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SIMULTANEOUS DIOPHANTINE APPROXIMATIONS AND HERMITE'S METHOD

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In this paper we generalize a result of Mahler on rational approximations of the exponential function at rational points by proving the following theorem: let $n \in \mathbb{N}^*$ and $\alpha_1, \ldots, \alpha_n$ be distinct non-zero rational numbers; there exists a constant $c = c(n, \alpha_1, \ldots, \alpha_n) > 0$ such that

$$q^{n+(c/\log\log q)}\left|q_0 + q_1e^{\alpha_1} + \ldots + q_ne^{\alpha_n}\right| \geq 1$$

for every non-zero integer point $(q_0, q_1, \ldots, q_n)$ and $q = \max\{|q_1|, \ldots, |q_n|, 3\}$.

1.

In 1873, Hermite gave his famous proof of the transcendence of e . Since then many improvements were introduced into Hermite's method which led to the deep results of Siegel. (For this development we refer to the survey paper by Fel'dman and Shidlovskii [5] and to the appendix of Mahler's book [10].) This method enabled Mahler [8] (see also [4]) to obtain a measure of irrationality of e . In fact, Mahler dealt with the problem of finding an effective lower bound for $|e^{\alpha}-\beta|$ with α and β rational numbers and thus determining explicitly the constants which appeared in the earlier results of Mahler [6], [7] and Popken [11]. In

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this paper, we apply the same method to the more general problem of determining an effective lower bound for $\left|\beta_0 + \beta_1 e^{\alpha_1} + \dots + \beta_n e^{\alpha_n}\right|$ where $\beta_0, \beta_1, \dots, \beta_n$ and $\alpha_1, \dots, \alpha_n$ are rational numbers. More precisely, we obtain

THEOREM. *Let* $n \in \mathbb{N}^*$ *and* $\alpha_1, \dots, \alpha_n$ *be distinct non-zero rational numbers. There exists a constant (easily computable)* $c = c(n, \alpha_1, \dots, \alpha_n) > 0$ *such that*

$$q^{n+(c/\log\log q)}\left|q_0 + q_1 e^{\alpha_1} + \dots + q_n e^{\alpha_n}\right| \geq 1$$

for every non-zero integer point $(q_0, q_1, \dots, q_n)$ *and* $q = \max\{|q_1|, \dots, |q_n|, 3\}$.

By means of a transference principle (see for example Cassels [3]), it is easy to derive from this theorem the following result:

COROLLARY. *Let* $n \in \mathbb{N}^*$ *and* $\alpha_1, \dots, \alpha_n$ *be distinct non-zero rational numbers. There exists a constant* $c_1 = c_1(n, \alpha_1, \dots, \alpha_n) > 0$ *such that*

$$\max_{1 \leq j \leq n} \|qe^{\alpha_j}\| \geq q^{-(1/n)-(c_1/\log\log q)} ,$$

for any integer $q \geq 3$. *(Here* $\|x\|$ *denotes the distance from a real number* x *to the nearest integer.)*

REMARK. This result is to be compared with the following theorem of Baker ([1] and [2], Theorem 10.1): let $\alpha_0, \alpha_1, \dots, \alpha_n$ be distinct non-zero rational numbers; there exist two constants $c = c(n, \alpha_0, \dots, \alpha_n) > 0$ and $\delta = \delta(n, \alpha_0, \dots, \alpha_n) > 0$ such that for any non-zero integer point $(q_0, q_1, \dots, q_n)$ with

$$q = \max_{0 \leq j \leq n} (|q_j|) \geq c$$

one has

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$$q_0^* q_1^* \ldots q_n^* \left| q_0 e^{\alpha_0} + q_1 e^{\alpha_1} + \ldots + q_n e^{\alpha_n} \right| \geq q^{1-\left(\delta/(\log\log q)^{\frac{1}{2}}\right)}$$

where $q_j^* = \max\left(1, |q_j|\right)$, $j = 0, 1, \ldots, n$.

Baker dealt with Siegel's method. (This method was likewise used by Mahler [9] for determining explicitly the constants c and δ .) This enabled him to obtain a lower bound which depends on the size of all the coefficients of the linear form, but with the exponent $(\log\log q)^{-\frac{1}{2}}$ in place of $(\log\log q)^{-1}$. It remains to find a method which succeeds in combining the two results.

2. Proof of the theorem

1. Denote by $v \geq 1$ an integer such that $v\alpha_j = u_i \in \mathbb{Z}$ for $j = 1, \ldots, n$ and put $u_0 = 0$. Let $q_0, q_1, \ldots, q_n$ be integers, not all zero, and $q = \max\{|q_1|, \ldots, |q_n|, 3\}$. It will be shown that

$$q^{n+(c/\log\log q)} \left| q_0 e^{u_0/v} + q_1 e^{u_1/v} + \ldots + q_n e^{u_n/v} \right| \geq 1 ,$$

where $c > 0$ denotes a constant that does not depend on q .

