ON A CLASS OF GENERALIZED FUNCTIONS

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1. Introduction. Let $C_0^\infty(J)$ be the complex linear space of all infinitely differentiable functions φ on the interval $J=(a,b)$ $(-\infty \leqq a<0<b\leqq +\infty)$ such that $\varphi^{(k)}(0)=0$ for all non-negative integers k. Krabbe ([2], [3]) has defined a class of generalized functions on J as an algebra $\mathscr{M}$ of linear operators on $C_0^\infty(J)$ and has developed an operational calculus for these operators†. Shultz ([6], Theorem 2.18) has recently shown that $\mathscr{M}$ is isomorphic to $\mathscr{D}'_+ \times \mathscr{D}'_-$, where $\mathscr{D}'_+$ (resp. $\mathscr{D}'_-$) is the set of all distributions on J whose supports are contained in $J_+=[0,b)$ $(J_-=(a,0])$. In this paper we combine some of the ideas developed in [4] with results established in an earlier paper by Shultz to give an easier proof of the above result. Our methods also give a more direct proof of the main result (Theorem 1.22) of [2].

In the sequel, unless there is ambiguity, we shall denote $C_0^\infty(J)$ simply by C_0^∞.

2. The algebra of generalized functions. We give C_0^∞ the topology of uniform convergence for all derivatives on compact subsets of J; this topology is defined by the semi-norms

$$r_{m,n}(\varphi)=\max_{k=0,\ldots,m}\left(\max_{a_n\leqq u\leqq b_n}\left|\varphi^{(k)}(u)\right|\right)\quad (m=0,1,\ldots;\ n=1,2,\ldots),$$

where

$$a_n=\begin{cases}a(1-1/n) & \text{if } a>-\infty\\ -n & \text{if } a=-\infty,\end{cases}\quad\text{and}\quad b_n=\begin{cases}b(1-1/n) & \text{if } b<+\infty\\ n & \text{if } b=+\infty.\end{cases}$$

In this way C_0^∞ becomes a Fréchet space. In the sequel, the topology defined by the semi-norms $\{r_{m,n}\}$ will be referred to as the C^∞ topology on C_0^∞.

In ([2], [3]), Krabbe has defined the product of $\varphi,\psi\in C_0^\infty$ to be the function $\varphi\wedge\psi$ given by

$$(\varphi\wedge\psi)(t)=\int_0^t\varphi(t-u)\psi(u)du\quad (t\in J).\tag{1}$$

For each φ in C_0^∞, define φ_+ (resp. φ_-) to be the function which coincides with φ on J_+ (J_-) and vanishes on J_- (J_+), and let $(C_0^\infty)_+$ $((C_0^\infty)_-)$ be the set of all such functions. Then $C_0^\infty=(C_0^\infty)_+\oplus(C_0^\infty)_-$. Also, it is easy to see that

$$\varphi\wedge\psi=\varphi_+*\psi_+-\varphi_-*\psi_-,\tag{2}$$

where $*$ denotes the convolution product, so that properties of the multiplicative operation $\wedge$ may be deduced from the corresponding properties of the convolution product. Thus, in view of the relationship between the operations $\wedge$ and $*$ given by (2), there seems to be no advantage in using both symbols, and so in the sequel we shall denote the product $\varphi\wedge\psi$ by $\varphi*\psi$.

† It should be mentioned that Krabbe's theory of generalized functions has some resemblances to an earlier algebraic theory of generalized functions due to Weston [9].

