A REMARK ON FINITELY GENERATED MODULES, II.

TADASI NAKAYAMA

The following small remark on the relationship of the (quasi-)regularity and the zero-divisor property may be more or less known but does not seem to the writer to have been explicitly stated in a literature.

PROPOSITION 1. *Let* $\mathfrak{r}$ *be a right-ideal in a ring* R. *If and only if* $\mathfrak{r}$ *is not contained in the radical* N *of* R (*or, what is equivalent,* $\mathfrak{r}$ *is not quasi-regular*), R *can be imbedded into a ring* S *with unit element 1 such that there is an element* a *in* $\mathfrak{r}$ *for which* $1-a$ *is a right zero-divisor in* S.

Proof. The "only if" part is clear. For, if $\mathfrak{r} \subseteqq N$ then every element a in $\mathfrak{r}$ is quasi-regular and $1-a$ is regular. To prove the "if" part, let $\mathfrak{r} \subseteqq\!\!\!\!/\, N$ and let $\mathfrak{m}$ be a non-zero cyclic R-right-module with a generator u_0 such that $\mathfrak{m} = u_0\mathfrak{r}$; for the existence of $\mathfrak{m}$ cf. (the second half of) V in our first note.[1] Let R^* be the ring $R \oplus Z$ (Z being the ring of rational integers) in which $x1 = 1x = x$ for every $x \in R$. $\mathfrak{m}$ may be considered, in natural manner, as a right-module of R.* In the module $\mathfrak{m} \oplus R^*$ we introduce distributive multiplication by

$$\mathfrak{m}^2 = 0, \quad R\mathfrak{m} = 0, \quad 1u = u \qquad (u \in \mathfrak{m});$$

the product of two elements in R^* and the product of an element in $\mathfrak{m}$ and an element of R,* in this order, are defined as they already are. The multiplication is associative, and $\mathfrak{m} \oplus R^*$ becomes a ring, which we want to denote by S. Now, since $u_0 \in \mathfrak{r} = u_0\mathfrak{r}$, there is an element a in $\mathfrak{r}$ such that $u_0 = u_0 a$. In S we have $u_0(1-a) = u_0 - u_0 a = 0$. So $1-a$ is a right zero-divisor in S. Our proposition is thus proved.

We have analogously also

PROPOSITION 2. *Let* a *be an element of a ring* R. *If, and only if,* a *is not right quasi-regular* (*in* R), *we can imbed* R *into a ring* S *with a unit element* 1 *such that* $1-a$ *is a right zero-divisor in* S.

Received December 30, 1954.

1) A remark on finitely generated modules, Nagoya Math. J. 3 (1951), 139–140. Thus, we have merely to consider $\mathfrak{m} = R/\mathfrak{s}$ with a maximal right-ideal $\mathfrak{s}$ with left modulo-unit in which $\mathfrak{r}$ is not contained.

Proof. The "only if" part is again clear, since if a is right quasi-regular in R then a is so in any extension S of R. Let, to prove the converse, a be a right non-quasi-regular element in R. Then the right-ideal $\mathfrak{r} = \{x - ax \mid x \in R\}$ does not coincide with R. Put $\mathfrak{m} = R/\mathfrak{r}$. Let R^* be as in the proof to the preceding proposition. We consider the module $\mathfrak{m} \oplus R^*$ with our present $\mathfrak{m}$, and introduce in it a multiplication just in the same manner in the proof to the preceding proposition, $\mathfrak{m}^2 = 0$, $R\mathfrak{m} = 0$ and $1u = u$ $(u \in \mathfrak{m})$. Denote the ring so obtained again by S. Let u_0 be the residue-class of a modulo $\mathfrak{r}$; $u_0 \in \mathfrak{m}$ and $u_0 \neq 0$. Then $u_0(1-a) = u_0 - u_0 a = 0$, since $a - a^2 \in \mathfrak{r}$. Hence $1 - a$ is a right zero-divisor in S. This proves our proposition.

The motivation and the relationship to our first note,[2] of the above considerations, are the following, besides that we have referred to it. In our first note we generalized namely a theorem of Azumaya and combined the theorem thus obtained with the Jacobson-Kaplansky theory of radicals to produce several propositions which may be summarized as follows: Let $\{\mathfrak{r}\}$ be a certain family of right-ideals in a ring R. The following properties of $\{\mathfrak{r}\}$ imply each other:

(A) For every maximal right-ideal $\mathfrak{r}_0$ with left modulo-unit of R, there exists (at least) a right-ideal $\mathfrak{r}$ in the family $\{\mathfrak{r}\}$ such that $\mathfrak{r} \subseteqq \mathfrak{r}_0$;

(B) If $\mathfrak{m}$ is a finitely generated right-module of R such that $\mathfrak{m} = \mathfrak{m}R$ and $\mathfrak{m} = u_1\mathfrak{r} + \dots + u_n\mathfrak{r}$ for every finite generating system $u_1, \dots, u_n$ of $\mathfrak{m}$ and for every $\mathfrak{r} \in \{\mathfrak{r}\}$, then $\mathfrak{m} = 0$[3].

---

[2] S. foot note 1).

[3] The line "the family of right-ideals of III," before the proposition V of the first note, should read "the family of all maximal right-ideals with modulo-unit."