Put

$$\theta = \sum_{j=0}^{n} q_j e^{u_j/v} .$$

Let $N \in \mathbb{N}^*$. We define polynomials $f_j(x, z)$ $(0 \leq j \leq n)$, $P_{ij}(z)$ $(0 \leq i, j \leq n , \ i \neq j)$ and $Q_j(z)$ $(0 \leq j \leq n)$ by

$$f_j(x, z) = \frac{v^{1-N}}{(N-1)!} \left(vx-u_j z\right)^{N-1} \prod_{k=0}^{n} \left(vx-u_k z\right)^N ,$$

$$P_{ij}(z) = \sum_{m=N}^{(n+1)N-1} \frac{m!}{(N-1)!} v^{m-N+1}$$

$$\cdot \left[\sum_{\substack{\nu_0+\ldots+\nu_n=m \\ (\nu_i=N)}} \binom{N-1}{\nu_j} \left(u_i-u_j\right)^{N-1-\nu_j} \prod_{\substack{k=0 \\ k\neq i,j}}^{n} \binom{N}{\nu_k} \left(u_i-u_k\right)^{N-\nu_k} \right] z^{(n+1)N-m-1} ,$$

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$$Q_j(z) = \sum_{m=N-1}^{(n+1)N-1} \frac{m!}{(N-1)!} v^{m-N+1} \cdot \left[\sum_{\substack{\nu_0+\ldots+\nu_n=m \\ (\nu_j=N-1)}} \prod_{\substack{k=0 \\ k\neq j}}^{N} \binom{N}{\nu_k} \left(u_j-u_k\right)^{N-\nu_k} \right] z^{(n+1)m-N-1} .$$

Furthermore put

$$T_{ij}(z) = \left(1-\delta_{ij}\right)P_{ij}(z) + \delta_{ij}Q_j(z) , \quad 0 \le i,\ j \le n ,$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{jj} = 1$.

The coefficients of T_{ij} are rational integers and T_{ij} is of degree exactly $nN - 1$ if $i \neq j$ and of degree exactly nN if $i = j$.

It follows from the definition of T_{ij} that

$$\sum_{m\geq 0} \frac{\partial^m}{\partial x^m} f_j\left(\left(u_i/v\right)z,\ z\right) = T_{ij}(z) , \quad 0 \le i,\ j \le n .$$

Then, by Hermite's identity, we obtain for $z \in \mathbb{R}$ and $0 \le i,\ j \le n$,

$$(1) \quad e^{u_i z/v} T_{0j}(z) - T_{ij}(z)$$

$$= e^{u_i z/v} \int_0^{u_i z/v} e^{-t} f_j(t,\ z)dt = z^{(n+1)N} e^{u_i z/v} \int_0^{u_i/v} e^{-zt} \tilde{f}_j(t)dt$$

where $\tilde{f}_j$ is the polynomial defined by

$$\tilde{f}_j(t) = \frac{v^{1-N}}{(N-1)!} \left(vt-u_j\right)^{N-1} \prod_{\substack{k=0 \\ k\neq j}}^{n} \left(vt-u_k\right)^N .$$

Therefore, the determinant

$$\Delta(z) = \det_{0\le i,j\le n} \left(T_{ij}(z)\right)$$

is a polynomial in z of the exact degree $n(n+1)N$ which has at $z = 0$ a zero of order $n(n+1)N$. Then $\Delta(z) \neq 0$ if $z \neq 0$ and thus

$$\Delta(1) \neq 0 .$$

Now, we easily obtain from (1),

$$\theta T_{0j}(1) - \sum_{i=0}^{n} q_i T_{ij}(1)$$

$$= \sum_{i=0}^{n} q_i e^{u_i/v} \int_0^{u_i/v} e^{-t} f_j(t, 1)dt , \quad j = 0, 1, \ldots, n .$$

Since $\Delta(1) \neq 0$, at least one of the integers

$$\sum_{i=0}^{n} q_i T_{ij}(1) , \quad j = 0, \ldots, n ,$$

is distinct from zero. It follows that

$$\text{(2)} \qquad 1 \leq |\theta||T_{0j_0}(1)| + \left|\sum_{i=0}^{n} q_i e^{u_i/v} \int_0^{u_i/v} e^{-t} f_{j_0}(t, 1)dt\right|$$

for a suitable $j_0 \in \{0, 1, \ldots, n\}$.

