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Abstract

In this paper we study persistence features of topological entropy and periodic orbit growth of Hamiltonian diffeomorphisms on surfaces with respect to Hofer's metric. We exhibit stability of these dynamical quantities in a rather strong sense for a specific family of maps studied by Polterovich and Shelukhin. A crucial ingredient comes from enhancement of lower bounds for the topological entropy and orbit growth forced by a periodic point, formulated in terms of the geometric self-intersection number and a variant of Turaev's cobracket of the free homotopy class that it induces. Those bounds are obtained within the framework of Le Calvez and Tal's forcing theory.

1. Introduction and results

Prototypical examples of maps that define a dynamical system of chaotic nature are horseshoe maps, introduced by Smale [Sma67]. In dimension 2, a horseshoe model is given by stretching a square horizontally and squeezing it vertically before folding it back in the shape of a horseshoe. Iterating a horseshoe T has the features of a topologically chaotic system: there is a rich symbolic dynamics on a compact invariant set and, in particular, there is exponential growth of periodic orbits and positive topological entropy. A remarkable property of horseshoes is their local stability. Structural stability asserts that a diffeomorphism T' that is sufficiently C^1-close to T contains the same symbolic dynamics as that of T, and has at least the topological entropy of T.[1] Horseshoes are prevalent in dynamical systems of complex orbit structure. This is particularly the case for surface diffeomorphisms: By a celebrated result of Katok [Kat80], any C^2 surface diffeomorphism of positive topological entropy on a compact surface has a hyperbolic fixed point with a transverse homoclinic point and has a horseshoe in some iterate. Besides these local stability features of surface maps, there exist 'global' analogues in surface dynamics, by restricting to certain isotopy classes of homeomorphisms: a pseudo-Anosov homeomorphism is a factor of a subsystem of any homeomorphism isotopic to it [Han85]. In particular, it minimizes the topological entropy in its isotopy class.

In this paper we exhibit stability phenomena for Hamiltonian diffeomorphisms on surfaces that have flavors of local as well as global nature. We motivate and briefly explain here some

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[1] In fact, a similar result holds for C^0-perturbations of T (see [Nit71]).

results and refer to § 1.2 below for definitions and precise statements. The group of Hamiltonian diffeomorphisms carries a remarkable conjugation-invariant norm $\|\cdot\|_{\text{Hofer}}$ inducing a bi-invariant metric d_{Hofer}, the Hofer metric. From a point of view that is concerned with C^k topology, $k \geq 0$, this metric is rather flexible, in particular diffeomorphisms close with respect to d_{Hofer} are in general not C^0 close, and at least part of the dynamics of some horseshoe might not survive under perturbation. Nevertheless, in terms of lower bounds of some dynamical quantities, remarkable stability features are displayed, even under relatively large perturbations (in terms of Hofer's metric). We exhibit such phenomena in the case of surfaces of genus $g \geq 2$ by considering a specific construction from [PS16] of the so-called eggbeater maps or eggbeaters, and its generalization by the first author [Cho22]. Eggbeaters are Hamiltonian variants of linked twist maps, and the latter were studied by various authors (e.g. by Thurston [Thu88] as a family of examples of pseudo-Anosov homeomorphisms). Eggbeaters have positive topological entropy and exponential homotopical orbit growth (cf. [Dev78] for the detection of horseshoes in linked-twist maps). We are interested in the extent to which dynamical properties of eggbeaters persist under perturbation. To that end, fix some $0 < \delta < 1$. We say that some dynamical property of ϕ δ-persists if it holds for all ψ with $d_{\text{Hofer}}(\psi, \phi) < \delta\|\phi\|_{\text{Hofer}}$. One of our main results is that on closed surfaces of genus $g \geq 2$ and for some fixed $\delta > 0$, any given lower bound on the topological entropy ('$h_{\text{top}} \geq E$') δ-persists for an unbounded family of eggbeater diffeomorphisms (cf. Theorem 1.4). An analogous result holds for the exponential homotopical orbit growth. Moreover, we exhibit minimality of h_{top} up to a fixed finite error on a family of eggbeaters (cf. Corollary 1.5), which is reminiscent of the minimality property of h_{top} for pseudo-Anosov homeomorphisms.

To obtain our persistence results we will use certain lower bounds on the topological entropy for surface homeomorphisms in the presence of orbit types with specific properties (cf. Theorems 1.1–1.3), and the first part of the paper is devoted to their proofs. We will now motivate and state these results as well as define relevant dynamical notions, and then, below in § 1.2, turn to the persistence results mentioned above.

1.1 Free homotopy classes of periodic orbits and forcing

Let M be a closed oriented surface and $f : M \to M$ a homeomorphism isotopic to the identity. The topological entropy $h_{\text{top}}(f)$ is one of the most widely used measures for the complexity of the dynamics of f and can be defined as follows: for a fixed metric d on M, denote by $N_{\text{top}}(f, n, \epsilon)$ the maximal cardinality of a set X in M such that $\sup_{0 \leq k \leq n} d(f^k(x), f^k(y)) \geq \epsilon$ for all $x, y \in X$ with $x \neq y$. The *topological entropy* of f is then

$$h_{\text{top}}(f) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(N_{\text{top}}(f, n, \epsilon))}{n},$$

which is well defined and independent of the metric d. A related measure for dynamical complexity is the growth of periodic points: if $N_{\text{per}}(f, n)$ denotes the number of periodic points of f of period at most n, then the (exponential) growth rate of periodic points is defined as $\text{Per}^{\infty}(f) := \limsup_{n \to \infty}(\log(N_{\text{per}}(f, n))/n)$.

Although it is in general very hard to calculate or estimate the topological entropy, remarkably, the existence of certain types of periodic points of f implies lower bounds on $h_{\text{top}}(f)$. The following result in dimension 1 is a typical example of this forcing phenomenon: if a continuous mapping g on $[0, 1]$ admits a periodic point of period n which is not a power of 2, then $h_{\text{top}}(g) > (1/n) \log(2)$ (see [BF76]). Passing to dimension 2, in order to deduce some interesting dynamical information of f, the period alone is not sufficient and one has to take into account further characteristics of a periodic point. It is natural to require that the desired

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bounds are invariant under any isotopy of f relative to the periodic orbit, which amounts to considering the braiding information, or braid type, of the periodic orbit in the suspension flow. Thurston–Nielsen theory provides a natural framework for understanding the dynamics that is forced by a given braid type, namely, dynamically minimal maps for a braid type are given by the Thurston–Nielsen canonical form. There are algorithmic methods to obtain the latter, although it sometimes becomes difficult to apply them in practice, especially when passing to families of braid types. Entropy bounds for specific (families of) braid types have been investigated by many researchers (see, for example, [Hal94, Son05, HK06] and references therein). The complete braiding information of an orbit, in particular if one considers orbits of higher complexity, is somewhat hard to overlook, and one simplification is to project braids back to the surface, in other words, to look for braid-type invariants that can be presented purely in terms of the homotopy class of the curve that the orbit traces on M through an identity isotopy of f. It is that kind of invariants that we will consider in this paper. Especially for dealing with bounds related to that kind of invariants, the recently developed forcing theory of Le Calvez and Tal [LeCT18] building upon Le Calvez's theory of transverse foliations [LeC05] is perfectly suited. Lower bounds for $h_{\mathrm{top}}(f)$ and $\mathrm{Per}^\infty(f)$ of that kind are contained in [LeCT18, LeCT22, DaS19], as applications of their methods, and in [Dow11] using Thurston–Nielsen theory. Our results improve and extend part of these bounds, and we work closely within the theory in [LeCT18]. For related results see also the recent work [GM22]. It contains a proof of the existence of topological horseshoes if the homotopy class induced by the orbit cannot be represented by a multiple of a simple curve, with additional properties of the horseshoe depending on further specifics of that class.

Let us introduce some essential notions. Let $I = (I_t)_{t\in[0,1]}$ be an identity isotopy for f (i.e. a continuous path from the identity to f), and consider the set $\mathrm{dom}(I) = M \setminus \mathrm{Fix}(I)$, where $\mathrm{Fix}(I) = \{x \in M \mid I_t(x) \equiv x \text{ for all } t \in [0,1]\} \subset \mathrm{Fix}(f)$ are the points that are fixed throughout the isotopy I. We denote by $\widehat{\pi}(X)$ the set of free homotopy classes of loops in a space X, and for any loop $\Gamma : S^1 \to X$, by $[\Gamma] = [\Gamma]_{\widehat{\pi}(X)}$ its free homotopy class in $\widehat{\pi}(X)$. We say that $\alpha \in \widehat{\pi}(\mathrm{dom}(I))$ is *primitive* if there is no representative of α that multiply covers another loop. For $m \in \mathbb{N}$, denote by $m\alpha$ the free homotopy class that is represented by the m-fold iteration of a loop that represents α. For any periodic point $x \in \mathrm{dom}(I)$ of f, say of period $q \in \mathbb{N}$, consider the loop $I^q(x)$ given by the concatenation of the paths $I_t(x)_{t\in[0,1]}$, $I_t(f(x))_{t\in[0,1]}, \dots,$ and $I_t(f^{q-1}(x))_{t\in[0,1]}$. The loop $I^q(x)$ defines a free homotopy class $[I^q(x)] = \alpha \in \widehat{\pi}(\mathrm{dom}(I))$ of loops in $\mathrm{dom}(I)$. An identity isotopy I is *maximal* if, for any fixed point $y \in \mathrm{dom}(I)$ of f, the loop $I(y) = I^1(y)$ is not contractible in $\mathrm{dom}(I)$ (cf. § 2.3). Let $\Gamma : S^1 \to \mathrm{dom}(I)$ be a loop and denote by $\mathcal{S}(\Gamma) := \{y \in \mathrm{dom}(I) \mid y = \Gamma(t) = \Gamma(t'), t \neq t'\}$ the set of self-intersections of Γ, and by $\mathrm{si}(\Gamma) = \#\mathcal{S}$ its cardinality. The *geometric self-intersection number of* α is defined as $\mathrm{si}_{\mathrm{dom}(I)}(\alpha) := \min \mathrm{si}(\Gamma)$, where the minimum is taken over all smooth loops Γ with $[\Gamma] = \alpha$ in $\widehat{\pi}(\mathrm{dom}(I))$ that are in *general position*, that is, each self-intersection $y \in \mathcal{S}(\Gamma)$ is transverse and $\#\Gamma^{-1}(y) = 2$.

Theorem 1.1. *Let M be a closed oriented surface, $f : M \to M$ a homeomorphism isotopic to the identity, and I a maximal identity isotopy for f. Let $\alpha \in \widehat{\pi}(\mathrm{dom}(I))$ be primitive with $\mathrm{si}_{\mathrm{dom}(I)}(\alpha) \neq 0$. If there is a q-periodic point x of f in $\mathrm{dom}(I)$ with $[I^q(x)] = m\alpha \in \widehat{\pi}(\mathrm{dom}(I))$ for some $m \in \mathbb{N}$, then both $\mathrm{Per}^\infty(f)$ and $h_{\mathrm{top}}(f)$ are at least equal to*

$$\frac{m}{q} \max\left\{ \frac{\log(\mathrm{si}_{\mathrm{dom}(I)}(\alpha) + 1)}{16}, \frac{\log(2)}{2} \right\}.$$

If $M = S^2$ and $\mathrm{si}_{\mathrm{dom}(I)}(\alpha) \neq 0$, a fixed positive lower bound for $h_{\mathrm{top}}(f)$ and $\mathrm{Per}^\infty(f)$ was obtained in [LeCT18, Theorem 41] and improved in [LeCT22, DaS19]. Theorem 1.1 improves

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these lower bounds, the main addition here being that the bounds grow with the complexity of α. We note that for many choices of M and α, similar lower bounds were obtained by Dowdall in [Dow11] using Thurston–Nielsen theory.

Theorem 1.1 implies similar lower bounds that are independent of the isotopy I. If M has genus greater than or equal to 2, then, since the space of homeomorphisms isotopic to the identity is contractible, the free homotopy class $\alpha := [I^q(x)]_{\widehat{\pi}(M)}$ in $\widehat{\pi}(M)$ for a q-periodic point $x \in M$ of f does not depend on the choice of identity isotopy I, and we may say that x is a *q-periodic point of class α*.[2] Theorem 1.1 and a result in [BCLR20] about existence of maximal isotopies (cf. § 2.3) yield the following theorem (see § 4).

THEOREM 1.2. *Let M be a closed oriented surface of genus $g \geq 2$ and α a primitive class in $\widehat{\pi}(M)$ with $\mathrm{si}_M(\alpha) \neq 0$. Let $f : M \to M$ be a homeomorphism that is isotopic to the identity. If f has a q-periodic point of class α, then $\mathrm{Per}^\infty(f)$ and $h_{\mathrm{top}}(f)$ are at least equal to*

$$1/q \max\left\{\frac{\log(\mathrm{si}_M(\alpha)+1)}{16}, \frac{\log(2)}{2}\right\}.$$

In other words, a periodic point of class α that admits geometric self-intersections forces lower bounds for topological entropy and exponential orbit growth, and the more self-intersections the higher the bounds.

Theorem 1.2 raises the following question. What can be said about the classes in $\widehat{\pi}(M)$ of those periodic points that are forced by the periodic point x of class α? We address this question in connection with growth and consider the *exponential homotopical orbit growth rate* given by

$$\mathrm{H}^\infty(f) := \limsup_{n\to\infty} \frac{\log(N_{\mathrm{h}}(f,n))}{n},$$

where $N_{\mathrm{h}}(f,n)$ is the number of distinct classes of periodic points of period less than n. Do $\mathrm{H}^\infty(f)$ and $\mathrm{Per}^\infty(f)$ satisfy similar bounds? While the proof of Theorem 1.2 does not give a positive lower bound on $\mathrm{H}^\infty(f)$ in terms of $\mathrm{si}_M(\alpha)$, we can bound $\mathrm{H}^\infty(f)$ in terms of another invariant of α. We adapt a construction of Turaev [Tur78, Tur91] and then use it to define a growth rate $\mathrm{T}^\infty(\alpha)$. We explain this construction briefly here (see § 5 for details).

Denote by $\widehat{\pi}(M)^*$ the set of non-trivial classes in $\widehat{\pi}(M)$. A loop $\Gamma : [0,1] \to M$ in general position representing a class $\alpha \in \widehat{\pi}(M)^*$ splits at each self-intersection point $y \in \mathcal{S}(\Gamma)$ into two (oriented) closed loops u_1^y and u_2^y based at y, representing two classes $\alpha_1^y, \alpha_2^y \in \widehat{\pi}(M)$, where we choose the labeling such that the initial tangent points of u_1^y and u_2^y define the orientation of Σ. Denoting by $\mathcal{Y}$ the intersection points for non-trivial α_1^y and α_2^y,

$$v(\alpha) = \sum_{y\in\mathcal{Y}} \alpha_1^y \otimes \alpha_2^y - \alpha_2^y \otimes \alpha_1^y \in \mathbb{Z}[\widehat{\pi}(M)^*] \otimes \mathbb{Z}[\widehat{\pi}(M)^*]$$

defines (by linear extension) *Turaev's cobracket* on the free $\mathbb{Z}$-module over $\widehat{\pi}(M)^*$. We consider a variant of this construction, and assume additionally that $x_0 = \Gamma(0) = \Gamma(1)$ is not an intersection point of Γ. For each self-intersection point $y = \Gamma(t) = \Gamma(t'), t < t'$, by composing with the paths $\Gamma_{[0,t]}$ and its inverse $\overline{\Gamma_{[0,t]}}$ we obtain loops $\Gamma_{[0,t]} u_i^y \overline{\Gamma_{[0,t]}}$, $i = 1, 2$, based at $x_0 = \Gamma(0) = \Gamma(1)$. These loops define elements $a_i^y \in \pi_1(M, x_0)$, $i = 1, 2$ (see Figures 12 and 13).

