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APPROXIMATION OF L_p-CONTRACTIONS BY ISOMETRIES

BY

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ABSTRACT. We construct a positive linear contraction T of all $L_p(X, \mu)$-spaces, $1 \leqq p \leqq \infty$, $\mu(X) = 1$ such that $T1 = 1$, $T^*1 = 1$ and also $Tf > 0$ a.e. for all $f \geqq 0$ a.e., $f \not\equiv 0$ but for which there is an $f \in L_\infty$ such that $(T^n f - \int f d\mu)$ does not converge in L_1-norm. We also show that if T is a contraction of a Hilbert space H, there exists an isometry Q and a contraction R such that $\|T^n x - Q^n R x\| \longrightarrow 0$ as $n \longrightarrow \infty$ for all x in H.

0. **Introduction.** Let (X, Σ, μ) be a measure space and $\mu(X) = 1$. Let T be a positive linear transformation acting on the (equivalent classes of) measurable functions on X such that $T1 = 1$ and T is contraction of all L_p-spaces with μ an invariant measure; i.e. $\int |Tf|^p d\mu \leqq \int |f|^p d\mu$ for all $p, 1 \leqq p \leqq \infty$, and for all measurable $f : X \longrightarrow \mathbf{R}$. It is known [2], [3] that there is a sub σ-algebra $\Sigma_d \subset \Sigma$ such that if $1 < p < \infty$, then

$$L_p(\Sigma_d) = \{f \text{ in } L_p : \|T^n f\|_p = \|f\|_p \text{ for all } n \geqq 0\}$$

and $T^n f$ converges weakly to $E(f|\Sigma_d)$ in $L_2(\mu)$ for all f in L_2, where $E(\cdot|\Sigma_d)$ is the conditional expectation with respect to Σ_d.

If, moreover, T is recurrent in the sense of Harris and aperiodic (see [2]), then $T^n f$ converges to $\int f d\mu$ in L_1-norm. In [5, pp. 113–115], Rosenblatt gave an example where Σ_d is trivial but $T^n f$ does not converge to $\int f d\mu$ in norm for some indicator functions, I_A. However, in section 1, we show that for any contraction T of a Hilbert space H, there is an isometry Q and a contraction R such that for all f in H, $(T^n f - Q^n R f)$ converges in norm to 0 as $n \longrightarrow \infty$.

If T, as in the beginning, is also Harris recurrent then it has the following property [4, p. 160]: there is a set C, $\mu(C) \neq 0$ and an integer $N > 0$ such that if $f \geqq 0$, $f I_C \not\equiv 0$ then $T^n f > 0$ on C for all $n \geqq N$. Then one might ask whether $T^n f$ converges to $\int f d\mu$ in L_1-norm for all f in L_1 for a contraction T with the property: $Tf > 0$ for all $f \geqq 0, f \not\equiv 0$. In section 2, we show that this is not so. Rosenblatt's example does not satisfy this property. Note that if T is Harris recurrent with this property then it must be aperiodic.

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1. **Contractions on Hilbert Spaces.** Let H be a Hilbert space. Recall that a linear operator $S : H \longrightarrow H$ is called a contraction if $\|S\| \leqq 1$, an isometry if $\|Sf\| = \|f\|$ for all $f \in H$, self-adjoint if $S = S^*$ and positive definite if $(Sf, f) \geqq 0$ for all $f \in H$. Here, as usual, $S^* : H \longrightarrow H$ denotes adjoint operator of S and $(\cdot, \cdot)$ is the inner product in H. Hence, by definition, $(Sf, g) = (f, S^*g)$ for all $f, g \in H$. Our purpose in this section is to obtain the following theorem.

THEOREM 1.1. *Given a contraction $T : H \longrightarrow H$ there is an isometry $Q : H \longrightarrow H$ and another contraction $R : H \longrightarrow H$ such that* $\lim_n \|T^n f - Q^n Rf\| = 0$ *for all $f \in H$.*

The proof will be given after several lemmas.