2. Let $u = \max_{1 \leq j \leq n} \{|u_j|\}$. For $j \neq 0$ we have

$$|T_{0j}(1)| \leq v \cdot \sum_{m=N}^{(n+1)N-1} \frac{m!}{(N-1)!} \binom{nN-1}{m-N} v^{m-N} u^{(n+1)N-1-m} < \frac{[(n+1)N]!}{N!} (u+v)^{nN} ,$$

and, for $j = 0$,

$$|T_{00}(1)| \leq \sum_{m=N-1}^{(n+1)N-1} \frac{m!}{(N-1)!} \binom{nN}{m-N+1} v^{m-N+1} u^{(n+1)N-m-1} < \frac{[(n+1)N]!}{N!} (u+v)^{nN} .$$

Next, for $j = 0, 1, \ldots, n$,

$$\sup_{|x| \leq u/v} |f_j(x, 1)| \leq \frac{v^{1-N}}{(N-1)!} (2u)^{(n+1)N-1} < \frac{(2u)^{(n+1)N-1}}{(N-1)!} .$$

Then, by (2), we can write

$$\text{(3)} \qquad 1 \leq |\theta| \frac{[(n+1)N]!}{N!} (u+v)^{nN} + qne^u \frac{(2u)^{(n+1)N-1}}{(N-1)!} .$$

3. Denote by m_0 the smallest integer which satisfies

$$\text{(4)} \qquad qne^u (2u)^{(n+1)(m_0+1)} \leq m_0! .$$

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From the definition of m_0, it follows that

(5) $$(m_0-1)! < qne^u(2u)^{(n+1)m_0} .$$

Since

$$(Np)! \le N^{Np}(p!)^N \quad \text{for} \quad p \ge 1 \quad \text{and} \quad N \ge 1 ,$$

we have by (4), (5) and (3), with $N = m_0 + 1$,

$$1 \le 2|\theta|(n+1)^{(n+1)(m_0+1)}[(m_0+1)!]^n(u+v)^{n(m_0+1)}$$
$$\le 2|\theta|(n+1)^{(n+1)(m_0+1)}[(m_0+1)m_0]^n \cdot q^n n^n e^{nu}(2u)^{n(n+1)m_0} \cdot (u+v)^{n(m_0+1)} .$$

Hence

(6) $$1 \le |\theta| q^n c_1^{m_0}$$

with some constant $c_1 > 0$ which does not depend on q and m_0.

We now require an upper estimate for m_0. By (5) we have

$$m_0^{m_0+\frac{1}{2}} e^{-m_0} < m_0! < qne^u(2u)^{(n+1)m_0} m_0$$

and thus

$$m_0^{m_0} < qc_2^{m_0}$$

with

$$c_2 = ne^{u+1}(2u)^{n+2} .$$

From this it follows that

$$m_0 \le c_2 \frac{\log q}{\log\log q}$$

provided $\log\log q > 0$; that is, $q \ge 3$.

Finally we obtain, by (6),

$$1 \le |\theta| q^{n+(c/\log\log q)}$$

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with $c = c_2 \log c_1$.

REMARK. The estimates occurring in the above proof are mostly quite trivial and it is clear that the constant c can be greatly improved.

References

[1] A. Baker, "On some Diophantine inequalities involving the exponential function", *Canad. J. Math.* **17** (1965), 616-626.

[2] Alan Baker, *Transcendental number theory* (Cambridge University Press, Cambridge, 1975).

[3] J.W.S. Cassels, *An introduction to Diophantine approximation* (Cambridge Tracts in Mathematics and Mathematical Physics, 45. Cambridge University Press, Cambridge, 1957).

[4] Alain Durand, "Note on rational approximations of the exponential function at rational points", *Bull. Austral. Math. Soc.* **14** (1976), 449-455.

[5] Н.И. Фельдман, А.Б. Шидловский [N.I. Fel'dman and A.B. Shidlovskii], "Развитие и современное состояние теории трансцендентных чисел" [The development and present state of the theory of transcendental numbers], *Uspehi Mat. Nauk* **22** (1967), No. 3, 3-81; *Russian Math. Surveys* **22** (1967), No. 3, 1-79.

[6] Kurt Mahler, "Zur Approximation der Exponentialfunktion und des Logarithmus. I", *J. reine angew. Math.* **166** (1932), 118-136.

[7] Kurt Mahler, "Zur Approximation der Exponentialfunktion und des Logarithmus. II", *J. reine angew. Math.* **166** (1932), 137-150.

[8] Kurt Mahler, "On rational approximations of the exponential function at rational points", *Bull. Austral. Math. Soc.* **10** (1974), 325-335.

[9] Kurt Mahler, "On a paper by A. Baker on the approximation of rational powers of e ", *Acta Arith.* **27** (1975), 61-87.

[10] Kurt Mahler, *Lectures on transcendental numbers* (Lecture Notes in Mathematics, **546**. Springer-Verlag, Berlin, Heidelberg, New York, 1976).

https://doi.org/10.1017/S0004972700006274 Published online by Cambridge University Press

[11] J. Popken, "Zur Transzendenz von e", *Math. Z.* **29** (1929), 525-541.

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