Let $g \in \pi_1(M, x_0)$ be the element represented by Γ. Denote by $\pi_1(M, x_0)_g^*$ the set of g-equivalence classes of the non-trivial elements in $\pi_1(M, x_0)$, where we say that two elements are *g-equivalent* if one is a conjugation of the other by a multiple of g, and denote the equivalence

[2] This partitions q-periodic points into equivalence classes which coincide with f^q-Nielsen classes of those points (see, for example, [Jia96]).

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class by $[\cdot]_g$. The element

$$\mu(g) = \sum_{y \in \mathcal{Y}} [a_1^y]_g \otimes [a_2^y]_g - [a_2^y]_g \otimes [a_1^y]_g \in \mathbb{Z}[\pi_1(M, x_0)_g^*] \otimes \mathbb{Z}[\pi_1(M, x_0)_g^*]$$

is independent of the choice of the loop Γ based at x_0 that represents g; moreover, $\mu(g)$ behaves well with respect to conjugation (see Lemma 5.1). One actually obtains via μ an invariant of $\alpha \in \widehat{\pi}(M)$ that is a (strict) refinement of the invariant v. We continue to define a growth rate as follows. First, for a subset $S = \{s_1, \dots, s_m\} \subset \pi_1(M, x_0)$, let $\widehat{N}(n, S)$ be the number of distinct conjugacy classes of elements in $\pi_1(M, x_0)$ that can be written as a product of at most n factors from S, and define $\Gamma(S) := \limsup_{n\to\infty}(\log(\widehat{N}(n, S))/n)$. Moreover, given $g \in \pi_1(M, x_0)$ and a set $\mathfrak{S}$ of g-equivalence classes of elements in $\pi_1(M, x_0)$, we define $\Gamma(\mathfrak{S}, g) = \inf \Gamma(S)$, where the infimum is taken over all sets $S \subset \pi_1(M, x_0)$ with $\#S = \#\mathfrak{S}$ and such that each element in S represents a different g-equivalence class in $\mathfrak{S}$. Now, writing $\mu(g) = \sum_{\mathfrak{a}, \mathfrak{b} \in \pi_1(M, x_0)_g^*} k_{\mathfrak{a},\mathfrak{b}}(\mathfrak{a} \otimes \mathfrak{b})$, with $k_{\mathfrak{a},\mathfrak{b}} \in \mathbb{Z}$, we define the complexity $\mathrm{Comp}(\mu(g))$ as the collection of terms $\mathfrak{a} \otimes \mathfrak{b}$ with $k_{\mathfrak{a},\mathfrak{b}} > 0$. Let $\mathrm{Comp}_+(\mu(g)) = \{\mathfrak{a} \mid \exists \mathfrak{b} : \mathfrak{a} \otimes \mathfrak{b} \in \mathrm{Comp}(\mu(g))\}$ and $\mathrm{Comp}_+(\mu(g)) = \{\mathfrak{b} \mid \exists \mathfrak{a} : \mathfrak{a} \otimes \mathfrak{b} \in \mathrm{Comp}(\mu(g))\}$, and define

$$\Gamma^g := \min_{\pm} \min_{\mathfrak{S}} \left\{ \Gamma(\mathfrak{S} \cup [g]_g, g) \,\middle|\, \mathfrak{S} \subset \mathrm{Comp}_{\pm}(\mu(g)),\ \#\mathfrak{S} = \left\lceil \tfrac{1}{2} \#\mathrm{Comp}(\mu(g)) \right\rceil \right\}.$$

One shows that Γ^g is actually invariant under conjugation, and hence defines a growth rate $\mathrm{T}^\infty(\alpha) \in [0, +\infty)$ associated to each free homotopy class α of loops in M.

We obtain the following result.

THEOREM 1.3. *Let M be a closed oriented surface of genus $g \geq 2$ and $\alpha \in \widehat{\pi}(M)$ be a primitive class. Let $f : M \to M$ be a homeomorphism isotopic to the identity. If f has a q-periodic point x of class α, then* $\mathrm{H}^\infty(f) \geq (1/q)\mathrm{T}^\infty(\alpha)$.

By a version of Ivanov's inequality we have that $h_{\mathrm{top}}(f) \geq \mathrm{H}^\infty(f)$ ([Iva82, Jia96]; see also [Alv16a]). Hence, Theorem 1.3 provides another lower bound on $h_{\mathrm{top}}(f)$ in terms of the complexity of α. In § 7.3 we exhibit an infinite family of classes α in $\widehat{\pi}(M)$ such that $\mathrm{T}^\infty(\alpha) \geq \log(\mathrm{si}_M(\alpha)/4)$ (see Lemmas 7.8 and 7.9). In particular, the bound in Theorem 1.3 sometimes turns out to be significantly better than that in Theorem 1.2.

We outline the general strategy for the proofs of Theorems 1.1–1.3. Assume there is a q-periodic point x of f that is in a primitive class α with $\mathrm{si}_M(\alpha) > 0$. By the results in [BCLR20] one can choose a maximal identity isotopy I for f. Le Calvez's theory of transverse foliations yields a singular foliation $\mathcal{F}$ on M that is transverse to I (see § 2). This means, in particular, that there is a loop Γ transverse to $\mathcal{F}$ which is freely homotopic to $I^q(x)$ in $\mathrm{dom}(I)$, and we say that Γ is associated to x. Le Calvez and Tal's results provide methods to manipulate Γ in order to obtain other transverse loops associated to periodic points of f. Roughly speaking, in good situations such loops might be created by starting at $\Gamma(0)$, following Γ positively transverse to $\mathcal{F}$, stopping at some intersection point y which is reached for the first time, and finally continuing along Γ in positive direction after a turn at y. The turn is to the left or to the right, depending on which direction is positively transverse to the foliation. In other words, one creates a shortcut of Γ at y. Moreover, one might iterate this procedure and create shortcuts at several self-intersection points of Γ. The question whether the created loops are associated to periodic points is a bit subtle and one crucial assumption is that the intersections where the shortcuts are created are $\mathcal{F}$-transverse (see § 2 for the definition). Moreover, especially if several self-intersections are considered, it is important to understand what the maximal 'length' of a subpath of a lift of Γ to the universal cover of $\mathrm{dom}(I)$ is such that this subpath is $\widetilde{\mathcal{F}}$-equivalent to a subpath of another

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lift of Γ (see § 2 for the definitions). A better upper bound on this length generally leads to a better lower bound on the topological entropy. We address this and related problems in § 3 and establish some results (e.g. Lemma 3.8), which might also be of independent interest. The proof of Theorem 1.1 is then given in § 4. Finally, let us outline the argument for Theorem 1.3, and refer to § 5 for details. If $\mathrm{T}^\infty(\alpha) > 0$ and g denotes the element represented by Γ in $\pi_1(M, \Gamma(0))$, then for each element $\mathfrak{a} \otimes \mathfrak{b}$ in $\mathrm{Comp}(\mu(g))$ there is an $\mathcal{F}$-transverse self-intersection point y of Γ for which the shortcut at y represents either $\mathfrak{a}$ or $\mathfrak{b}$, and one can deduce the existence of a periodic point of a class that is induced by $\mathfrak{a}$ or $\mathfrak{b}$. Moreover, one can create several shortcuts to an iterate of Γ. We show that for (sufficiently large) $n > 0$ the loops $\Gamma' = \gamma_{\rho_1}\gamma_{\rho_2}\cdots\gamma_{\rho_n}$ are associated to periodic points of period nq, where Γ' is a concatenation of n loops γ_{ρ_i}, each of which either is equal to Γ or is obtained from Γ by a shortcut at some of the above intersection points y with one additional assumption: the turns at the chosen intersection points have to be either all to the left or all to the right. These observations allow us to conclude the inequality $\mathrm{H}^\infty(f) \geq (1/q)\mathrm{T}^\infty(\alpha)$.

1.2 Hamiltonian diffeomorphisms and persistence of topological entropy

We now turn to applications of Theorem 1.1 in the context of Hamiltonian diffeomorphisms.

Let us consider a closed symplectic manifold (M, ω) and denote by $\mathrm{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms on M, that is, the group of those diffeomorphisms that are the time-1 map of the (time-dependent) flow generated by a Hamiltonian vector field X_{H_t} of some $H : S^1 \times M \to \mathbb{R}$ (cf. § 6.1). The group $\mathrm{Ham}(M, \omega)$ carries a distinctive bi-invariant metric d_{Hofer}, the Hofer metric, which plays a central role in the study of rigidity phenomena of Hamiltonian diffeomorphisms. The distance $d_{\mathrm{Hofer}}(\varphi, \psi)$ for any $\varphi, \psi \in \mathrm{Ham}(M, \omega)$ may be defined to be $\|\varphi \circ \psi^{-1}\|_{\mathrm{Hofer}}$, where

$$\|\theta\|_{\mathrm{Hofer}} := \inf_H \int_0^1 \Big(\max_M H_t - \min_M H_t \Big)\, dt,$$

with the infimum taken over all Hamiltonian functions $H : [0,1] \times M \to \mathbb{R}$ whose associated Hamiltonian flow has θ as time-1 map. It follows directly from the definition that small perturbations in the sense of Hofer's metric are not necessarily small in the C^0 metric.[3] The geometry of $(\mathrm{Ham}(M, \omega), d_{\mathrm{Hofer}})$ and its interplay with dynamics has been thoroughly studied since its discovery by Hofer and his work in the early 1990s (see [Pol01] for an extensive account with many results and references).

Hofer's metric plays an important role in the stability features of Floer homology. Let us briefly explain this in our context (more details can be found in § 6.1). Given that (M, ω) is symplectically aspherical and atoroidal, then for a free homotopy class α the action functional of a Hamiltonian H on the smooth representatives $\mathcal{L}^\alpha(M)$ of α is given by $\mathcal{A}_H(x) = \int_0^1 H(t, x(t))\, dt - \int_{\bar{x}} \omega$, where $\bar{x} : S^1 \times [0,1] \to M$ is a smooth map from an annulus to M, where one boundary component $S^1 \times \{1\}$ maps to x and the other $S^1 \times \{0\}$ to a fixed representative of α. If H is non-degenerate one defines the Floer homology as the homology of a chain complex generated by 1-periodic orbits of X_{H_t}, and chain maps given by counting zero-dimensional moduli spaces of solutions $u : \mathbb{R} \times S^1 \to M$ of a certain perturbed nonlinear Cauchy–Riemann equation $\bar{\partial}_{H,J}(u) = 0$ whose asymptotes $\lim_{s \to \pm\infty} u(s,t) = x_\pm(t)$ are 1-periodic orbits $x_\pm(t)$ of X_{H_t}. Since action decreases along such solutions, the Floer homology can be filtered by action, taking into account only periodic orbits of action less than a, for varying $a \in \mathbb{R}$. The full Floer homology for a non-trivial free homotopy class α vanishes, but one can take the filtration into account in

[3] More subtle is the fact that also the converse fails in general (see, for example, [EPP12]).

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order to understand the structure and existence of orbits in α. An elegant and fruitful way to keep track of the filtered Floer homology is to use the theory of persistent modules and barcodes [PRSZ20]. With this terminology, any Hamiltonian gives rise to a barcode (i.e. a multiset of intervals in $\mathbb{R}$). The barcode, in this setting, only depends on the Hamiltonian diffeomorphism that is the time-1 map of X_{H_t}. Action estimates on continuation maps in Floer homology lead to stability properties of barcodes with respect to Hofer's metric, and hence in particular to the persistence of certain fixed points.

A special family $\mathcal{E}_g$ of Hamiltonian diffeomorphisms on surfaces $M = \Sigma_g$ of genus g, called eggbeaters, was introduced by Franjione and Ottino in [FO92], and used in a symplectic setting by Polterovich and Shelukhin in [PS16]. The construction was carried out for surfaces of genus $g \geq 4$ and later extended by the first author in [Cho22] to surfaces of genus $g \geq 2$. Computations of certain barcode invariants were carried out and it was proved that $\mathcal{E}_g$ provide a class of Hamiltonian diffeomorphisms whose Hofer distance to the space of autonomous Hamiltonian diffeomorphisms can be arbitrarily large. The same results pass to some products $\Sigma_g \times N$ [PS16, Zha19], and the constructions allow the free group of two generators to be embedded into the asymptotic cone of the group of Hamiltonian diffeomorphisms equipped with Hofer's metric [AGK+19, Cho22].

One may formulate the above results as (a consequence of) persistence of certain dynamical properties of eggbeaters. Eggbeaters are prototypical for a chaotic dynamical system. While this chaotic behavior, as opposed to integrable behavior, served as motivation to investigate these maps in [PS16], the question of the persistence of chaos, that is, the persistence of entropy, exponential orbit complexity etc., has not yet been addressed. The following result gives some answers in the setting of surfaces of genus $g \geq 2$.

Theorem 1.4. *Let (Σ_g, σ_g) be a surface of genus $g \geq 2$ with an area form σ_g. There is a sequence $\phi_l \in \mathcal{E}_g$ of eggbeaters on Σ_g with $M_l := \|\phi_l\|_{\mathrm{Hofer}} \to \infty$ and constants $\delta, C > 0$, such that for all $l \in \mathbb{N}$ and all $\psi \in \mathrm{Ham}(M, \omega)$ with $d_{\mathrm{Hofer}}(\psi, \phi_l) < \delta M_l$,*

$$\mathrm{H}^{\infty}(\psi) \geq \log(C M_l^2)$$

and, in particular,

$$h_{\mathrm{top}}(\psi) \geq \log(C M_l^2). \tag{1}$$

In particular, using the terminology at the beginning of this introduction, for any constant E there is an unbounded family in $(\mathrm{Ham}(\Sigma_g, \sigma_g), d_{\mathrm{Hofer}})$ on which ('$h_{\mathrm{top}} \geq E$') δ-persists.

The sequence ϕ_l satisfies $h_{\mathrm{top}}(\phi_l) \leq \log(C' M_l^2)$ for some constant $C' > C$ (see Remark 7.4), and hence the lower bounds in (1) are 'optimal' from an asymptotic viewpoint, or, in other words, eggbeaters are 'almost' minimal points for the topological entropy on $\mathrm{Ham}(\Sigma_g, \sigma_g)$. More precisely, we shall prove the following corollary in § 7.

Corollary 1.5. *Let (Σ_g, σ_g) be as above. Then there are constants $K > 0$ and $\delta > 0$, and a sequence of $\phi_l \in \mathcal{E}_g$ with $M_l := \|\phi_l\|_{\mathrm{Hofer}} \to \infty$ and $h_{\mathrm{top}}(\phi_l) \to \infty$, such that all $\psi \in \mathrm{Ham}(\Sigma_g, \sigma_g)$ with $d(\psi, \phi_l) < \delta M_l$ satisfy*

$$h_{\mathrm{top}}(\psi) \geq h_{\mathrm{top}}(\phi_l) - K.$$

The sequence ϕ_l defines an element in the asymptotic cone of $(\mathrm{Ham}(\Sigma_g, \sigma_g), d_{\mathrm{Hofer}})$. An interesting question is whether similar minimality properties hold also for defining sequences of most elements in the image of the embedding of F_2 into the asymptotic cone of $(\mathrm{Ham}(\Sigma_g, \sigma_g), d_{\mathrm{Hofer}})$ obtained in [AGK+19], and this question motivates efforts to further strengthen the bounds obtained in Theorem 1.3.

Finally, we will also observe that the lower bounds obtained in Theorem 1.4 are Hofer generic and we show the following theorem.