LEMMA 1.2. *If $T : H \longrightarrow H$ is a contraction then*

$$\|T^{*k}T^{n+k}f - T^n f\|^2 \leqq \|T^n f\|^2 - \|T^{n+k}f\|^2$$

for any $f \in H$ and for any integers $n, k \geqq 0$.

PROOF. Since T^* is also a contraction,

$$\begin{aligned}\|T^{*k}T^{n+k}f - T^n f\|^2 &= \|T^{*k}T^{n+k}f\|^2 + \|T^n f\|^2 - 2(T^{*k}T^{n+k}f, T^n f)\\ &\leqq \|T^{n+k}f\|^2 + \|T^n f\|^2 - 2\|T^{n+k}f\|^2\\ &= \|T^n f\|^2 - \|T^{n+k}f\|^2.\end{aligned}$$

LEMMA 1.3. *Given a contraction $T : H \longrightarrow H$ there is a self adjoint and positive definite $P : H \longrightarrow H$ such that* $\lim_n \|T^{*n}R^n f - Pf\| = 0$ *for any $f \in H$.*

PROOF. Lemma 3.2 shows that

$$\|T^{*n+k}T^{n+k}f - T^{*n}T^n f\|^2 \leqq \|T^{*k}T^{n+k}f - T^n f\| \leqq \|T^n f\|^2 - \|T^{n+k}f\|^2.$$

Since $\|T^n f\|^2$ is a non increasing sequence of non negative numbers, we see that $T^{*n}T^n f$ is a Cauchy sequence in H. Define Pf as the norm limit if $T^{*n}T^n f$. Then it is clear that $P : H \longrightarrow H$ is linear, self adjoint and positive definite.

REMARKS 1.4. A similar but a much better known result is the convergence of $(T^*T)^n f$. Also, Lemma 1.3 is a special case of a result in [1], obtained later.

LEMMA 1.5. *Let T and P be as in Lemma 1.3. Then* $\lim_n \|PT^n f - T^n f\| = 0$ *for each $f \in H$.*

PROOF. Lemmas 1.3 and 1.2 show that

$$\begin{aligned}\|PT^n f - T^n f\|^2 &= \lim_k \|T^{*k}T^k T^n f - T^n f\|^2\\ &\leqq \|T^n f\|^2 - \lim_k \|T^{n+k}f\|\|^2,\end{aligned}$$

from which the proof follows.

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LEMMA 1.6. *If $R : H \longrightarrow H$ is positive definite then $\|Rf - f\| \leqq \|R^2 f - f\|$ for any $f \in H$.*

PROOF. Since R is positive definite,

$$\begin{aligned} 0 &\leqq (R(Rf - f), (Rf - f)) \\ &= ((R^2 f - f) - (Rf - f), (Rf - f)) \\ &= (R^2 f - f, Rf - f) - \|Rf - f\|^2 \\ &\leqq \|Rf - f\|(\|R^2 f - f\| - \|Rf - f\|), \end{aligned}$$

which gives the proof. □

PROOF OF THEOREM 1.1. Let $T : H \longrightarrow H$ be a contraction and let P be as in lemma 1.3. Since P is positive definite and self adjoint, it is known that there is a positive definite and self adjoint operator $R : H \longrightarrow H$ such that $P = R^2$. Note that

$$\begin{aligned} \|Rf\|^2 &= (Rf, Rf) = (R^2 f, f) \\ &= (Pf, f) = \lim_n (T^{*n} T^n f, f) \\ &= \lim_n (T^n f, T^n f) = \lim_n \|T^n f\|^2. \end{aligned}$$

Let $H_0 = \{f \mid f \in H, \lim_n \|T^n f\| = 0\}$, which is a closed linear subspace of H. Let $H_1 = H_0^{\perp}$ be the orthogonal complement of H_0. Hence $Rf = 0$ whenever $f \in H_0$ and $Rf \neq 0$ if $f \in H_1 - \{0\}$. Finally let $M = RH (= RH_1)$ be the range of R, which is a linear subspace of H.