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It is straightforward to prove that, for any non-negative integer k,

$$(\varphi*\psi)^{(k)}(t) = (\varphi*\psi^{(k)})(t) \quad (t \in J). \tag{3}$$

With multiplication of functions defined according to (1), C_0^∞ becomes a linear, associative and commutative algebra; in fact, C_0^∞ is an ideal in the algebra L_1^{loc} of all complex-valued functions on J which are Lebesgue-integrable over every compact subinterval of J. Multiplication is jointly continuous in C_0^∞; this follows immediately from the inequality

$$r_{m,n}(\varphi*\psi) \leqq (b_n + |a_n|) r_{0,n}(\varphi) r_{m,n}(\psi).$$

We define a sequence $\{\varphi_n\}$ of *odd* functions in $C_0^\infty(-\infty, \infty)$ by the formula

$$\varphi_n(t) = \begin{cases} \dfrac{1}{\sqrt{(4\pi n t^3)}} e^{-1/4nt} & (t > 0), \\ 0 & (t = 0), \end{cases}$$

and note the following properties of $\{\varphi_n\}$:

(i) $\displaystyle\int_0^\infty \varphi_n(t)\,dt = 1 \quad (n = 1, 2, \ldots);$

(ii) for any $\delta > 0$, $\displaystyle\lim_{n\to\infty} \int_\delta^\infty \varphi_n(t)\,dt = 0.$

Let $\{\eta_n\}$ denote the restriction of φ_n to J. Then we have the following

LEMMA 1. *The sequence $\{\eta_n\}$ is an approximate identity for C_0^∞.*

Proof. We have to show that, for any ψ in C_0^∞,

$$\psi = \lim_{n\to\infty} \psi * \eta_n. \tag{4}$$

By (2), to prove (4) it is sufficient to show that

$$\psi_+ = \lim_{n\to\infty} \psi_+ * (\eta_n)_+, \tag{5}$$

and

$$\psi_- = \lim_{n\to\infty} \psi_- * (\eta_n)_-. \tag{6}$$

We shall prove only (6); (5) may be proved using a similar method.

Let $\varepsilon > 0$, p be any positive integer, and m any non-negative integer. Since $\psi^{(k)}(0) = 0\,(k = 0, 1, \ldots)$, there exists a positive number δ_k' such that $|\psi_-^{(k)}(v)| < \varepsilon$ whenever $-\delta_k' \leqq v \leqq 0$. Let $\delta' = \min(\delta_0', \ldots, \delta_m')$. By uniform continuity, there exists a positive number δ_k'' such that

$$|\psi_-^{(k)}(u - v) - \psi_-^{(k)}(u)| < \varepsilon$$

whenever $a_p \leqq u \leqq 0$ and $-\delta_k'' \leqq v \leqq 0$. Let $\delta'' = \min(\delta_0'', \ldots, \delta_m'')$, and let $\delta = \min(\delta', \delta'')$.

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Now, since $(\eta_n)_-(t) = \varphi_n(t)$ for t in J_-, we have

$$r_{m,p}(\psi_-*(\eta_n)_- + \psi_-) = \max_{k=0,\ldots,m}\left(\max_{a_p \leqq u \leqq 0}\left|\int_u^0 \psi_-^{(k)}(u-v)\varphi_n(v)dv + \psi_-^{(k)}(u)\right|\right).$$

If $-\delta \leqq u \leqq 0$, then

$$\left|\int_u^0 \psi_-^{(k)}(u-v)\varphi_n(v)dv + \psi_-^{(k)}(u)\right| < 2\varepsilon$$

for $k = 0, \ldots, m$. If $a_p \leqq u \leqq -\delta$, then

$$\left|\int_u^0 \psi_-^{(k)}(u-v)\varphi_n(v)dv + \psi_-^{(k)}(u)\right| = \left|\int_u^0 \psi_-^{(k)}(u-v)\varphi_n(v)dv - \int_{-\infty}^0 \psi_-^{(k)}(u)\varphi_n(v)dv\right|$$

$$\leqq \int_{-\delta}^0 |\psi_-^{(k)}(u-v) - \psi_-^{(k)}(u)||\varphi_n(v)|\,dv + \int_{a_p}^{-\delta} |\psi_-^{(k)}(u-v)||\varphi_n(v)|\,dv$$