Theorem 1.6. *Let (Σ_g, σ_g) be a surface of genus $g \geq 2$, and let $M \geq 0$. There is an open and dense set $\mathcal{U} \subset \mathrm{Ham}(\Sigma_g, \sigma_g)$ with respect to the topology induced by Hofer's metric d_{Hofer} such that $h_{\mathrm{top}}(\psi) \geq M$ for all $\psi \in \mathcal{U}$.*

The proofs of Theorems 1.4 and 1.6 and Corollary 1.5 are given in § 7.

A recent result in [Kha21], building on different methods (cf. [BM19]), asserts that for surfaces Σ_g of genus $g \geq 1$ there is for any $C \geq 0$ a class of Lagrangian pairs (L, L'), $L, L' \subset \Sigma_g$, such that $h_{\mathrm{top}}(\varphi) > C$ for all φ with $\varphi(L) = L'$. It would be interesting to understand whether these sets of pairs may satisfy some stability properties similar to those of Theorems 1.4 and 1.6.

Very recently, the authors of [ÇGG21] show that the topological entropy of a Hamiltonian diffeomorphism φ on a closed surface coincides with its barcode entropy $\hbar(\varphi)$ which they introduce, and which measures the growth of the number of certain bars in the barcode of the iterates of φ. Hence, together with the results in this paper, this shows that the results obtained in Theorems 1.4, 1.6 and Corollary 1.5 hold additionally for $\hbar$. This is noteworthy, since it is a priori not clear which stability properties hold for $\hbar$ with respect to Hofer's metric.

Also recently, motivated by the present work, stability properties with respect to d_{Hofer} on the braid types of periodic orbits of Hamiltonian surface diffeomorphisms have been studied by Alves and the second author (see [AM21]). One dynamical consequence is that h_{top} is lower semi-continuous on $(\mathrm{Ham}(\Sigma, \omega), d_{\mathrm{Hofer}})$ for closed surfaces Σ.

Related questions of global robustness of positive entropy for families of contactomorphisms on contact manifolds were studied extensively and fruitfully in recent years by various methods. A large class of contactomorphisms are those that arise via Reeb flows, and there is an abundance of contact manifolds for which the topological entropy or the exponential orbit growth rate is positive for all Reeb flows. Examples and dynamical properties of those manifolds are investigated in [AASS21, Alv16a, Alv16b, Alv19, ACH19, AM19, Côt21, FS06, MS11, Mei18]. Some of these results generalize to positive contactomorphisms [Dah18], and results on the dependence of some lower bounds on the topological entropy with respect to their positive contact Hamiltonians have been obtained in [Dah21]. Forcing results for Reeb flows are obtained in [AP22]. A related discussion and results on questions of C^0-stability of the topological entropy of geodesic flows can be found in [ADMM22]. While the approach in our paper is suited to dimension 2, different methods yield higher-dimensional symplectic manifolds for which conclusions similar to that of Theorems 1.4 and 1.6 hold. This will be discussed elsewhere.

1.3 Structure of the paper

In § 2 we recall the theory of transverse foliations on surfaces, which is the setting in which Theorems 1.1–1.3 are proved. In § 3 we restrict our attention to loops with so-called $\mathcal{F}$-transverse self-intersections, and prove a few key claims.

In § 4 we prove Theorems 1.1 and 1.2, using the tools developed in the previous sections. In § 5 we define the growth rate T^∞ of a free homotopy class and prove Theorem 1.3.

Section 6 gives a short background on Floer theory and persistence modules, which is then used in § 7 to derive bounds on entropy and periodic orbit growth of large perturbations with respect to Hofer's metric of eggbeater maps. This will, in particular, prove Theorems 1.4 and 1.6.

2. Transverse foliations and transverse intersections

In this section we will give the definitions and results from the theory of transverse foliations that are relevant for this paper. We follow mainly [LeCT18].

2.1 Surface foliations and transverse paths

In the following let M be an oriented surface. The plane $\mathbb{R}^2$ will be endowed with the usual orientation. A *path* on M is a continuous map $\gamma : J \to M$, defined on an interval $J \subset \mathbb{R}$. A path γ is *proper* if J is open and the preimage of every compact subset of M is compact. A *line* is an injective and proper path $\lambda : J \to M$; it inherits a natural orientation induced by the usual orientation of $\mathbb{R}$. If $M = \mathbb{R}^2$, the complement of λ has two connected components, one on the right, $R(\lambda)$, and one on the left, $L(\lambda)$. A *loop* is a continuous map $\Gamma : S^1 = \mathbb{R}/\mathbb{Z} \to M$. It lifts to a path $\gamma : \mathbb{R} \to M$ with $\gamma(t+1) = \gamma(t)$ for all $t \in \mathbb{R}$, the *natural lift* of Γ. For two closed finite intervals $J = [a, b], J' = [a', b']$ and paths $\gamma : J \to M$, $\gamma' : J' \to M$ with $\gamma(b) = \gamma'(a')$, we denote by $\gamma\gamma'$ the usual concatenation of paths. In particular, Γ^m, for $m \in \mathbb{N}$, is the m-fold iteration of a loop Γ (i.e. $\Gamma^m(t) := \gamma(mt)$). The path obtained by reverse parametrization of γ is denoted by $\overline{\gamma}$.

A *singular oriented foliation* on M is an oriented topological foliation $\mathcal{F}$ defined on an open set of M. This set is called the domain of $\mathcal{F}$ and is denoted by $\mathrm{dom}(\mathcal{F})$. The complement $M \setminus \mathrm{dom}(\mathcal{F})$ is the *singular set*, denoted by $\mathrm{sing}(\mathcal{F})$. We denote by ϕ_z the leaf passing through $z \in \mathrm{dom}(\mathcal{F})$, and by ϕ_z^+ the positive and by ϕ_z^- the negative half-leaf. A path $\gamma : J \to M$ is *(positively) transverse to* $\mathcal{F}$ or $\mathcal{F}$*-transverse* if its image does not meet the singular set and if, for every $t_0 \in J$, there is a continuous chart $h : W \to (0, 1)^2$ at $\gamma(t_0)$ compatible with the orientation and sending the foliation $\mathcal{F}|_W$ onto the vertical foliation oriented downwards such that the map $\pi_1 \circ h \circ \gamma$ is increasing in a neighborhood of t_0, where π_1 is the vertical projection. If $\mathrm{dom}(\mathcal{F})$ is connected we denote by $\widetilde{\mathrm{dom}(\mathcal{F})}$ the universal covering space of the surface $\mathrm{dom}(\mathcal{F})$, otherwise we denote by $\widetilde{\mathrm{dom}(\mathcal{F})}$ the disjoint union of the universal coverings of its connected components. Then $\mathcal{F}|_{\mathrm{dom}(\mathcal{F})}$ lifts to a (non-singular) foliation $\widetilde{\mathcal{F}}$ on $\widetilde{\mathrm{dom}(\mathcal{F})}$. Note that since there is no non-singular foliation on S^2, $\widetilde{\mathrm{dom}(\mathcal{F})}$ is always homeomorphic to $\mathbb{R}^2$, or to several copies of $\mathbb{R}^2$ if $\mathrm{dom}(\mathcal{F})$ is disconnected. Every lift of an $\mathcal{F}$-transverse path γ is an $\widetilde{\mathcal{F}}$-transverse path on $\widetilde{\mathrm{dom}(\mathcal{F})}$. We say that $\widetilde{\gamma} : \mathbb{R} \to \widetilde{\mathrm{dom}(\mathcal{F})}$ is a *lift of the loop* Γ (*to* $\widetilde{\mathrm{dom}(\mathcal{F})}$) if it is a lift of its natural lift $\gamma : \mathbb{R} \to \mathrm{dom}(\mathcal{F})$.

If $\mathcal{F}$ is a *non-singular* foliation on $\mathbb{R}^2$, then two $\mathcal{F}$-transverse paths $\gamma : J \to \mathbb{R}^2$ and $\gamma' : J' \to \mathbb{R}^2$ are *equivalent for* $\mathcal{F}$ if there exists an increasing homeomorphism $h : J \to J'$ such that $\phi_{\gamma'(h(t))} = \phi_{\gamma(t)}$ for every $t \in \mathbb{R}$. In general, if $\mathcal{F}$ is a (possibly *singular*) foliation on an oriented surface M, two transverse paths $\gamma : J \to M$ and $\gamma' : J' \to M$ are called *equivalent for* $\mathcal{F}$ if they can be lifted to the universal covering space $\widetilde{\mathrm{dom}(\mathcal{F})}$ of $\mathrm{dom}(\mathcal{F})$ as paths that are equivalent for the lifted foliation $\widetilde{\mathcal{F}}$. A loop $\Gamma : S^1 \to \mathrm{dom}(\mathcal{F})$ is called *positively transverse* to $\mathcal{F}$ if this holds for the natural lift $\gamma : \mathbb{R} \to \mathrm{dom}(\mathcal{F})$. Two $\mathcal{F}$-transverse loops Γ and Γ' are *equivalent* if there exist two lifts $\widetilde{\gamma} : \mathbb{R} \to \widetilde{\mathrm{dom}(\mathcal{F})}$ and $\widetilde{\gamma}' : \mathbb{R} \to \widetilde{\mathrm{dom}(\mathcal{F})}$ of Γ and Γ' respectively, a deck transformation T and an orientation-preserving homeomorphism $h : \mathbb{R} \to \mathbb{R}$ invariant by $t \mapsto t + 1$ and such that for all $t \in \mathbb{R}$,

$$\widetilde{\gamma}(t+1) = T(\widetilde{\gamma}(t)), \quad \widetilde{\gamma}'(t+1) = T(\widetilde{\gamma}'(t)), \quad \phi_{\widetilde{\gamma}'(h(t))} = \phi_{\widetilde{\gamma}(t)}.$$

2.2 $\mathcal{F}$-transverse intersection

We now recall the definition of $\mathcal{F}$-transverse intersection, which is a central notion in [LeCT18]. Many details and illustrating figures can also be found in [LeCT18, §§ 2 and 3].

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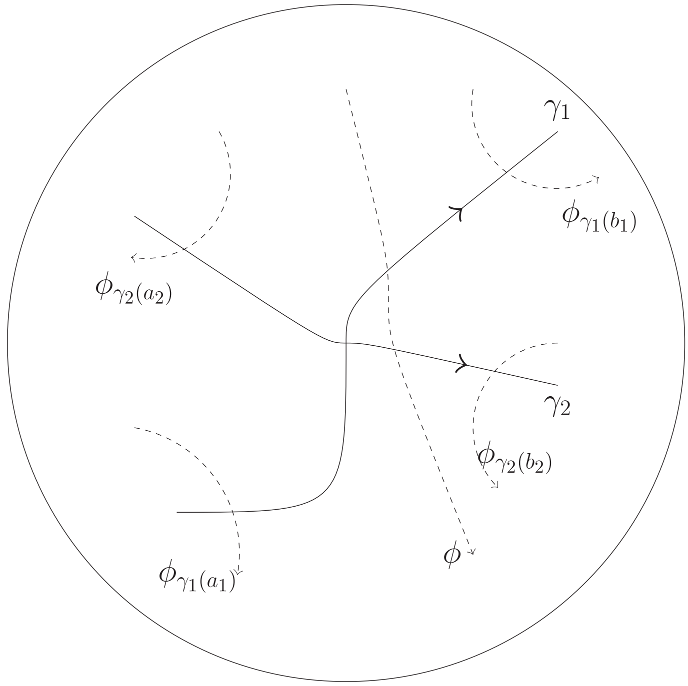

FIGURE 1. γ_1 and γ_2 intersect $\mathcal{F}$-transversally and positively at ϕ.

Let λ_0, λ_1, and λ_2 be three lines in $\mathbb{R}^2$. The line λ_2 is *above* λ_1 *relative to* λ_0 (or λ_1 is *below* λ_2 *relative to* λ_0) if:

- the three lines are pairwise disjoint;
- none of the lines separates the two others;
- if γ_1, γ_2 are two disjoint paths that join $z_1 = \lambda_0(t_1)$ (respectively, $z_2 = \lambda_0(t_2)$) to $z_1' \in \lambda_1$ (respectively, $z_2' \in \lambda_2$) and do not meet the three lines except at the ends, then $t_2 > t_1$.

Assume $\mathcal{F}$ is a non-singular foliation on $\mathbb{R}^2$. Let $\gamma_1 : J_1 \to \mathbb{R}^2$ and $\gamma_2 : J_2 \to \mathbb{R}^2$ be two transverse paths such that $\phi_{\gamma_1(t_1)} = \phi_{\gamma_2(t_2)} = \phi$. The paths γ_1 and γ_2 *intersect $\mathcal{F}$-transversally and positively at ϕ* (see Figure 1) if there exist a_1, b_1 in J_1 and $a_2, b_2 \in J_2$ satisfying $a_1 < t_1 < b_1$, $a_2 < t_2 < b_2$, such that

- $\phi_{\gamma_2(a_2)}$ is below $\phi_{\gamma_1(a_1)}$ relative to ϕ, and
- $\phi_{\gamma_2(b_2)}$ is above $\phi_{\gamma_1(b_1)}$ relative to ϕ.

In this situation, we also say that γ_1 intersects γ_2 $\mathcal{F}$-transversally and positively, γ_2 and γ_1 intersect $\mathcal{F}$-transversally and negatively, or γ_2 intersects γ_1 $\mathcal{F}$-transversally and negatively at ϕ.

Remark 2.1. One has the following transitivity property. Let $\gamma_1 : J_1 \to \mathbb{R}^2, \gamma_2 : J_2 \to \mathbb{R}^2$, and $\gamma_3 : J_3 \to \mathbb{R}^2$ be transverse paths with $\phi_{\gamma_1(t_1)} = \phi_{\gamma_2(t_2)} = \phi_{\gamma_3(t_3)} = \phi$. If γ_1 and γ_2 intersect $\mathcal{F}$-transversally and positively at ϕ, and γ_2 and γ_3 intersect $\mathcal{F}$-transversally and positively at ϕ, then γ_1 and γ_3 intersect $\mathcal{F}$-transversally and positively at ϕ.

Now let $\mathcal{F}$ be a (possibly singular) foliation on an oriented surface M. Let $\gamma_1 : J_1 \to M$ and $\gamma_2 : J_2 \to M$ be two transverse paths such that $\phi_{\gamma_1(t_1)} = \phi_{\gamma_2(t_2)} = \phi$. We say that γ_1 and γ_2 *intersect $\mathcal{F}$-transversally and positively at ϕ (respectively, negatively at ϕ)* if there exist paths

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$\widetilde{\gamma}_1 : J_1 \to \widetilde{\mathrm{dom}(\mathcal{F})}$ and $\widetilde{\gamma}_2 : J_2 \to \widetilde{\mathrm{dom}(\mathcal{F})}$, lifting γ_1 and γ_2, with a common leaf $\widetilde{\phi} = \phi_{\widetilde{\gamma}_1(t_1)} = \phi_{\widetilde{\gamma}_2(t_2)}$ that lifts ϕ such that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ intersect $\widetilde{\mathcal{F}}$-transversally and positively at $\widetilde{\phi}$ (respectively, negatively at $\widetilde{\phi}$). If two paths γ_1 and γ_2 intersect $\mathcal{F}$-transversally, there exist t_1' and t_2' such that $\gamma_1(t_1') = \gamma_2(t_2')$ and such that γ_1 and γ_2 intersect $\mathcal{F}$-transversally at $\phi_{\gamma_1(t_1')} = \phi_{\gamma_2(t_2')}$. We say that γ_1 and γ_2 intersect $\mathcal{F}$-transversally at $\gamma_1(t_1') = \gamma_2(t_2')$. A transverse path γ has a *(positive) $\mathcal{F}$-transverse self-intersection at* $\gamma(t_1) = \gamma(t_2)$, $t_1 < t_2$, if for every lift $\widetilde{\gamma}$ there is a deck transformation U such that $\widetilde{\gamma}$ and $U\widetilde{\gamma}$ have a (positive) $\widetilde{\mathcal{F}}$-transverse intersection at $\widetilde{\gamma}(t_1) = U\widetilde{\gamma}(t_2)$. A transverse loop Γ has an *$\mathcal{F}$-transverse self-intersection at* $\Gamma(t_1) = \Gamma(t_2)$ if its natural lift γ has an $\mathcal{F}$-transverse self-intersection at $\gamma(t_1) = \gamma(t_2)$.