We now define an operator $Q : M \longrightarrow M$ as follows. If $f \in M$ then there is a unique $g \in H_1$ such that $f = Rg$. We then let $Qf = RTg$. It is clear that Q is a linear operator. Also, $\|Qf\| = \|RTg\| = \lim_n \|T^n(Tg)\| = \lim_n \|T^n g\| = \|Rg\| = \|f\|$ for any $f \in M$. This shows that Q is an isometry on M. Hence it can be extended to an isometry on $\bar{M}$, the closure of M. We can also extend Q to an isometry on the whole space H by defining it, for example, as the identity operator on the orthogonal complement of $\bar{M}$. Let $Q : H \longrightarrow H$ denote this extended operator. Now it is easy to see that $Q : H \longrightarrow H$ is an isometry such that $QRf = RTf$ for all $f \in H$. Hence, by induction, $Q^n Rf = RT^n f$ for each $f \in H$ and for each integer $n \geqq 0$.

We now claim that $\lim_n \|T^n f - Q^n Rf\| = 0$ for all $f \in H$. In fact,

$$\begin{aligned} \|T^n f - Q^n Rf\| &= \|T^n f - RT^n f\| \\ &\leqq \|T^n f - R^2 T^n f\| = \|T^n f - PT^n f\| \longrightarrow 0 \end{aligned}$$

as $n \longrightarrow \infty$. Here the inequality follows from Lemma 1.3 and the convergence of $\|T^n f - PT^n f\|$ to zero follows from Lemma 1.3.

2. **An example.** We give in this section, an example of a positive linear contraction of all L_p, $1 \leqq p \leqq \infty$, such that $T1 = 1$ and such that the σ-algebra Σ_d defined in 1.1 is trivial but the limit distribution ν of $T^n f$ is not a point mass for some function f.

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Construction of T. Let $(Z, \mathcal{H}) = (X \times Y, \mathcal{F} \times \mathcal{G})$, $X = Y = [0,1]$ with the Lebesgue product measure μ. $Q : Z \longrightarrow Z$ is the well-known Baker's transformation: $Q(x,y) = (x/2, 2y) \pmod 1$ if $2y < 1$ and $Q(x,y) = ((x+1)/2, 2y) \pmod 1$ if $2y > 1$. Put $P_0 = \{(x,y) : 0 < x < 1/2\}$ and $P_1 = P_0^c$.

$$\mathcal{P} = \{P_0, P_1\} \quad \text{and} \quad \mathcal{P}_n = \bigvee_{i=0}^{n-1} Q^i \mathcal{P}.$$

A set of the form $\{(x,y) : a < y < b\}$ will be called a a strip and a set of the form $\{(x,y) : a < x < b\}$ will be called a column. The intersection of a strip and a column will be called a section. The members of $\mathcal{P}_n$ are the columns

$$C_k^n = \left\{(x,y) : \frac{k-1}{2^n} < x < \frac{k}{2^n}\right\}, 1 \leqq k \leqq 2^n.$$

A column of $\mathcal{P}_n$ is called even if k is even and odd if k is odd.

For $n \geqq 1$, the transformation τ_n defined on a strip B_n, $\tau_n : B_n \longrightarrow B_n$ permutes two by two, by translating the sections of $\mathcal{P}_n \cap B_n$, that is $\tau_n x = x + (-1)^{k+1}/2^n$ if $x \in C_k^n \cap B_n$. The operator T will be constructed inductively. Fix ρ, $0 < \rho < 1/4$. $B_1 = B_{1,1}$ is any strip such that $\mu(B_{1,1}) = \rho$. R_1 is τ_1 on B_1 and is identity elsewhere. Suppose $B_{n,i}$, $i = 1, \ldots, n$, B_n and R_n defined, then $B_{n+1,i}$, $i = 1, \ldots, n+1$ are strips such that $\mu(B_{n+1,i}) = \rho^{n+1}$, $B_{n+1,i} \subset B_{n,i}$, $i = 1, \ldots, n$ and $B_{n+1,n+1} \subset B_n^c$, $B_{n+1} = \bigcup_{i=1}^{n+1} B_{n+1,i}$, R_{n+1} is τ_{n+1} on each $B_{n+1,i}$, $i = 1, \ldots, n+1$ and is identity elsewhere. Put $R = \lim_{n \to \infty} R_n R_{n-1} \ldots R_2 R_1$. This is well defined for almost all $(x,y) \in Z$ since $\Sigma \mu(B_n) < \Sigma n \rho^n < \infty$. Finally, E is the conditional expectation with respect to $\mathcal{G}$, that is for $f \in \mathbf{L}_p(Z)$, $(Ef)(x) = \int_y f(x,y) dy$. By identifying $\mathbf{L}_p(X, \mathcal{F})$ with $\mathbf{L}_p(Z, \mathcal{F} \times \{\phi Y\})$, we define T by $T = E(RQ)^{-1}$ where $(RQ)^{-1} f(x,y) = f((RQ)^{-1}(x,y))$.