$$+ \int_{-\infty}^{-\delta} |\psi_-^{(k)}(u)||\varphi_n(v)|\,dv. \tag{7}$$

From (ii), it follows that there exists a positive integer n_0 such that, for $n \geqq n_0$,

$$\int_{-\infty}^{-\delta} |\varphi_n(v)|\,dv < \varepsilon / r_{m,p}(\psi).$$

Therefore, for $n \geqq n_0$ and $k = 0, \ldots, m$, the right-hand side of (7) is less than 3ε. Hence, for $n \geqq n_0$,

$$r_{m,p}(\psi_- + \psi_-*(\eta_n)_-) < 3\varepsilon,$$

and so, since ε is arbitrary,

$$\psi_- = \lim_{n\to\infty} \psi_-*(\eta_n)_-,$$

as required.

We note that, since C_0^∞ has an approximate identity, C_0^∞ has no non-zero annihilators.

In ([2], [3]), Krabbe has defined a generalized function on J to be a *multiplier* on C_0^∞†; that is, a mapping A of C_0^∞ into itself such that

$$A(\varphi*\psi) = A\varphi*\psi$$

for all φ, ψ in C_0^∞. Let $\mathscr{M}$ denote the set of multipliers on C_0^∞. It is well-known that the multipliers on a commutative algebra with no non-zero annihilators are linear, and that, if linear combinations and products of multipliers are defined in the usual way, then $\mathscr{M}$ is a commutative algebra ([8], §4). Also, every element of $\mathscr{M}$ is C^∞ continuous ([1], Theorem 1).

It follows from (3) that the differential operator $D \in \mathscr{M}$. Also, each f in L_1^{loc} determines an element F in $\mathscr{M}$ according to the equation

$$F\varphi = f*\varphi. \tag{8}$$

† Krabbe did not use the term multiplier, but this seems now to be the standard terminology for this class of operators (see, for example, ([4], p. 51)).

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Krabbe ([2], (1.12)–(1.13)) has defined the *operator of the function* f to be the operator $\{f(t)\}$ given by the equation

$$\{f(t)\}\varphi = f*\varphi'.$$

In our notation $\{f(t)\}$ is DF, and so, in particular, $\{(f_1*f_2)(t)\}$ is DF_1F_2. If $f, g \in L_1^{\text{loc}}$ and $F = G$, then it follows from a theorem of Titchmarsh ([7], p. 327, Theorem 152) that $f = g$ a.e. By the same argument we can show that $DF = DG$ implies that $f = g$ a.e.

We have now proved the following theorem, which is the main result of [2].

THEOREM 1 ([2], Theorem 1.22). *The algebra* $\mathcal{M}$ *is commutative and each element of* $\mathcal{M}$ *is* C^∞ *continuous. The operator* $D \in \mathcal{M}$, *and each* f *in* L_1^{loc} *determines an element* F *of* $\mathcal{M}$ *according to* (8). *Also, for* f, g *in* L_1^{loc}, *the following properties hold:*

(a) $D\{(f*g)(t)\} = \{f(t)\}\{g(t)\}$;
(b) *if* $\{f(t)\} = \{g(t)\}$, *then* $f = g$ *a.e.*

For any $u > 0$, the shift operator I_u is the mapping of C_0^∞ into itself defined by the equation

$$I_u\varphi(t) = \begin{cases} \varphi(t-u) & (u \leqq t < b), \\ 0 & (-u < t < u), \\ \varphi(t+u) & (a < t \leqq -u). \end{cases}$$

Then $I_u \in \mathcal{M}$ for all $u > 0$.

Suppose $A \in \mathcal{M}$, and let φ_+ be any element of $(C_0^\infty)_+$. By Lemma 1,

$$\varphi_+ = \lim_{n\to\infty} \varphi_+ * \eta_n,$$

and so, since A is C^∞ continuous,

$$A\varphi_+ = \lim_{n\to\infty} \varphi_+ * A\eta_n.$$

It follows that A maps $(C_0^\infty)_+$ into itself. Similarly, A maps $(C_0^\infty)_-$ into itself. Thus every element of $\mathcal{M}$ is linear, C^∞ continuous, commutes with I_u $(u > 0)$, and leaves invariant $(C_0^\infty)_+$ and $(C_0^\infty)_-$. Conversely, we have the following result.