Finally, we say that a transverse path $\gamma : [a, b] \to \mathbb{R}^2$ (for a regular foliation $\mathcal{F}$ on $\mathbb{R}^2$) has a *leaf on its right* (respectively, *a leaf on its left*), if there is a leaf ϕ such that ϕ is above (respectively, below) $\phi_{\gamma(a)}$ relative to $\phi_{\gamma(b)}$. We say that $\gamma : [a, b] \to M$ has *a leaf on its right* (respectively, *a leaf on its left*), if a lift of γ to $\widetilde{\mathcal{F}}$ has a leaf on its right (respectively, a leaf on its left).

2.3 Identity isotopies

In the following, let f be a homeomorphism on M that is isotopic to the identity. Let $\mathcal{I}$ be the set of isotopies $I = (f_t)_{t\in[0,1]}$ between the identity and f. Here, isotopy means a continuous path of homeomorphisms with respect to the topology defined by the uniform convergence of maps and their inverses on compact sets. For $I \in \mathcal{I}$, the *trajectory* $I(z)$ of a point $z \in M$ is defined to be the path $t \mapsto f_t(z)$.

Let $\mathrm{Fix}(I) := \bigcap_{t\in[0,1]} \mathrm{Fix}(f_t)$ and $\mathrm{dom}(I) = M \setminus \mathrm{Fix}(I)$. There is the following preorder on $\mathcal{I}$. We define $I < I'$ if

- $\mathrm{Fix}(I) \subset \mathrm{Fix}(I')$, and
- I' is homotopic to I relative to $\mathrm{Fix}(I)$.

For each $I \in \mathcal{I}$, there is $I' \in \mathcal{I}$ with $I < I'$ and such that I' is maximal with respect to $<$. This was proved in [Jau14] with certain restrictions and in [BCLR20] in full generality. See also [HLRS16] for the case of diffeomorphisms. Maximal elements are exactly those $I \in \mathcal{I}$ such that for every $z \in \mathrm{Fix}(f) \setminus \mathrm{Fix}(I)$ the loop $I(z)$ is not contractible in $\mathrm{dom}(I)$ (see [Jau14]). A foliation $\mathcal{F}$ on M is called *transverse to I* if

- the singular set $\mathrm{sing}(\mathcal{F})$ coincides with $\mathrm{Fix}(I)$, and
- for every $z \in \mathrm{dom}(I)$, the trajectory $I(z)$ is homotopic in $\mathrm{dom}(I)$ relative to the endpoints to a path γ positively transverse to $\mathcal{F}$.

One has the following fundamental result.

Theorem 2.2. *[LeC05] For any maximal isotopy $I \in \mathcal{I}$ there exists a singular oriented foliation $\mathcal{F}$ on M that is transverse to I.*

2.4 Admissible paths

Let I be a maximal isotopy to f and $\mathcal{F}$ be transverse to I. Let $I_{\mathcal{F}}(z)$ denote the class of paths that are positively transverse to $\mathcal{F}$, that join z and $f(z)$, and that are homotopic in $\mathrm{dom}(I)$ to $I(z)$ relative to the endpoints. Every path in this class is called a *transverse trajectory* to z. One defines for every n, $I^n_{\mathcal{F}}(z) := \Pi_{0\le k<n} I_{\mathcal{F}}(f^k(z))$, which means the class of paths that can be written as concatenation of paths in $I_{\mathcal{F}}(f^k(z))$, $0 \le k < n$.

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A path $\gamma : [a,b] \to \mathrm{dom}(I)$ positively transverse to $\mathcal{F}$ is called *admissible of order* n if it is equivalent to a path in $I^n_{\mathcal{F}}(z)$ for some $z \in \mathrm{dom}(I)$. A path γ is called *admissible of order at most* n if it is a subpath of a path that is admissible of order n. A transverse path that has a leaf on its right or a leaf on its left and that is admissible of order at most n, is also admissible of order n [LeCT18, Proposition 19].

The concept of $\mathcal{F}$-transverse intersections allows us to show admissibility for various transverse paths. Let us state the 'fundamental proposition' in [LeCT18].

PROPOSITION 2.3 [LeCT18, Proposition 20]. *Let* $\gamma_1 : [a_1, b_1] \to M$ *and* $\gamma_2 : [a_2, b_2] \to M$ *be transverse paths that intersect $\mathcal{F}$-transversally at* $\gamma_1(t_1) = \gamma_2(t_2)$. *If* γ_1 *is admissible of order* n_1 *and* γ_2 *is admissible of order* n_2, *then* $\gamma_1|_{[a_1,t_1]}\gamma_2|_{[t_2,b_2]}$ *and* $\gamma_2|_{[a_2,t_2]}\gamma_1|_{[t_1,b_1]}$ *are admissible of order* $n_1 + n_2$. *Furthermore, either one of these paths is admissible of order* $\min(n_1, n_2)$ *or both paths are admissible of order* $\max(n_1, n_2)$.

This proposition is essential for the following result.

PROPOSITION 2.4 [LeCT18, Proposition 23]. *Suppose that* $\gamma : [a,b] \to M$ *is a transverse path admissible of order* n *and that* γ *has an $\mathcal{F}$-transverse self-intersection at* $\gamma(s) = \gamma(t)$ *with* $s < t$. *Then* $\gamma|_{[a,s]}\gamma|_{[t,b]}$ *is admissible of order* n *and* $\gamma|_{[a,s]}(\gamma|_{[s,t]})^q\gamma|_{[t,b]}$ *is admissible of order* qn *for every* $q \geq 1$.

The second statement of Proposition 2.4 is a direct consequence of Proposition 2.3, whereas the first statement follows from Proposition 2.3 and from the fact that there is q such that $\gamma|_{[a,s]}(\gamma|_{[s,t]})^q\gamma|_{[t,b]}$ is not admissible of order n (see [LeCT18, Proof of Proposition 23]). Besides Proposition 2.4 we will use the following, slightly more general, statement.

PROPOSITION 2.5. *Let* $k > 0$. *Suppose that* $\gamma : [a,b] \to M$ *is a transverse path admissible of order* n *and that* γ *has positive $\mathcal{F}$-transverse self-intersections at* $\gamma(s_i) = \gamma(t_i)$, $i = 1, \dots, k$, *with* $a < s_1 < t_1 < s_2 < t_2 < \cdots < s_k < t_k < b$. *Then* $\gamma|_{[a,s_1]}\gamma|_{[t_1,s_2]}\gamma|_{[t_2,s_3]}\cdots\gamma|_{[t_k,b]}$ *is admissible of order* n. *The same holds true if all $\mathcal{F}$-transverse self-intersections above are negative.*

Proof. One obtains the proof by iterating the arguments for the proof of Proposition 2.4 contained in [LeCT18]. By Proposition 2.4, $\gamma|_{[a,s_1]}\gamma|_{[t_1,b]}$ is admissible of order n. We claim that $\gamma|_{[a,s_1]}\gamma|_{[t_1,b]}$ and γ have a positive $\mathcal{F}$-transverse intersection at $\gamma|_{[a,s_1]}\gamma|_{[t_1,b]}(s_2) = \gamma(t_2)$. Indeed, choose lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ of γ with $\widetilde{\gamma}_1(s_2) = \widetilde{\gamma}_2(t_2)$, and choose $a_1, a_2, b_1, b_2 \in [a,b]$ with $a_1 < s_2 < b_1$, $a_2 < t_2 < b_2$ such that $\phi_{\widetilde{\gamma}_1(a_1)}$ is above $\phi_{\widetilde{\gamma}_2(a_2)}$ relative to $\phi := \phi_{\widetilde{\gamma}_1(s_2)} = \phi_{\widetilde{\gamma}_2(t_2)}$ and $\phi_{\widetilde{\gamma}_2(b_2)}$ is above $\phi_{\widetilde{\gamma}_1(b_1)}$ relative to ϕ. Let $\widetilde{\gamma}_0$ be the lift of γ such that $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ intersect $\widetilde{\mathcal{F}}$-transversally and positively at $\widetilde{\gamma}_0(s_1) = \widetilde{\gamma}_1(t_1)$, and choose $a_0 \in [a,b]$ and decrease $a_1 \in [a,b]$ if necessary (which will not change the properties above) such that $\phi_{\widetilde{\gamma}_0(a_0)}$ is above $\phi_{\widetilde{\gamma}_1(a_1)}$ relative to $\phi_{\widetilde{\gamma}_1(t_1)}$. Since $\phi_{\widetilde{\gamma}_0(a_0)}$ and $\phi_{\widetilde{\gamma}_1(a_1)}$ are on the right of $\phi_{\widetilde{\gamma}_1(t_1)}$, $\phi_{\widetilde{\gamma}_0(a_0)}$ is also above $\phi_{\widetilde{\gamma}_1(a_1)}$ relative to ϕ. To ensure that the paths $\widetilde{\gamma}_0|_{[a,s_1]}\widetilde{\gamma}_1|_{[t_1,b]}$ and $\widetilde{\gamma}_2|_{[a,b]}$ intersect $\widetilde{\mathcal{F}}$-transversally and positively at ϕ, we need to check that $\phi_{\widetilde{\gamma}_0(a_0)}$ is above $\phi_{\widetilde{\gamma}_2(a_2)}$ relative to ϕ. That those three leaves are pairwise disjoint is clear. Moreover, $\phi_{\widetilde{\gamma}_0(a_0)}$ and $\phi_{\widetilde{\gamma}_2(a_2)}$ lie on the right of ϕ, and hence ϕ does not separate them. Since $\phi_{\widetilde{\gamma}_2(a_2)}$ does not separate $\phi_{\widetilde{\gamma}_1(a_1)}$ and ϕ, it does not intersect $\widetilde{\gamma}_1|_{[a_1,s_2]}$. Hence, $\phi_{\widetilde{\gamma}_2(a_2)}$ does not intersect the path $\widetilde{\gamma}_0|_{[a_0,s_1]}\widetilde{\gamma}_1|_{[t_1,s_2]}$, neither its subpath that is equivalent to a subpath of $\widetilde{\gamma}_1|_{[a_1,s_2]}$, nor the complementary subpath, which would contradict the fact that $\phi_{\widetilde{\gamma}_1(a_1)}$ is above $\phi_{\widetilde{\gamma}_2(a_2)}$ relative to ϕ. Hence, $\phi_{\widetilde{\gamma}_1(a_1)}$ does not separate $\phi_{\widetilde{\gamma}_0(a_0)}$ and ϕ, and similarly one sees that $\phi_{\widetilde{\gamma}_0(a_0)}$ does not separate $\phi_{\widetilde{\gamma}_1(a_1)}$ and ϕ. Finally, for any two disjoint paths η_0 (respectively, η_2) that join a point in $\phi_{\widetilde{\gamma}_0(a_0)}$ to a point y_0 in ϕ (respectively, a point in $\phi_{\widetilde{\gamma}_2(a_2)}$ to a point y_2 in ϕ), there is, by the non-separation property, a path η_1 that joins a point in $\phi_{\widetilde{\gamma}_1(a_1)}$ to a point

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y_1 which lies in the subpath of ϕ connecting y_2 and y_0. It follows by our assumptions that y_2 must be above y_0 in ϕ. We conclude that the paths $\widetilde{\gamma}_0|_{[a,s_1]}\widetilde{\gamma}_1|_{[t_1,b]}$ and $\widetilde{\gamma}_2|_{[a,b]}$ intersect $\widetilde{\mathcal{F}}$-transversally and positively at ϕ, which implies the claim above.

Hence, by Proposition 2.3, the path $\gamma|_{[a,s_1]}\gamma|_{[t_1,s_2]}\gamma|_{[t_2,b]}$ is admissible of order n or the path $\gamma|_{[a,t_2]}\gamma|_{[s_2,b]} = \gamma|_{[a,s_2]}(\gamma|_{[s_2,t_2]})^2\gamma|_{[t_2,b]}$ is admissible of order n. Repeating this argument inductively, that is, applying Proposition 2.3 to the paths $\gamma|_{[a,s_1]}\gamma|_{[t_1,b]}$ and $\gamma|_{[a,s_2]}(\gamma|_{[s_2,t_2]})^q\gamma|_{[s_2,b]}$, we get that $\gamma|_{[a,s_1]}\gamma|_{[t_1,s_2]}\gamma|_{[t_2,b]}$ is admissible of order n or $\gamma|_{[a,s_2]}(\gamma|_{[s_2,t_2]})^q\gamma|_{[t_2,b]}$ is admissible of order n for all $q \geq 1$. That the latter is impossible is shown in [LeCT18, Proof of Proposition 23]. Hence, repeating this argument for $i = 3, \ldots, k$ shows the assertion. $\square$

Certain assumptions on a transverse loop Γ guarantee the existence of periodic points of f. An $\mathcal{F}$-transverse loop Γ with natural lift γ is *linearly admissible of order q* if there exist two sequences $(r_k)_{k\geq 0}$ and $(s_k)_{k\geq 0}$ of natural numbers with

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} s_k = +\infty, \quad \limsup_{k\to\infty} r_k/s_k \geq 1/q,$$

and such that $\gamma|_{[0,r_k]}$ is admissible of order at most s_k.

If z is a periodic point of period q, then there exists a transverse loop Γ' whose natural lift satisfies $\gamma'|_{[0,1]} = I^q_{\mathcal{F}}(z)$. A transverse loop Γ is said to be *associated* to z if it is $\mathcal{F}$-equivalent to Γ'. Note that Γ is then linearly admissible of order q. The following important realization result asserts a partial converse.

Proposition 2.6 [LeCT18, Proposition 26]. *Let Γ be a linearly admissible transverse loop of order q that has an $\mathcal{F}$-transverse self-intersection. Then for every rational number $r/s \in (0, 1/q]$ written in lowest terms, Γ^r is associated to a periodic point of period s.*

3. Non-simple free homotopy classes and $\mathcal{F}$-transverse self-intersections

Throughout this section let M be an oriented closed surface and $\mathcal{F}$ a (singular) oriented foliation on M. In this section we study $\mathcal{F}$-transverse loops in $\mathrm{dom}(\mathcal{F})$ that are not freely homotopic to a multiple of a simple loop and derive some useful properties. In particular, we prove Lemma 3.8 which provides some upper bounds on the length of intervals along which different lifts of such loops to the universal cover can be $\widetilde{\mathcal{F}}$-equivalent. First, we show that the existence of an $\mathcal{F}$-transverse self-intersection is sufficient for a loop to be of this type.