Consider $f = 1_{P_0}$. On $Q^n P_0$, $n \geqq 1$,

$$T^n 1_{P_0} > 1 - \sum_1^n k\rho^k = 1 - \frac{\rho}{(1-\rho)^2} > \frac{5}{9}.$$

Thus, $T^n f$ does not converge to a constant function.

Lemma 2.1. *For all $n \geqq 1$ and for all pair of columns, $C_k^n, C_l^n \in \mathcal{P}_n$, $1 \leqq k, l \leqq 2^n$, $\mu((RQ)C_k^n \cap C_l^n) > \rho^n/2^n$.*

Proof. For $n = 1$, by construction of R_1,

$$\mu((R, Q)C_k^1 \cap C_l^1) = \begin{cases} \rho & \text{if } k \neq l \\ 1 - \rho & \text{if } k = l \end{cases}.$$

Suppose that $\mu((R_{n-1}Q)C_k^{n-1} \cap C_l^{n-1}) > \rho^{n-1}/2^{n-1}$. Let $C_k^n, C_l^n \in \mathcal{P}_n$ where k and l are odd, $k \neq l$, then $C_k^n \cup C_{k+1}^n = D_k \in \mathcal{P}_{n-1}$ and $C_l^n \cup C_{l+1}^n = D_l \in \mathcal{P}_{n-1}$.

Hence $(R_{n-1}\dots R_1Q)D_k\cap D_l$ contains a section of a strip B_{n-1} of measure $>\rho^{n-1}/2^{n-1}$. Since $B_{n,i}\subset B_{n-1,i}$ and since R_n permutes the sections of $B_{n,i}\cap \mathcal{P}_n$, $(R_n\dots R_1)C_k,\cap C_l$, where $k'=k$ or $k+1$ and $l'=l$ or $l+1$ contains a section of a strip B_n of measure $(\rho/2)^n$. And since at least $(\rho/2)^{n-1}-(\rho/2)^n>\rho^n/2^n$ is invariant for all R_m, $m>n$, the lemma is true for n.

If $l=k+1$ or $l=k-1$, the lemma is true because of the action of R_n on the strip $B_{n,n}$.

Then if $\|Tf\|_p=\|f\|_p$ for some p, $1<p<\infty$ and some $f\in \mathbf{L}_p(X)$ then from the preceding lemma, $f(x+1/2^n)=f(x)$ a.e. for all $n\geqq 1$. Hence $E(f|\mathcal{P}_n)=c$ but $E(f|\mathcal{P}_n)\longrightarrow f$ a.e. since $\mathcal{P}_n\uparrow\mathcal{F}$. Then $f=c$.

Notice also that for all $f\geqq 0$ a.e. then $Tf>0$ a.e. Let $B\in\mathcal{F}$, $\mu(B)>0$. Put $E=\{k/2^n : -2^n<k<2^n,\ n=0,1,2,\dots\}$ and let $A=B+E$ (mod 1). Then $T1_A<1_A$, which implies that $T1_A=1_A$ and thus $1_A=c1$. Then $T1_B>0$ a.e. □

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