Further, the writer was perhaps too hasty when he wrote, in the proof to the proposition V there, that the implication of (B) from (A) was "evident." The "evident" needs an explanation. It is indeed evident if we take into account (the proposition 1 there and) the fact that for every maximal right-ideal $\mathfrak{r}^*$ of the ring $R^* = R \oplus Z$ as above (and as there) the intersection $\mathfrak{r}^* \cap R$ is either R itself or a maximal right-ideal of R with left modulo-unit; this fact can readily be seen from that if $\mathfrak{r}^* \cup R \neq R$ then $\mathfrak{r}^* + R = R^*$ whence there is an element $c \in R$ with $1 - c \in \mathfrak{r}^*$ (whence $x \equiv cx \bmod \mathfrak{r}^* \cap R$ for all $x \in R$).

However, what is perhaps better is to prove the implication directly, and the proof is merely to repeat the argument of our proof to the proposition 1 of the first note. Thus, assume (A). From $\mathfrak{m} = \mathfrak{m}R$ we have $\mathfrak{m} = u_1R + \dots + u_nR$, for any generating system $u_1, \dots, u_n$ of R. So u_1 can be expressed in a form $u_1 = u_1c_1 + \dots + u_nc_n$ $(c_i \in R)$. Let $\mathfrak{r}_1$ be the right-ideal of R consisting of elements x such that $u_1x \in u_2R + \dots + u_nR$; in case $m = 1$ the void sum in the right-hand side stands for 0. As $u_1x = u_1c_1x + u_2c_2x + \dots + u_nc_nx$, whence $x - c_1x \in \mathfrak{r}_1$, for any element x of R, c_1 is a left modulo-unit for $\mathfrak{r}_1$. Suppose here $c_1 \notin \mathfrak{r}_1$, i.e. $\mathfrak{r}_1 \neq R$. Then there is a maximal right-ideal $\mathfrak{r}_0$ of R containing $\mathfrak{r}_1$, which evidently possesses c_1 as a left modulo-unit. By our assumption we have $\mathfrak{m} = u_1\mathfrak{r}_0 + \dots + u_n\mathfrak{r}_0$, whence much the more $\mathfrak{m} = u_1\mathfrak{r}_0 + u_2R + \dots + u_nR$. Expressing u_1c_1 accordingly in a form $u_1c_1 = u_1a + \dots$ with $a \in \mathfrak{r}_0$, we have $c_1 - a \in \mathfrak{r}_1$ $(a \in \mathfrak{r}_0)$. This is however a contradiction, since $\mathfrak{r}_1 \subseteqq \mathfrak{r}_0$, $c_1 \notin \mathfrak{r}_0$. Thus necessarily $\mathfrak{r}_1 = R$, or, what is the same, $u_1R \subseteqq u_2R + \dots + u_nR$. Now our assertion $\mathfrak{m} = 0$ can be obtained by an easy induction on the minimum number of generating elements.

(We may restrict ourselves in (B) to cyclic modules and their single generators; cf. the end of the first note.)

The two-sided analogy of this is the following fact, due essentially to Jacobson and Kaplansky (cf. II, III of the first note): Let N be the radical of a ring R. The following properties concerning a (two-sided) ideal M are equivalent to each other: (A_0) $M \subseteqq N$; (B_0) If $\mathfrak{m}$ is a finitely generated right-module of R with $\mathfrak{m} = \mathfrak{m}M$, then necessarily $\mathfrak{m} = 0$.

The implication $(A_0) \Rightarrow (B_0)$ may be used in proving another theorem of Jacobson that the radical of the matrix ring R (contains and in fact) is N (and indeed a somewhat more general theorem) as Azumaya observes. It is also true that we can prove the implication $(A_0) \Rightarrow (B_0)$ by means of the last matrix ring theorem. For if a right-module $\mathfrak{m}$ is generated by $u_1, \ldots, u_n$ and satisfies $\mathfrak{m} = \mathfrak{m}N$, then there are n elements a_{ij} in N such that $(u_1, \ldots, u_n) = (u_1, \ldots, u_n)(a_{ij})$, or $(u_1, \ldots, u_n)(I - (a_{ij})) = 0$, where I is the unit matrix of degree n considered as the unit element of the matrix ring $(R^*)_n$ over the ring R^* obtained from R by the adjunction of a unit element 1. If we know that[4] N_n is contained in the radical of $(R^*)_n$, then we can conclude that $(I - (a_{ij})$ is regular in $(R^*)_n$ whence$)(u_1, \ldots, u_n) = 0$.

This argument may be modified to show the following fact: Let n be a natural number. Let M be an ideal of R. Suppose that, for every set of n^2 elements a_{ij} of M, the element $I - (a_{ij})$ of $(R^*)_n$ is a right non-zero-divisor in $(R^*)_n$. Then, if $\mathfrak{m}$ is a right-*ideal* in R generated by some n elements and satisfies $\mathfrak{m} = \mathfrak{m}M$, then necessarily $\mathfrak{m} = 0$.

These considerations seem to show the necessity of clarifying the relationship of the (quasi-)regularity and the non-zero-divisor property, and our answer is what we proved above.

Nagoya University and University of Hamburg

---

4) Observe that N is contained in (and coincides with, in fact) the radical of R^*.