Lemma 3.1. *If Γ is an $\mathcal{F}$-transverse loop in* $\mathrm{dom}(\mathcal{F})$ *that has an $\mathcal{F}$-transverse self-intersection, then Γ is not freely homotopic in* $\mathrm{dom}(\mathcal{F})$ *to a multiple of a simple loop.*

Proof. Choose a Riemannian metric g on $\mathrm{dom}(\mathcal{F})$ and let $\widetilde{g}$ be the lift to the universal cover $\widetilde{\mathrm{dom}(\mathcal{F})}$, $\widetilde{d}(\cdot,\cdot)$ its induced metric. For a lift $\widetilde{\gamma}$ of γ and for $\epsilon > 0$ denote by $\mathcal{U}(\widetilde{\gamma},\epsilon) := \{x \in \widetilde{\mathrm{dom}(\mathcal{F})} \mid \exists t \in \mathbb{R} \text{ with } \widetilde{d}(\widetilde{\gamma}(t), x) < \epsilon\}$ the ϵ-neighborhood of $\widetilde{\gamma}$. Since Γ is $\mathcal{F}$-transverse we can choose $\epsilon > 0$ so small that, for any lift $\widetilde{\gamma}$ of γ, every leaf of $\widetilde{\mathcal{F}}$ that intersects $\mathcal{U}(\widetilde{\gamma},\epsilon)$ also intersects $\widetilde{\gamma}$. Now let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be two lifts of γ that intersect $\mathcal{F}$-transversally and positively at $\widetilde{\gamma}_1(t_1) = \widetilde{\gamma}_2(t_2)$. We show that

$$\begin{aligned} &\forall C > 0, \exists a, b \in \mathbb{R} \text{ such that } \widetilde{\gamma}_1(a) \in R(\widetilde{\gamma}_2),\ \widetilde{\gamma}_1(b) \in L(\widetilde{\gamma}_2), \text{ and} \\ &\widetilde{\gamma}_1(a),\ \widetilde{\gamma}_1(b) \text{ do not lie in the } C\text{-neighborhood } \mathcal{U}(\widetilde{\gamma}_2, C) \text{ of } \widetilde{\gamma}_2. \end{aligned} \tag{2}$$

If Λ is a loop freely homotopic to Γ, then (2) still holds for the lifts λ_1 and λ_2 of Λ that are obtained by lifting a homotopy of Γ to Λ to homotopies that extend $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, respectively.

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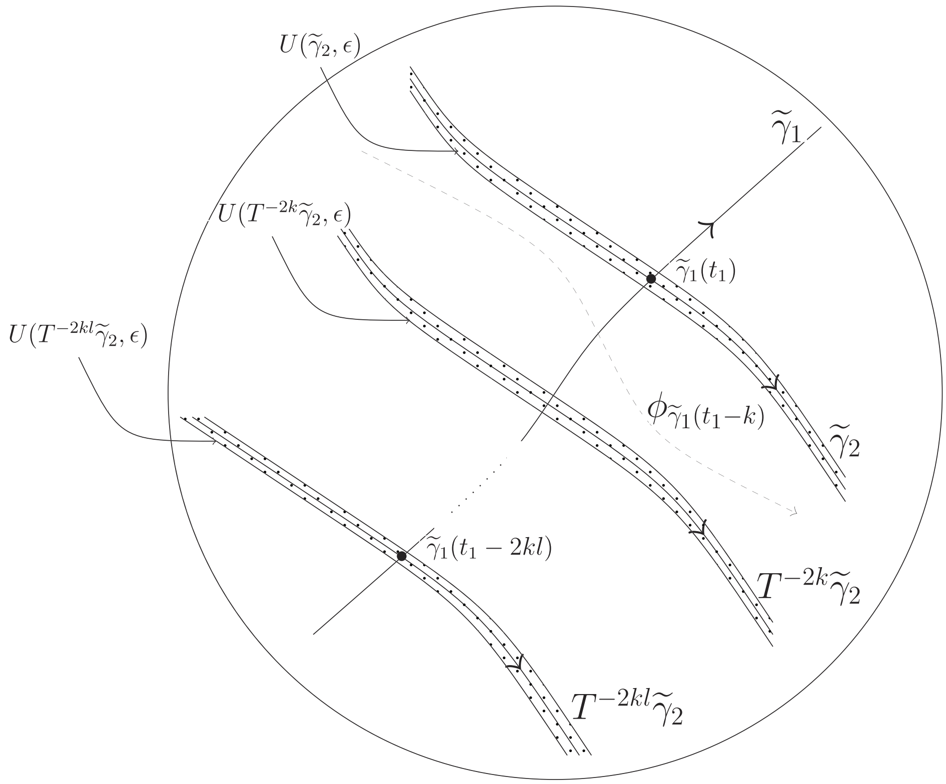

FIGURE 2. Situation in the proof of Lemma 3.1. The points $\widetilde{\gamma}_1(t_1-2kl)$ are on the right of $\widetilde{\gamma}_2$ for all $l>0$ and are not contained in $U(\widetilde{\gamma}_2,C)$ if $l>C/2\epsilon$.

In particular, the images of λ_1 and λ_2 are non-identical and so Λ cannot be a multiple of a simple loop.

Let $T:\widetilde{\mathrm{dom}(\mathcal{F})}\to\widetilde{\mathrm{dom}(\mathcal{F})}$ be the deck transformation that is given by $T(\widetilde{\gamma}_1(t))=\widetilde{\gamma}_1(t+1)$. By the assumptions, there exist $a_1,b_1,a_2,b_2\in\mathbb{R}$ with $a_1<t_1<b_1$ and $a_2<t_2<b_2$ such that $\phi_{\widetilde{\gamma}_1(a_1)}$ is above $\phi_{\widetilde{\gamma}_2(a_2)}$ relative to $\phi:=\phi_{\widetilde{\gamma}(t_1)}=\phi_{\widetilde{\gamma}(t_2)}$ and $\phi_{\widetilde{\gamma}_2(b_2)}$ is above $\phi_{\widetilde{\gamma}_1(b_1)}$ relative to ϕ. Since the lifts of γ intersect the foliation $\widetilde{\mathcal{F}}$ positively and since ϕ and $\phi_{\widetilde{\gamma}_2(b_2)}$ are on the right of $\phi_{\widetilde{\gamma}_1(b_1)}$, the lift $\widetilde{\gamma}_2$ is on the right of $\phi_{\widetilde{\gamma}_1(b_1)}$. Similarly, $\widetilde{\gamma}_2$ is on the left of $\phi_{\widetilde{\gamma}_1(a_1)}$. Choose $k\in\mathbb{N}$ with $t_1+k\geq b_1$ and $t_1-k\leq a_1$. Then $\widetilde{\gamma}_2$ also lies on the right of $\phi_{\widetilde{\gamma}_1(t_1+k)}=T^k(\phi_{\widetilde{\gamma}_1(t_1)})$ and $\widetilde{\gamma}_2$ lies on the left of $\phi_{\widetilde{\gamma}_1(t_1-k)}=T^{-k}(\phi_{\widetilde{\gamma}_1(t_1)})$. Consider the lifts $T^{2kl}\widetilde{\gamma}_2(t):=T^{2kl}(\widetilde{\gamma}_2(t))$, $l\in\mathbb{Z}$, of γ. The lifts $\widetilde{\gamma}_1$ and $T^{2kl}\widetilde{\gamma}_2$ intersect $\widetilde{\mathcal{F}}$-transversally and positively at $\widetilde{\gamma}_1(t_1+2kl)=T^{2kl}\widetilde{\gamma}_2(t_2)$. The line $T^{2kl}\widetilde{\gamma}_2$ is on the right of $\phi_{\widetilde{\gamma}_1(t+2kl+k)}=T^{2kl+k}(\phi_{\widetilde{\gamma}_1(t)})$ and on the left of $\phi_{\widetilde{\gamma}_1(t+2kl-k)}=T^{2kl-k}(\phi_{\widetilde{\gamma}_1(t)})$. Moreover, $\phi_{\widetilde{\gamma}_1(t+2kl+k)}$ is on the left of $T^{2kl}\widetilde{\gamma}_2$ and $\phi_{\widetilde{\gamma}_1(t+2kl-k)}$ is on the right of $T^{2kl}\widetilde{\gamma}_2$. It follows that for all $l_1,l_2\in\mathbb{Z}, l_1<l_2$, the line $T^{2kl_2}\widetilde{\gamma}_2$ is on the left of $T^{2kl_1}\widetilde{\gamma}_2$ and $T^{2kl_1}\widetilde{\gamma}_2$ is on the right of $T^{2kl_2}\widetilde{\gamma}_2$. No leaf of $\widetilde{\mathcal{F}}$ intersects both $T^{2kl_1}\widetilde{\gamma}_2$ and $T^{2kl_2}\widetilde{\gamma}_2$ for $l_1\neq l_2$, hence the sets $\mathcal{U}(T^{2kl}\widetilde{\gamma}_2,\epsilon)$, $l\in\mathbb{Z}$, are pairwise disjoint. Since any path from $T^{2kl_1}\widetilde{\gamma}_2$ to $T^{2kl_2}\widetilde{\gamma}_2$, $l_1<l_2\in\mathbb{Z}$, has to cross the lifts $T^{2k(l_1+1)}\widetilde{\gamma}_2,T^{2k(l_1+2)}\widetilde{\gamma}_2,\ldots,T^{2k(l_2-1)}\widetilde{\gamma}_2$, the $\widetilde{d}$-distance of the images of $T^{2kl_1}\widetilde{\gamma}_2$ and $T^{2kl_2}\widetilde{\gamma}_2$ in $\widetilde{\mathrm{dom}(\mathcal{F})}$ is bounded from below by $2\epsilon(l_2-l_1-1)+2\epsilon=2\epsilon(l_2-l_1)$. Hence, for any $C>0$ we have for $l>C/2\epsilon$ that $\widetilde{\gamma}_1(t_1+2kl)\in L(\widetilde{\gamma}_2)$, $\widetilde{\gamma}_2(t_1-2kl)\in R(\widetilde{\gamma}_2)$, and both points do not lie in $\mathcal{U}(\widetilde{\gamma}_2,C)$ (see Figure 2). $\square$

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Now, and throughout the section if not explicitly stated otherwise, let $\Gamma : S^1 \to \mathrm{dom}(\mathcal{F})$ be an *$\mathcal{F}$-transverse loop that is not freely homotopic in* $\mathrm{dom}(\mathcal{F})$ *to a multiple of a simple loop*, and denote by $\gamma : \mathbb{R} \to \mathrm{dom}(\mathcal{F})$ its natural lift. We will assume that $\mathrm{dom}(\mathcal{F})$ is connected, otherwise consider instead of $\mathrm{dom}(\mathcal{F})$ the connected component that contains Γ. In the remainder of the section we will prove a few properties of lifts of Γ to $\widetilde{\mathrm{dom}(\mathcal{F})}$. In particular, we will see that the converse of Lemma 3.1 holds, that is, that Γ has an $\mathcal{F}$-transverse self-intersection.

Since there is a primitive non-simple free homotopy class of loops in $\mathrm{dom}(\mathcal{F})$, $\mathrm{dom}(\mathcal{F})$ is not homeomorphic to a sphere, to a sphere minus one or two points, or to the torus. Equip $\mathrm{dom}(\mathcal{F})$ with a complex structure. By the uniformization theorem, we can identify the universal cover $\widetilde{\mathrm{dom}(\mathcal{F})}$ with the unit disc. It follows that $\mathrm{dom}(\mathcal{F})$ admits a complete hyperbolic metric g_{hyp} that lifts to the Poincaré metric on $\widetilde{\mathrm{dom}(\mathcal{F})}$. We obtain a circle compactification $\widetilde{\mathrm{dom}(\mathcal{F})} \cup S_\infty$ by adding the boundary at infinity S_∞ to $(\widetilde{\mathrm{dom}(\mathcal{F})}, \widetilde{g_{\mathrm{hyp}}})$. In the hyperbolic surface $(\mathrm{dom}(\mathcal{F}), g_{\mathrm{hyp}})$ there is a sequence of compact subsurfaces, $C_1 \subset \cdots C_k \subset C_{k+1} \cdots \subset M$, such that their boundary ∂C_k is a finite union of simple closed geodesics and every non-trivial closed curve in $\mathrm{dom}(\mathcal{F})$ is freely homotopic to a closed curve that is contained in C_k for sufficiently large k (see, for example, [BK08, Proposition 2.3]). In particular, every free homotopy class that is not a multiple of a simple class has a geodesic representative.

For a lift $\widetilde{\gamma}$ of γ to the universal cover $\widetilde{\mathrm{dom}(\mathcal{F})}$ and a deck transformation U on $\widetilde{\mathrm{dom}(\mathcal{F})}$, we write $U\widetilde{\gamma} : \mathbb{R} \to \widetilde{\mathrm{dom}(\mathcal{F})}$ for the lift satisfying $U\widetilde{\gamma}(t) = U(\widetilde{\gamma}(t))$ for every $t \in \mathbb{R}$. We call the deck transformation T with $T\widetilde{\gamma}(t) = \widetilde{\gamma}(t+1)$ the *shift* for $\widetilde{\gamma}$. Note that every deck transformation U extends to a homeomorphism $\overline{U}$ on $\widetilde{\mathrm{dom}(\mathcal{F})} \cup S_\infty$ and, since Γ has a geodesic representative in $\mathrm{dom}(\mathcal{F})$, any shift T for some lift $\widetilde{\gamma}$ of Γ is a hyperbolic transformation; in particular, $\overline{T}$ admits exactly two fixed points $\widetilde{\gamma}^+ = \lim_{t\to+\infty} \widetilde{\gamma}(t) = \lim_{n\to+\infty} T^n\widetilde{\gamma}(t) \in S_\infty$ and $\widetilde{\gamma}^- = \lim_{t\to-\infty} \widetilde{\gamma}(t) = \lim_{n\to+\infty} T^{-n}\widetilde{\gamma}(t) \in S_\infty$. For any lift $\widetilde{\gamma}$ we get two non-empty arcs $\widehat{L}(\widetilde{\gamma}) := (\widetilde{\gamma}^+, \widetilde{\gamma}^-)$ and $\widehat{R}(\widetilde{\gamma}) := (\widetilde{\gamma}^-, \widetilde{\gamma}^+)$ on S_∞, where we equip S_∞ with the counterclockwise orientation. We say that two lifts $\widetilde{\gamma}_1 : \mathbb{R} \to \widetilde{\mathrm{dom}(\mathcal{F})}$ and $\widetilde{\gamma}_2 : \mathbb{R} \to \widetilde{\mathrm{dom}(\mathcal{F})}$ are *translates of each other* if the shifts T_1 and T_2 for $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, respectively, coincide. If the free homotopy class of Γ is primitive, then this holds if and only if $\widetilde{\gamma}_1(t+k) = \widetilde{\gamma}_2(t)$ for some $k \in \mathbb{Z}$ and all $t \in \mathbb{R}$. Using that Γ has a closed geodesic representative in $(\mathrm{dom}(\mathcal{F}), g_{\mathrm{hyp}})$, one obtains that two lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are not translates of each other if and only if $\widetilde{\gamma}_1^\pm$ and $\widetilde{\gamma}_2^\pm$ are four pairwise distinct points in S_∞. Since Γ is not freely homotopic to a multiple of a simple loop, there are two lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ such that $\widetilde{\gamma}_1^\pm$ *separate* $\widetilde{\gamma}_2^\pm$ in S_∞ (i.e. $\widehat{L}(\widetilde{\gamma}_1) \cap \widehat{R}(\widetilde{\gamma}_2) \neq \emptyset$ and $\widehat{R}(\widetilde{\gamma}_1) \cap \widehat{L}(\widetilde{\gamma}_2) \neq \emptyset$).

The following lemma states that the foliation separates two lifts with different asymptotics.

