times G(R_2)$, it suffices to prove the corollary only for directly indecomposable rings. So, assume that R is directly indecomposable and that G is n-insertive. Let us prove that R is n-insertive. Assume that $ab = n$ for some $a, b \in R$ and choose $r \in R$. By Theorem 2.1, R/J is a field and therefore R/J is generated by its units. But then R is also generated by its units, as was proved in [**2**, Lemma 4.5]. Thus, $r = u_1 + \cdots + u_k$ and $arb = au_1b + \cdots + au_kb = n(u_1 + \cdots + u_k) = nr$, because G is n-insertive. ∎

3. The n-insertiveness of the Jacobson group

In this section, we examine if the n-insertiveness of R is perhaps also equivalent to the n-insertiveness of the Jacobson group $1 + J$. Obviously, in general, the answer is negative, because the Jacobson group of a full matrix ring over some finite field is trivial, and therefore 1-insertive, but the ring itself is non-commutative and therefore not 1-insertive. However, we shall examine this question in the class of all finite local rings and find that the answer is positive, at least for those integers n that are units in R.

For a subset $S \subseteq R$, let $C(S) = \{x \in R; xs = sx \text{ for every } s \in S\}$ denote the centraliser of S in R.

Lemma 3.1. *Let R be an arbitrary finite ring and n an arbitrary integer. If n is a unit in R, then $1 + J$ is n-insertive if and only if $J \subseteq C(G)$.*

Proof: Assume $1 + J$ is n-insertive and choose $a \in G$. Then $naa^{-1} = n$, therefore $na(1 + j)a^{-1} = n(1 + j)$ for every $j \in J$, thus $n(aja^{-1} - j) = 0$. Since n is a unit, we can conclude that $aj = ja$ for every $j \in J$.

Conversely, if $J \subseteq C(G)$, then $a(1 + j)b = (1 + j)ab$ for every $a, b \in G$ and every $j \in J$, so $1 + J$ is indeed n-insertive. ∎

Theorem 3.2. *Let R be a finite local ring and n an arbitrary integer. If n is a unit in R, then the following are equivalent:*

1. *R is n-insertive.*
2. *$1 + J$ is n-insertive.*
3. *R is commutative.*

PROOF: If n is a unit and R is n-insertive, then R is also 1-insertive and thus commutative by [**3**, Lemma 1]. So, it suffices to prove that the n-insertiveness of $1+J$ implies the commutativity of R. Let us therefore assume that $1+J$ is n-insertive. We know that, since R is a finite local ring, the units of the factor field R/J form a cyclic group, generated by some element $g+J$ of order k. Then $G = \bigcup_{i=1}^{k}(g^i+J)$. By the previous lemma we conclude that all elements in J are also in the centraliser of G, thus $1+J$ is a commutative group, so J is commutative as well. Thus G is an Abelian group and therefore R is a commutative ring by the corollary of [**1**, Theorem 3.2]. ∎

The next example shows that this theorem does not hold if n is not a unit.

EXAMPLE 3.3. If S is a ring, then let $S\{x,y,z\}$ denote the polynomial ring over S in non-commuting variables. Let us examine the ring

$$R = \frac{\mathbb{Z}_3\{x,y,z\}}{(x^2+1, y^3, z^3, yz, zy, yx-xz, zx-xy)}.$$

Clearly, this is a finite ring, such that all of its non-units form the unique maximal ideal $J=(y,z)$, therefore R is a local ring. We notice that $J^3=0$, therefore $1+J$ is a 0-insertive group, since $ab=0$ implies $a,b\in J$. However, R is not a 0-insertive ring, because we have $yz=0$, but $yxz = xz^2 \neq 0$, because x is a unit and $z^2\neq 0$. The same argument also implies that $1+J$ is n-insertive and R is not n-insertive for every integer n which is a non-unit in R.

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