THEOREM 2. *Let* T *be a* C^∞ *continuous linear mapping of* C_0^∞ *into itself with the following properties:*

(a) $TI_u = I_uT$ *for all* $u > 0$;
(b) T *leaves invariant* $(C_0^\infty)_+$ *and* $(C_0^\infty)_-$.†

Then $T \in \mathcal{M}$.

Proof. Let $\varphi, \psi \in C_0^\infty$. Then

$$\varphi*\psi = \varphi_+*\psi_+ - \varphi_-*\psi_-.$$

† There are C^∞ continuous linear mappings of C_0^∞ into itself which have property (a) but not (b); for example, the mapping B given by

$$B\varphi(t) = \varphi(-t) \qquad (\varphi \in C_0^\infty,\ t \in J)$$

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Now $\varphi_+ * \psi_+$ may be approximated, in C^∞ sense, by a function of the form

$$\sum_{j=1}^{n} \alpha_j I_{u_j} \psi_+, \tag{9}$$

where $\alpha_j (j = 1, \ldots, n)$ are complex-numbers (see the proof of ([4], Theorem 4)). Similarly, the function $\varphi_- * \psi_-$ may be approximated, in the same sense, by a function of the form

$$\sum_{j=1}^{m} \beta_j I_{w_j} \psi_-. \tag{10}$$

Hence $\varphi * \psi$ may be approximated by $\sum_{j=1}^{n} \alpha_j I_{u_j} \psi_+ - \sum_{j=1}^{m} \beta_j I_{w_j} \psi_-$. Since T is linear,

$$T\psi = T\psi_+ + T\psi_-,$$

and, since T has property (b), it follows that $T\psi_+ = (T\psi)_+$ and $T\psi_- = (T\psi)_-$. Consequently

$$\varphi * T\psi = \varphi_+ * T\psi_+ - \varphi_- * T\psi_-.$$

The function in (9) (resp. (10)) may be chosen in such a way that while $\varphi_+ * \psi_+ (\varphi_- * \psi_-)$ can be approximated by (9) ((10)), $\varphi_+ * T\psi_+ (\varphi_- * T\psi_-)$ can be approximated by $\sum_{j=1}^{n} \alpha_j I_{u_j} T\psi_+$ $\left(\sum_{j=1}^{m} \beta_j I_{w_j} T\psi_-\right)$. Hence $\varphi * T\psi$ may be approximated by the function

$$\sum_{j=1}^{n} \alpha_j I_{u_j} T\psi_+ - \sum_{j=1}^{m} \beta_j I_{w_j} T\psi_-. \tag{11}$$

Since T is linear and commutes with the shift operators, (11) is the same as

$$T\left(\sum_{j=1}^{n} \alpha_j I_{u_j} \psi_+ - \sum_{j=1}^{m} \beta_j I_{w_j} \psi_-\right).$$

Finally, since T is C^∞ continuous, we have

$$T(\varphi * \psi) = \varphi * T\psi,$$

as required.

With convolution as multiplication, $\mathscr{D}'_+$ (resp. $\mathscr{D}'_-$) is a linear, associative, and commutative algebra. Let $\mathscr{M}_+$ ($\mathscr{M}'_-$) denote the algebra of multipliers on $(C_0)^\infty_+ ((C_0^\infty)_-)$. Shultz ([5], Theorem 4.3) has proved that $\mathscr{M}_+$ is algebraically isomorphic to $\mathscr{D}'_+$. Moreover, his proof may be easily modified to show that $\mathscr{M}_-$ is isomorphic to $\mathscr{D}'_-$. We may state Shultz's result as follows.