Lemma 3.2. *Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be two lifts of γ and assume there are $t_1, t_2 \in \mathbb{R}$ such that $\phi_{\widetilde{\gamma}_1(t_1)} = \phi_{\widetilde{\gamma}_2(t_2)} =: \phi$.*

- *If $\widetilde{\gamma}_1^- \in \widehat{R}(\widetilde{\gamma}_2)$ (respectively, $\widetilde{\gamma}_1^- \in \widehat{L}(\widetilde{\gamma}_2)$), then there is $a_1 < t_1$ and $a_2 < t_2$ such that $\phi_{\widetilde{\gamma}_1(a_1)}$ is above (respectively, below) $\phi_{\widetilde{\gamma}_2(a_2)}$ relative to ϕ.*
- *If $\widetilde{\gamma}_1^+ \in \widehat{L}(\widetilde{\gamma}_2)$ (respectively, $\widetilde{\gamma}_1^+ \in \widehat{R}(\widetilde{\gamma}_2)$), then there is $b_1 > t_1$ and $b_2 > t_2$ such that $\phi_{\widetilde{\gamma}_1(b_1)}$ is below (respectively, above) $\phi_{\widetilde{\gamma}_2(b_2)}$ relative to ϕ.*

Proof. Since Γ is not freely homotopic to a multiple of a simple loop, we can choose for any given lift $\widetilde{\gamma}$ of γ a deck transformation S on $\widetilde{\mathrm{dom}(\mathcal{F})}$ such that $(S\widetilde{\gamma})^+ \in \widehat{L}(\widetilde{\gamma})$ and $(S\widetilde{\gamma})^- \in \widehat{R}(\widetilde{\gamma})$. For all $n \in \mathbb{Z}$, we then have more generally that $(T^nS\widetilde{\gamma})^+ \in \widehat{L}(\widetilde{\gamma})$, $(T^nS\widetilde{\gamma})^- \in \widehat{R}(\widetilde{\gamma})$, $(T^nS^{-1}\widetilde{\gamma})^+ \in \widehat{R}(\widetilde{\gamma})$, and $(T^nS^{-1}\widetilde{\gamma})^- \in \widehat{L}(\widetilde{\gamma})$, where T denotes the shift for $\widetilde{\gamma}$.

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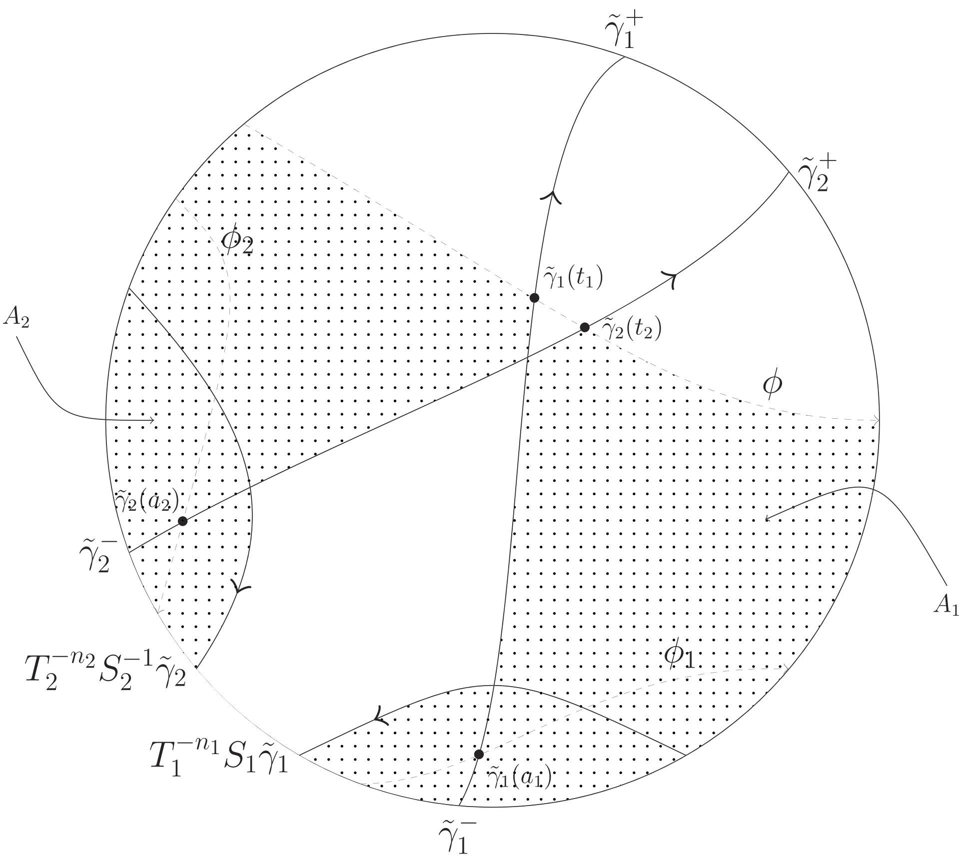

FIGURE 3. Lifts and leaves in the proof of Lemma 3.2. The dotted areas are $A_1 = L(T_1^{-n_1}S_1\tilde{\gamma}_1) \cup (R(\tilde{\gamma}_1) \cap R(\tilde{\gamma}_2) \cap R(\phi))$ and $A_2 = R(T_2^{-n_2}S_2^{-1}\tilde{\gamma}_2) \cup (L(\tilde{\gamma}_1) \cap L(\tilde{\gamma}_2) \cap R(\phi))$.

Now let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be the two lifts considered in the lemma, T_1 and T_2 their shifts, and S_1 and S_2 the deck transformations considered above. We show that if $\widetilde{\gamma}_1^- \in \widehat{R}(\widetilde{\gamma}_2)$ then there exist $a_1 < t_1$ and $a_2 < t_2$ such that $\phi_{\widetilde{\gamma}_1(a_1)}$ is above $\phi_{\widetilde{\gamma}_2(a_2)}$ relative to ϕ. The other three statements of the lemma are proved similarly.

So assume $\widetilde{\gamma}_1^- \in \widehat{R}(\widetilde{\gamma}_2)$. Since $\widetilde{\gamma}_1^- \neq \widetilde{\gamma}_2^-$ are the unique repelling fixed points in $\mathrm{dom}(\mathcal{F}) \cup S_\infty$ of $\overline{T_1}$ and $\overline{T_2}$ respectively, there are $n_1, n_2 \in \mathbb{N}$ such that $L(T_1^{-n_1}S_1\widetilde{\gamma}_1) \cap R(T_2^{-n_2}S_2^{-1}\widetilde{\gamma}_2) = \emptyset$. Choose $a_1 < t_1$ and $a_2 < t_2$ such that $\widetilde{\gamma}_1(a_1) \in L(T_1^{-n_1}S_1\widetilde{\gamma}_1)$ and $\widetilde{\gamma}_2(a_2) \in R(T_2^{-n_2}S_2^{-1}\widetilde{\gamma}_2)$ (see Figure 3). Let $\phi_i := \phi_{\widetilde{\gamma}_i(a_i)}$ for $i = 1, 2$. Since the lifts are positively transverse to $\mathcal{F}$ we have that $\phi_1^- \in L(T_1^{-n_1}S_1\widetilde{\gamma}_1)$ and $\phi_2^+ \in R(T_2^{-n_2}S_2^{-1}\widetilde{\gamma}_2)$, as well as $\phi_1^+ \in R(\widetilde{\gamma}_1) \cap R(\widetilde{\gamma}_2)$ and $\phi_2^- \in L(\widetilde{\gamma}_1) \cap L(\widetilde{\gamma}_2)$. Hence, $A_1 := L(T_1^{-n_1}S_1\widetilde{\gamma}_1) \cup (R(\widetilde{\gamma}_1) \cap R(\widetilde{\gamma}_2) \cap R(\phi))$ contains ϕ_1 and $A_2 := R(T_2^{-n_2}S_2^{-1}\widetilde{\gamma}_2) \cup (L(\widetilde{\gamma}_1) \cap L(\widetilde{\gamma}_2) \cap R(\phi))$ contains ϕ_2. The sets A_1 and A_2 are disjoint, are both connected, are both contained in $R(\phi)$, and the points on ϕ that are on the boundary of A_1 lie above the points of ϕ that lie on the boundary of A_2. It follows that ϕ_1 is above ϕ_2 relative to ϕ. $\square$

COROLLARY 3.3. *Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be two lifts of γ such that $\widetilde{\gamma}_1^\pm$ separate $\widetilde{\gamma}_2^\pm$ on S_∞. Then $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ intersect $\widetilde{\mathcal{F}}$-transversally. In particular, Γ has at least one $\mathcal{F}$-transverse self-intersection.*

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Proof. The first statement follows directly from Lemma 3.2 and the definition of an $\widetilde{\mathcal{F}}$-transverse intersection. Since Γ lies in a non-simple free homotopy class, there are at least two such lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ of γ. □

We also note the following result.

Lemma 3.4. *If two lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ of γ intersect $\widetilde{\mathcal{F}}$-transversally, then $\widetilde{\gamma}_1^{\pm}$ separate $\widetilde{\gamma}_2^{\pm}$ in S_∞.*

Proof. Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ two lifts that intersect $\widetilde{\mathcal{F}}$-transversally. Since Γ has a closed geodesic representative and since the boundary at infinity S_∞ of $(\widetilde{\mathrm{dom}(\mathcal{F})}, \widetilde{g_{\mathrm{hyp}}})$ can be identified with equivalence classes of geodesic rays, where two geodesic rays are equivalent if and only if they stay at finite distance from each other, it suffices to show that $\widetilde{d}(\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)) \to +\infty$ as $t \to \pm\infty$, where $\widetilde{d}$ is the metric on $\widetilde{\mathrm{dom}(\mathcal{F})}$ induced by the Poincaré metric $\widetilde{g_{\mathrm{hyp}}}$. But this follows directly from the proof of Lemma 3.1 by considering in that proof the hyperbolic metric g_{hyp} as a metric on $\mathrm{dom}(\mathcal{F})$. □

We proceed with a few further observations.

Lemma 3.5. *There are two lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ of γ such that*

- *$\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ intersect $\widetilde{\mathcal{F}}$-transversally and positively,*
- *there are no lifts $\widetilde{\gamma}$ of γ that intersect both $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ $\widetilde{\mathcal{F}}$-transversally and positively, and*
- *there are no lifts $\widetilde{\gamma}$ of γ that intersect both $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ $\widetilde{\mathcal{F}}$-transversally and negatively.*

Remark 3.6. Note that if $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ satisfy the properties stated in the lemma, then so do $U\widetilde{\gamma}_1$ and $U\widetilde{\gamma}_2$ for any deck transformation U on $\widetilde{\mathrm{dom}(\mathcal{F})}$, as well as $\widetilde{\gamma}_1$ and $T_1\widetilde{\gamma}_2$, where T_1 is the shift for $\widetilde{\gamma}_1$.

Proof of Lemma 3.5. Recall that two lifts are translates of each other if their shifts coincide. In this proof we denote by $[\widetilde{\gamma}]$ the equivalence class of a lift $\widetilde{\gamma}$ with respect to this equivalence relation. Consider in the following all pairs $([\widetilde{\gamma}], [\widetilde{\gamma}'])$, where $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ are lifts of γ that intersect $\widetilde{\mathcal{F}}$-transversally and positively. For each pair $\mathfrak{p} = ([\widetilde{\gamma}], [\widetilde{\gamma}'])$ consider the (well-defined) arcs $C(\mathfrak{p}) := (\widetilde{\gamma}^+, \widetilde{\gamma}'^-)$ and $D(\mathfrak{p}) := (\widetilde{\gamma}^-, \widetilde{\gamma}'^+)$ in S_∞ (see Figure 4). We say $\mathfrak{p}_1 = ([\widetilde{\gamma}_1], [\widetilde{\gamma}_1']) \leq \mathfrak{p}_2 = ([\widetilde{\gamma}_2], [\widetilde{\gamma}_2'])$ if $C(\mathfrak{p}_2) \subset C(\mathfrak{p}_1)$ and $D(\mathfrak{p}_2) \subset D(\mathfrak{p}_1)$. '$\leq$' defines a partial order on the set of pairs as above.

For a given pair $\mathfrak{p} = ([\widetilde{\gamma}], [\widetilde{\gamma}'])$ as above there are only finitely many equivalence classes $[\widetilde{\gamma}_*]$ such that for a representative $\widetilde{\gamma}_*$,

$$\widetilde{\gamma}_*^+ \in C(\mathfrak{p}) \quad \text{and} \quad \widetilde{\gamma}_*^- \in D(\mathfrak{p}), \tag{3}$$

as well as finitely many classes $[\widetilde{\gamma}_*]$ such that for a representative $\widetilde{\gamma}_*$,

$$\widetilde{\gamma}_*^- \in C(\mathfrak{p}) \quad \text{and} \quad \widetilde{\gamma}_*^+ \in D(\mathfrak{p}). \tag{4}$$

Indeed, note that if $\widetilde{\gamma}_*$ satisfies (3) or (4), then it intersects $\widetilde{\gamma}$ and $\widetilde{\gamma}'$. Also, the group of deck transformations act freely and cocompactly, so there are only finitely many deck transformations U such that $U\widetilde{\gamma}|_{[0,1]} \cap \widetilde{\gamma}|_{[0,1]} \neq \emptyset$. Hence, for every lift $\widetilde{\gamma}_*$ with $\widetilde{\gamma}_* \cap \widetilde{\gamma} \neq \emptyset$, there exist $r_*, r \in \mathbb{Z}$ such that $T^r U\widetilde{\gamma} = T_*^{r_*}\widetilde{\gamma}_*$, for one of the finitely many deck transformations U from above, where T and T_* denote the shift for $\widetilde{\gamma}$, and $\widetilde{\gamma}_*$, respectively. In particular, $[\widetilde{\gamma}_*] = [T^r U\widetilde{\gamma}]$. Assume $(U\widetilde{\gamma})^+ \in (\widetilde{\gamma}^+, \widetilde{\gamma}^-)$. Then, for all $n \in \mathbb{Z}$, the lift $T^n U\widetilde{\gamma}$ does not satisfy (4). Furthermore, note that $\overline{T}$ restricted to S_∞ has exactly two fixed points, one repelling fixed point, $\widetilde{\gamma}^-$, and one attracting fixed point, $\widetilde{\gamma}^+$. Hence there is $r < s$ such that no representative of $[T^n U\widetilde{\gamma}]$ satisfies (3) for $n \notin [r, s]$. Similarly, if $(U\widetilde{\gamma})^+ \in (\widetilde{\gamma}^-, \widetilde{\gamma}^+)$, then there is $r < s$ such that no representative of

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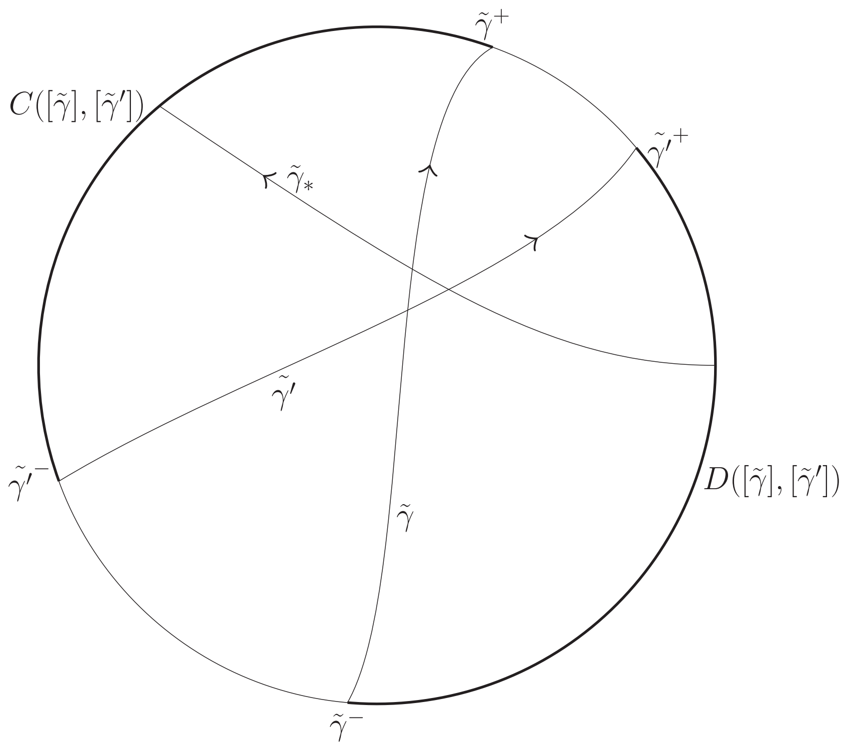

FIGURE 4. Lifts in the proof of Lemma 3.5. For each pair $([\widetilde{\gamma}],[\widetilde{\gamma}'])$, there are only finitely many $[\widetilde{\gamma}_*]$ such that $\widetilde{\gamma}_*^+ \in C([\widetilde{\gamma}],[\widetilde{\gamma}'])$ and $\widetilde{\gamma}_*^- \in D([\widetilde{\gamma}],[\widetilde{\gamma}'])$.

$[T^n U\widetilde{\gamma}]$ satisfies (3) or (4) for $n \notin [r,s]$. Hence, there are only finitely many equivalence classes $[\widetilde{\gamma}_*]$ such that its representatives satisfy (3) or (4).

We conclude that for every pair $\mathfrak{p}$ there are only finitely many pairs that are greater than or equal to $\mathfrak{p}$ and so there exist maximal elements. Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be lifts of γ such that $\mathfrak{p}_0 := ([\widetilde{\gamma}_1],[\widetilde{\gamma}_2])$ is maximal with respect to '$\leq$'. In particular, there is no $\widetilde{\gamma}_*$ that satisfies (3) or (4) for $\mathfrak{p}_0$, which means, using Lemma 3.4, that the properties stated in the lemma hold for the lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$. $\square$

We get the following consequence of Lemma 3.5.

LEMMA 3.7. *There are $\bar{t},\underline{t} \in [0,1)$ with $\Gamma(\bar{t}) = \Gamma(\underline{t})$ such that there exist two lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ of Γ to $\widetilde{\mathrm{dom}}(\mathcal{F})$ that intersect $\widetilde{\mathcal{F}}$-transversally and positively in $\widetilde{\gamma}_1(\bar{t}) = \widetilde{\gamma}_2(\underline{t})$, and such that*

(1) *if two lifts $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ intersect $\widetilde{\mathcal{F}}$-transversally and positively, then $\widetilde{\gamma}$ does not intersect the leaves $\phi_{\widetilde{\gamma}'(\bar{t}+k)}$, for all $k \in \mathbb{Z}$; and*
(2) *if two lifts $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ intersect $\widetilde{\mathcal{F}}$-transversally and negatively, then $\widetilde{\gamma}$ does not intersect the leaves $\phi_{\widetilde{\gamma}'(\underline{t}+k)}$, for all $k \in \mathbb{Z}$.*

Proof. Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be two lifts of γ that satisfy the properties in Lemma 3.5; in particular, $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ intersect $\widetilde{\mathcal{F}}$-transversally and positively in $\widetilde{\gamma}_1(\bar{t}) = \widetilde{\gamma}_2(\underline{t})$ for some $\bar{t},\underline{t} \in \mathbb{R}$. By Remark 3.6 we can choose $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ such that also $\bar{t},\underline{t} \in [0,1)$. Furthermore, by the same remark the properties of Lemma 3.5 are satisfied for $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_{k,2} := T_1^k\widetilde{\gamma}_2$, for any $k \in \mathbb{Z}$, where T_1 is the shift for $\widetilde{\gamma}_1$. Note that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_{k,2}$ intersect $\widetilde{\mathcal{F}}$-transversally in $\widetilde{\gamma}_1(\bar{t}+k) = \widetilde{\gamma}_{k,2}(\underline{t})$. Analogously, for all $k \in \mathbb{Z}$, the properties of Lemma 3.5 hold for $\widetilde{\gamma}_{k,1} = T_2^k\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, where T_2 is the shift for $\widetilde{\gamma}_2$.

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We now prove (1); the proof of (2) is analogous. So let $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ be two lifts of γ that intersect $\widetilde{\mathcal{F}}$-transversally and positively. Let U be the deck transformation such that $U\widetilde{\gamma}' = \widetilde{\gamma}_1$. Then, for all $k \in \mathbb{Z}$, $\widetilde{\gamma}' = U^{-1}\widetilde{\gamma}_1$ and $U^{-1}\widetilde{\gamma}_{k,2}$ satisfy the properties of Lemma 3.5, and they intersect $\widetilde{\mathcal{F}}$-transversally and positively in $\widetilde{\gamma}'(\bar{t}+k)$. Assume that $\widetilde{\gamma}$ intersects the leaf $\phi_{\widetilde{\gamma}'(\bar{t}+k)}$ for some $k \in \mathbb{Z}$. Then $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ also intersect $\mathcal{F}$-transversally and positively at $\phi_{\widetilde{\gamma}'(\bar{t}+k)}$. By the transitivity property (see Remark 2.1), $\widetilde{\gamma}$ intersects $U^{-1}\widetilde{\gamma}_{k,2}$ $\widetilde{\mathcal{F}}$-transversally and positively, which contradicts the properties of Lemma 3.5 for the lifts $\widetilde{\gamma}'$ and $U^{-1}\widetilde{\gamma}_{k,2}$. □

Lemma 3.8. *Let $a_0, a_1, b_0, b_1 \in \mathbb{R}$ with $a_0 < b_0$ and $a_1 < b_1$. Let $\widetilde{\gamma}_0, \widetilde{\gamma}_1$ be two lifts of γ which are not translates of each other and such that $\widetilde{\gamma}_0|_{[a_0,b_0]}$ is equivalent to $\widetilde{\gamma}_1|_{[a_1,b_1]}$. Then*

(1) $\max\{b_0 - a_0, b_1 - a_1\} \le 1$ *if $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ intersect $\widetilde{\mathcal{F}}$-transversally,*
(2) $\min\{b_0 - a_0, b_1 - a_1\} > \max\{\lfloor b_0 - a_0 \rfloor, \lfloor b_1 - a_1 \rfloor\} - 2$,
(3) $\max\{b_0 - a_0, b_1 - a_1\} < 6$.

Proof. Part (1) follows directly from Lemma 3.7.

For (2) it is sufficient by symmetry to show that $b_0 - a_0 > \lfloor b_1 - a_1 \rfloor - 2$, and to show this, we may assume $b_1 - a_1 \ge 3$. Denote by T_0 (respectively, T_1) the shift for $\widetilde{\gamma}_0$ (respectively, $\widetilde{\gamma}_1$), and let S be the deck transformation with $S\widetilde{\gamma}_0 = \widetilde{\gamma}_1$. Choose $\bar{t} \in [0, 1)$ according to Lemma 3.7. Let $k < l \in \mathbb{Z}$ with $a_1 \le \bar{t} + k < a_1 + 1$, and $b_1 - 1 < \bar{t} + l \le b_1$. Any lift $\widetilde{\gamma}$ that intersects $\widetilde{\gamma}_1$ $\widetilde{\mathcal{F}}$-transversally and positively in $\widetilde{\gamma}(t) = \widetilde{\gamma}_1(t_1)$ for some $t \in \mathbb{R}$ and $t_1 \in (\bar{t}+k, \bar{t}+l)$, already intersects $\widetilde{\gamma}_1|_{[\bar{t}+k,\bar{t}+l]}$ $\widetilde{\mathcal{F}}$-transversally and positively by Lemma 3.7, and hence also $\widetilde{\gamma}_0|_{[a_0,b_0]}$ $\widetilde{\mathcal{F}}$-transversally and positively in $\widetilde{\gamma}(t') = \widetilde{\gamma}_0(t_0)$ for some $t' \in \mathbb{R}$ and $t_0 \in (a_0, b_0)$. The latter holds since a subpath of $\widetilde{\gamma}_0|_{[a_0,b_0]}$ is equivalent to $\widetilde{\gamma}_1|_{[\bar{t}+k,\bar{t}+l]}$. There are finitely many (pairwise non-identical) images $\{\widetilde{\gamma}(t) \mid t \in \mathbb{R}\}$ of such lifts $\widetilde{\gamma}$, say M many. By Lemma 3.7, we can choose N lifts $\widetilde{\gamma}^1, \ldots, \widetilde{\gamma}^N$ of γ with pairwise non-identical image such that $\widetilde{\gamma}^i$ and $\widetilde{\gamma}_0|_{[\bar{t},\bar{t}+1]}$ intersect $\widetilde{\mathcal{F}}$-transversally and positively, and such that the image of a lift $\widetilde{\gamma}$ for which $\widetilde{\gamma}$ and $\widetilde{\gamma}_0$ intersect $\widetilde{\mathcal{F}}$-transversally and positively is of the form

$$\{T_0^m \widetilde{\gamma}^i(t) \mid t \in \mathbb{R}\}, \quad \text{for some } 1 \le i \le N,\ m \in \mathbb{Z}.$$

The same holds for $\widetilde{\gamma}_1$ instead of $\widetilde{\gamma}_0$ with $\widetilde{\gamma}^1, \ldots, \widetilde{\gamma}^N$ replaced by $S\widetilde{\gamma}^1, \ldots, S\widetilde{\gamma}^N$, and hence $M = (l-k)N$. Choose, for $i = 1, \ldots, N$, real numbers s^i and t^i with $0 < s^i < t^i < 1$ such that a subpath of $\widetilde{\gamma}^i$ is equivalent to $\widetilde{\gamma}_0|_{(\bar{t}+s^i,\bar{t}+t^i)}$. The interval $[a_0, b_0]$ contains at most $\lceil b_0 - a_0 \rceil N$ many intervals of the form $(\bar{t}+m+s^i, \bar{t}+m+t^i)$ with $1 \le i \le N$, $m \in \mathbb{Z}$. This means that $(l-k)N = M \le \lceil b_0 - a_0 \rceil N$, and hence $b_0 - a_0 > l - k - 1$. On the other hand, the non-negative integer $\lfloor b_1 - a_1 \rfloor - (l-k) = \lfloor b_1 - a_1 - (l-k) \rfloor = \lfloor (b_1 - l) + (k - a_1) \rfloor$ is strictly smaller than 2. Hence,

$$b_0 - a_0 > l - k - 1 \ge \lfloor b_1 - a_1 \rfloor - 2.$$

To show (3), we argue by contradiction. So assume the contrary, and, without loss of generality, that $b_1 - a_1 \ge 6$. By (1), we can assume that $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ do not intersect $\widetilde{\mathcal{F}}$-transversally. By (2), $b_0 - a_0 > 4$. Consider $\widetilde{\gamma}_2 := T_0\widetilde{\gamma}_1$ (see Figure 5). The path $\widetilde{\gamma}_2|_{[a_1,b_1]}$ is equivalent to $\widetilde{\gamma}_0|_{[a_0+1,b_0+1]}$. Let $c_1 \in (a_1, b_1)$ be the parameter such that $\widetilde{\gamma}_2|_{[a_1,c_1]}$ is equivalent to $\widetilde{\gamma}_0|_{[a_0+1,b_0]}$. So, again by (2), $c_1 - a_1 > \lfloor b_0 - (a_0+1) \rfloor - 2 \ge 1$. On the other hand, $\widetilde{\gamma}_2|_{[a_1,c_1]}$, which is equivalent to $\widetilde{\gamma}_0|_{[a_0+1,b_0]}$, is by assumption equivalent to a subpath of $\widetilde{\gamma}_1|_{[a_1,b_1]}$. Note now that $\widetilde{\gamma}_2$ intersects $\widetilde{\gamma}_1$ $\widetilde{\mathcal{F}}$-transversally. Indeed, since $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_0$ are not translates of each other, the attracting fixed point of $\overline{T_0}$ in S_∞ is disjoint from $\widetilde{\gamma}_1^+$. It follows that either $\widetilde{\gamma}_2^\pm = (T_0\widetilde{\gamma}_1)^\pm$ separate $\widetilde{\gamma}_1^\pm$, or these limit points have one of the following four orders on S_∞: (a) $\widetilde{\gamma}_1^-, \widetilde{\gamma}_1^+, \widetilde{\gamma}_2^+, \widetilde{\gamma}_2^-$; (b) $\widetilde{\gamma}_1^+, \widetilde{\gamma}_1^-, \widetilde{\gamma}_2^-, \widetilde{\gamma}_2^+$;

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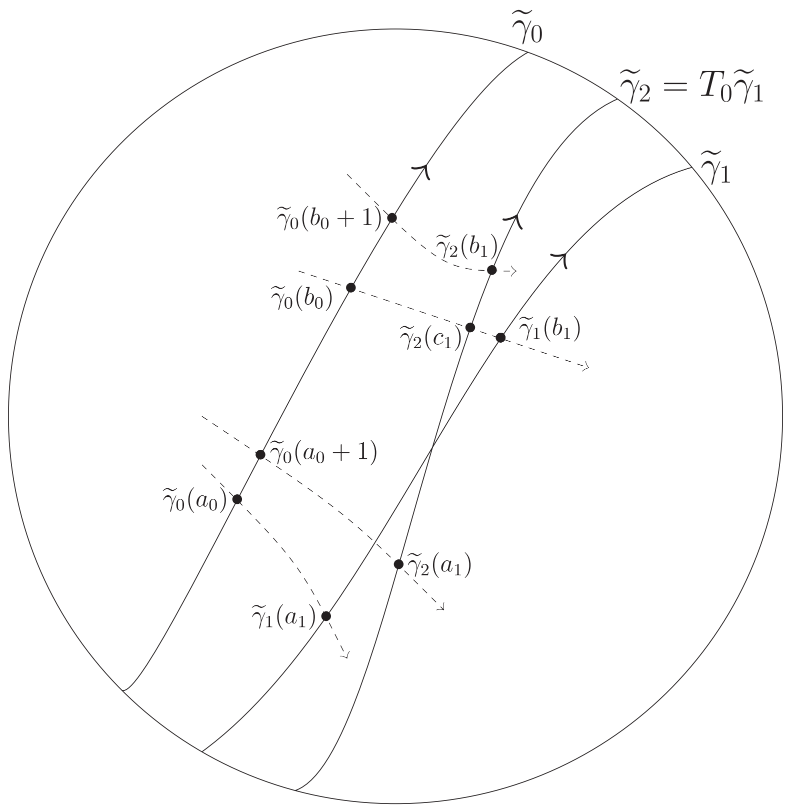

FIGURE 5. Lifts and leaves in the proof of Lemma 3.8(3).

(c) $\widetilde{\gamma}_1^-, \widetilde{\gamma}_1^+, \widetilde{\gamma}_2^-, \widetilde{\gamma}_2^+$; (d) $\widetilde{\gamma}_1^+, \widetilde{\gamma}_1^-, \widetilde{\gamma}_2^+, \widetilde{\gamma}_2^-$. Since the asymptotics of $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ do not separate in S_∞, $\widetilde{T_0}$ turns $\widetilde{\gamma}_1^-$ and $\widetilde{\gamma}_1^+$ to the same direction on S_∞ and so (a) and (b) can be excluded. Since there are leaves of $\widetilde{\mathcal{F}}$ that intersect both $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, (c) and (d) can be excluded. Hence $\widetilde{\gamma}_2^\pm$ separate $\widetilde{\gamma}_1^\pm$ and by Corollary 3.3, $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ intersect $\widetilde{\mathcal{F}}$-transversally. By (1), $c_1 - a_1 \le 1$, and we obtain a contradiction. □

We end this section with the following two lemmas about the existence of a convenient choice of loops equivalent to Γ.