THEOREM A ([5], Theorem 4.3). *Let $A \in \mathscr{M}_+$ (resp. $B \in \mathscr{M}_-$). Then there exists a unique $\Phi \in \mathscr{D}'_+$ ($\Psi \in \mathscr{D}'_-$) such that*

$$A\varphi = \Phi * \varphi \qquad (B\psi = \Psi * \psi)$$

for all $\varphi \in (C_0^\infty)_+$ ($\Psi \in (C_0^\infty)_-$). Conversely, each $\Phi \in \mathscr{D}'_+$ ($\Psi \in \mathscr{D}'_-$) determines an element A of $\mathscr{M}_+$ (B of $\mathscr{M}_-$) according to the above equation. The correspondence $A \leftrightarrow \Phi$ ($B \leftrightarrow \Psi$) is an isomorphism of $\mathscr{M}_+$ onto $\mathscr{D}'_+$ ($\mathscr{M}_-$ onto $\mathscr{D}'_-$).

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(In this connexion it is of interest to note that the above result of Shultz enables us to answer a question raised in ([4], p. 53), as follows. With the notation and terminology of [4], we have shown that the C^∞ continuous V-operators on $C_0^\infty[0,\infty)$ are precisely the multipliers on $C_0^\infty[0,\infty)$, and so, if T is a C^∞ continuous V-operator on $C_0^\infty[0,\infty)$, then, by Theorem A, there exists a $\Phi\in\mathscr{D}'_+$ such that

$$T\varphi = \Phi*\varphi \ (\varphi\in C_0^\infty[0,\infty)).$$

It follows that T satisfies (P_2), and so there are no C^∞ continuous V-operators on $C_0^\infty[0,\infty)$ which do not have property (P_2).)

We note that, if linear combinations of elements of $\mathscr{D}'_+\times\mathscr{D}'_-$ are defined in the usual way and multiplication is defined according to the equation

$$(\Phi_1,\Psi_1)\otimes(\Phi_2,\Psi_2) = (\Phi_1*\Phi_2, \Psi_1*\Psi_2),$$

then $\mathscr{D}'_+\times\mathscr{D}'_-$ becomes a linear, associative, and commutative algebra.

Finally, we give an easier proof of ([6], Theorem 2.18).

THEOREM 3. *The correspondence* $T\leftrightarrow(\Phi,\Psi)(T\in\mathscr{M},(\Phi,\Psi)\in\mathscr{D}'_+\times\mathscr{D}'_-)$, *where*

$$T\varphi = \Phi*\varphi_+ + \Psi*\varphi_- \ (\varphi\in C_0^\infty),$$

is an isomorphism of $\mathscr{M}$ *onto* $\mathscr{D}'_+\times\mathscr{D}'_-$.

Proof. Suppose that $T\in\mathscr{M}$, and let T_+ (resp. T_-) denote the restriction of T to $(C_0^\infty)_+$ $((C_0^\infty)_-)$. Then, since T_+ (T_-) leaves invariant $(C_0^\infty)_+$ $((C_0^\infty)_-)$, $T_+\in\mathscr{M}_+$ $(T_-\in\mathscr{M}_-)$ and

$$T\varphi = T_+\varphi_+ + T_-\varphi_-.$$

By Theorem A, there exists a unique (Φ,Ψ) in $\mathscr{D}'_+\times\mathscr{D}'_-$ such that

$$T\varphi = \Phi*\varphi_+ + \Psi*\varphi_-. \tag{12}$$

Conversely, each (Φ,Ψ) in $\mathscr{D}'_+\times\mathscr{D}'_-$ determines an element T of $\mathscr{M}$ according to (12). It is straightforward to show that the correspondence $T\leftrightarrow(\Phi,\Psi)$ is an isomorphism of the algebra $\mathscr{M}$ onto the algebra $\mathscr{D}'_+\times\mathscr{D}'_-$.

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