LEMMA 3.9. *Let Γ be any $\mathcal{F}$-transverse loop and assume that $[\Gamma]_{\widehat{\pi}(\mathrm{dom}(\mathcal{F}))} = m\alpha$, where α is a primitive free homotopy class and $m > 1$. Then there is an $\mathcal{F}$-transverse loop Γ' with $[\Gamma']_{\widehat{\pi}(\mathrm{dom}(\mathcal{F}))} = \alpha$ such that Γ is equivalent to $(\Gamma')^m$.*

LEMMA 3.10. *Let Γ be an $\mathcal{F}$-transverse loop such that $[\Gamma]_{\widehat{\pi}(\mathrm{dom}(\mathcal{F}))} \in \widehat{\pi}(\mathrm{dom}(\mathcal{F}))$ is primitive, and let $k := \mathrm{si}_{\mathrm{dom}(\mathcal{F})}([\Gamma])$. Then, up to a modification of Γ in its equivalence class, there are pairwise distinct points $x_1, \dots, x_k \in \mathrm{dom}(\mathcal{F})$, pairwise distinct parameters $t_1, \dots, t_k, t'_1, \dots, t'_k \in [0,1)$ with $t_i < t'_i$, for $i \in \{1, \dots, k\}$, and lifts $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_k$ of Γ, pairwise not translates of each other, such that, for all $i \in \{1, \dots, k\}$, $x_i = \Gamma(t_i) = \Gamma(t'_i)$ and $\widetilde{\gamma}$ and $\widetilde{\gamma}_i$ intersect $\widetilde{\mathcal{F}}$-transversally (positively or negatively) in $\widetilde{\gamma}(t_i) = \widetilde{\gamma}_i(t'_i)$.*

Proof of Lemma 3.9. Let $\widetilde{\gamma}$ be a lift of Γ and T the shift for $\widetilde{\gamma}$. There is a deck transformation S on $\widetilde{\mathrm{dom}}(\mathcal{F})$ such that $S^m = T$ and such that after identifying S with an element in the fundamental group of $\mathrm{dom}(\mathcal{F})$, its projection to $\widehat{\pi}(\mathrm{dom}(\mathcal{F}))$ is α. Consider the lifts $S^k\widetilde{\gamma}$ of Γ for $k = 0, 1, \dots$.

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We first show that these lifts are all pairwise equivalent in $\widetilde{\mathrm{dom}(\mathcal{F})}$. To this end it suffices to show that $S\widetilde{\gamma}$ is equivalent to $\widetilde{\gamma}$. Note that $S^k\widetilde{\gamma}$ intersects $\widetilde{\gamma}$ for all $k \in \mathbb{N}$, otherwise, for some $k \in \mathbb{N}$, $S^k\widetilde{\gamma}$ is completely on the left or completely on the right of $\widetilde{\gamma}$, and by induction this will hold for $S^{km}\widetilde{\gamma} = T^k\widetilde{\gamma} = \widetilde{\gamma}$, a contradiction. Let $s, t \in \mathbb{R}$ such that $S\widetilde{\gamma}(s) = \widetilde{\gamma}(t)$. Note that for every $k \in \mathbb{N}$, we have $S\widetilde{\gamma}(s+k) = ST^k\widetilde{\gamma}(s) = T^kS\widetilde{\gamma}(s) = T^k\widetilde{\gamma}(t) = \widetilde{\gamma}(t+k)$, so in fact, $S\widetilde{\gamma}$ and $\widetilde{\gamma}$ intersect infinitely many times. This implies that for any $k \in \mathbb{N}$, the paths $S\widetilde{\gamma}|_{[s+k,s+k+1]}$ and $\widetilde{\gamma}|_{[t+k,t+k+1]}$ are equivalent, otherwise there would be a leaf ϕ that meets one of these $\widetilde{\mathcal{F}}$-transverse paths twice, which is impossible since $\widetilde{\mathcal{F}}$ is non-singular. Therefore, the lifts $S\widetilde{\gamma}$ and $\widetilde{\gamma}$ are equivalent.

Let $Z \subset \widetilde{\mathrm{dom}(\mathcal{F})}$ be the union of the images of the lifts $S^k\widetilde{\gamma}$, $k = 0, \dots, m-1$. For each t, consider the unique point $x \in Z$ with the property that

- $x \in \phi_{\widetilde{\gamma}(t)}$, and
- there is no $y \in Z, y \neq x$ such that $y \in \phi_x^+$.

Since $S^k\widetilde{\gamma}, k = 0, \dots, m-1$ are pairwise equivalent, it is easy to see that this map defines a transverse line $t \mapsto \widetilde{\gamma}'(t)$ in $\widetilde{\mathrm{dom}(\mathcal{F})}$, equivalent to $\widetilde{\gamma}$. The deck transformation S leaves the image of $Z = \bigcup_{k=0}^{m-1} S^k\widetilde{\gamma}$ in $\widetilde{\mathrm{dom}(\mathcal{F})}$ invariant. Since S is orientation-preserving, and preserves Z and the foliation $\widetilde{\mathcal{F}}$, $\widetilde{\gamma}'$ is also invariant under S. Hence, $\widetilde{\gamma}'$ is a lift of a loop Γ' on $\mathrm{dom}(\mathcal{F})$ with the desired properties. The proof of the lemma is complete. $\square$

Proof of Lemma 3.10. By perturbing Γ in its equivalence class we may assume that for any $\widetilde{x} \in \widetilde{\mathrm{dom}(\mathcal{F})}$ there are at most two lifts $\widetilde{\gamma}$ of Γ that intersect in $\widetilde{x}$. Let $\Lambda : S^1 \to (\mathrm{dom}(\mathcal{F}), g_{\mathrm{hyp}})$ be a closed geodesic that is freely homotopic to Γ. We have $\mathrm{si}(\Lambda) = \mathrm{si}_{\mathrm{dom}(\mathcal{F})}([\Gamma]) = k$. Let $\widetilde{\lambda} : \mathbb{R} \to \widetilde{\mathrm{dom}(\mathcal{F})}$ be a lift of Λ, and T the shift for $\widetilde{\lambda}$. Since Λ is a primitive closed geodesic in a hyperbolic surface, we have lifts $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_k$ with $\widetilde{\lambda}_i^+ \in (\widetilde{\lambda}^+, \widetilde{\lambda}^-) \subset S_\infty$ and $\widetilde{\lambda}_i^- \in (\widetilde{\lambda}^-, \widetilde{\lambda}^+) \subset S_\infty$ such that for every $l \in \mathbb{Z}$ and $i \neq j$, $T^l\widetilde{\lambda}_i$ and $\widetilde{\lambda}_j$ are not translates of each other. Lifting a free homotopy from Λ to Γ to homotopies of the universal cover $\widetilde{\mathrm{dom}(\mathcal{F})}$ that extend $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_k$ respectively, we obtain lifts $\widetilde{\gamma}'_1, \dots, \widetilde{\gamma}'_k$, with the same properties. By applying multiples of the shifts $T'_1, \dots, T'_k$ for $\widetilde{\gamma}'_1, \dots, \widetilde{\gamma}'_k$ respectively, and applying multiples of T to them, we may additionally assume that $\widetilde{\gamma}'_i|_{[0,1)}$ and $\widetilde{\gamma}|_{[0,1)}$ intersect, for each $i \in \{1, \dots, k\}$. Choose $s_i, s'_i \in \mathbb{R}$ with $\widetilde{\gamma}(s_i) = \widetilde{\gamma}'_i(s'_i)$. Then let S_i, $i \in \{1, \dots, k\}$, be the deck transformation with $S_i\widetilde{\gamma} = \widetilde{\gamma}'_i$. With the following choice of lifts $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_k$ of γ and the parameters $t_1, \dots, t_n$, $t'_1 \dots, t'_n$, the properties of the lemma are satisfied. If $s_i < s'_i$ set $\widetilde{\gamma}_i := \widetilde{\gamma}'_i$, and $t_i := s_i$, $t'_i := s'_i$. If $s_i > s'_i$, set $\widetilde{\gamma}_i := S_i^{-1}\widetilde{\gamma}$, and $t_i := s'_i$, $t'_i := s_i$. $\square$

4. Geometric self-intersections, growth of periodic points and entropy

In this section let M be an oriented closed surface, $f : M \to M$ a homeomorphism isotopic to the identity, I a maximal identity isotopy for f, and $\mathcal{F}$ an oriented foliation transverse to I. As in the introduction, denote by $N_{\mathrm{per}}(f, n)$ the number of n-periodic points of f of period at most n, let $\mathrm{Per}^\infty(f) := \limsup_{n\to+\infty} \log(N_{\mathrm{per}}(f, n))/n$, and let $h_{\mathrm{top}}(f)$ denote the topological entropy of f. Let Γ be an $\mathcal{F}$-transverse loop that is not freely homotopic to a multiple of a simple loop in $\mathrm{dom}(I)$, say $[\Gamma]_{\widehat{\pi}(\mathrm{dom}(I))} = m\alpha$, where α is a non-simple primitive class in $\mathrm{dom}(I)$ and $m \in \mathbb{N}$. In this section we prove the following propositions.

PROPOSITION 4.1. *If* Γ *is linearly admissible of order* q, *then*

$$\mathrm{Per}^{\infty}(f) \geq \frac{m}{q} \max\left\{\frac{\log(\mathrm{si}_{\mathrm{dom}(I)}(\alpha)+1)}{16}, \frac{\log 2}{2}\right\}. \tag{5}$$

PROPOSITION 4.2. *If* Γ *is linearly admissible of order* q, *then*

$$h_{\mathrm{top}}(f) \geq \frac{m}{q} \max\left\{\frac{\log(\mathrm{si}_{\mathrm{dom}(I)}(\alpha)+1)}{16}, \frac{\log 2}{2}\right\}. \tag{6}$$

Remark 4.3. Note that the lower bound $m\log(2)/2q$ on the topological entropy of f is larger than the bound obtained in Proposition 38 in [LeCT18] and the bound obtained in Theorem N in [LeCT22].

Theorem 1.1 follows from Propositions 4.1 and 4.2.

Proof of Theorem 1.1. If x is a q-periodic point of f then $I^q(x)$ is homotopic with fixed endpoints to an $\mathcal{F}$-transverse loop Γ in $\mathrm{dom}(I)$. By assumption, $[\Gamma]_{\widehat{\pi}(\mathrm{dom}(I))} = m\alpha$. The natural lift γ of Γ satisfies that $\gamma|_{[0,k]}$ is admissible of order kq, for every $k \in \mathbb{N}$. In particular, Γ is linearly admissible of order q. Hence, the lower bounds on $\mathrm{Per}^{\infty}(f)$ and $h_{\mathrm{top}}(f)$ follow from Propositions 4.1 and 4.2. □

Before we proceed with the proofs of Propositions 4.1 and 4.2, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. Let x be a periodic point of class α. For any identity isotopy I for f there is by [BCLR20, Corollary 1.3] a maximal identity isotopy $\widehat{I}$ for f such that $\mathrm{Fix}(I) \subset \mathrm{Fix}(\widehat{I})$ and such that the loop $\widehat{I}(y)$ is homotopic to $I(y)$ in $\mathrm{dom}(I)$ for any $y \in \mathrm{dom}(I) \cap \mathrm{Fix}(f)$. In particular, $[\widehat{I}(x)] = \alpha$ in $\widehat{\pi}(M)$ and so $\widehat{I}(x)$ defines a free homotopy class $\widehat{\alpha}$ of loops in $\mathrm{dom}(\widehat{I})$ whose pushforward by the inclusion $\mathrm{dom}(\widehat{I}) \hookrightarrow M$ is α. The class $\widehat{\alpha}$ is primitive and $\mathrm{si}_M(\widehat{\alpha}) \geq \mathrm{si}_M(\alpha)$. So the statement follows from Theorem 1.1. □

The main steps in the proofs of Propositions 4.1 and 4.2 are adaptions of the main steps in the proofs of Propositions 31 and 38 in [LeCT18]. The notation and arguments are kept close to those in the corresponding proofs in [LeCT18].

Proof of Proposition 4.1. By modifying Γ in its equivalence class if necessary we may assume by Lemma 3.9 that there exist an $\mathcal{F}$-transverse loop Γ', primitive in $\mathrm{dom}(I)$, and $m \in \mathbb{N}$ such that $\Gamma = (\Gamma')^m$. We first give the proof of Proposition 4.1 for $m = 1$ (i.e. $\Gamma = \Gamma'$). It will be easy to adapt that proof for $m > 1$. Let $k = \mathrm{si}_{\mathrm{dom}(I)}([\Gamma])$, and fix a lift $\widetilde{\gamma}$ of the natural lift γ of Γ. By Lemma 3.10 there are, after modifying Γ further in its equivalence class, pairwise distinct points $x_1, \ldots, x_k \in \mathrm{dom}(\mathcal{F})$, pairwise distinct parameters $t_1, \ldots, t_k$, $t'_1, \ldots, t'_k \in [0,1)$ with $t_i < t'_i$, and lifts $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_k$ of Γ, pairwise not translates of each other, such that, for all $i = 1, \ldots, k$, $x_i = \Gamma(t_i) = \Gamma(t'_i)$ and $\widetilde{\gamma}$ and $\widetilde{\gamma}_i$ intersect $\widetilde{\mathcal{F}}$-transversally (positively or negatively) in $\widetilde{\gamma}(t_i) = \widetilde{\gamma}_i(t'_i)$.

By Lemma 3.7 there are $\overline{x} \in \{x_1, \ldots, x_k\}$ and $\overline{t}, \underline{t} \in [0,1)$ with $\overline{x} = \Gamma(\overline{t}) = \Gamma(\underline{t})$, such that for any lifts $\widetilde{\gamma}'$ and $\widetilde{\gamma}''$ of Γ that intersect $\mathcal{F}$-transversally and positively, (respectively, negatively), $\widetilde{\gamma}'$ does not intersect the leaves $\phi_{\widetilde{\gamma}''(\overline{t}+k)}$, (respectively, $\phi_{\widetilde{\gamma}''(\underline{t}+k)}$), for all $k \in \mathbb{Z}$. We may assume that $\overline{x} = x_1$ (i.e. $\{t_1, t'_1\} = \{\overline{t}, \underline{t}\}$). For each $i \in \{1, \ldots, k\}$ and $l \in \mathbb{N}$ let $\gamma_{i,l}$ be the transverse path $\gamma_{[0,t_i]}\gamma_{[t'_i,l]}$ in $\mathrm{dom}(\mathcal{F})$. Furthermore, let $\gamma_{0,l} := \gamma_{[0,l]}$. Let $n \geq 1$. For any $\rho = (\rho_1, \ldots, \rho_n)$ with $\rho_i \in \{0, \ldots, k\}$, and $l \geq 1$, consider the path $\gamma_{\rho,l} := \gamma_{\rho_1,l}\gamma_{\rho_2,l}\cdots\gamma_{\rho_n,l}$. This path defines a transverse loop $\Gamma_{\rho,l}$.

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Claim 1.

(1) *For each* $\rho = (\rho_1, \dots, \rho_n)$ *with* $\rho_i \in \{0, \dots, k\}$, $i = 1, \dots, n$ *and* $l \geq 2$, $\Gamma_{\rho,l}$ *is linearly admissible of order* lnq.
(2) *For each* $\rho = (\rho_1, \dots, \rho_n)$ *with* $\rho_i \in \{0, 1\}$, $i = 1, \dots, n$, $\Gamma_{\rho,1}$ *is linearly admissible of order* nq.

Proof. Let $l \geq 1$. Let $\rho = (\rho_1, \dots, \rho_n)$, with $\rho_i \in \{0, \dots, k\}$ if $l \geq 2$ and $\rho_i \in \{0, 1\}$ if $l = 1$. Consider the paths $\gamma|_{[0,l(pn+1)]}$, $p \in \mathbb{N}$. Since Γ is linearly admissible of order q, there is a sequence $s_p = s_p(n, l)$, $p \in \mathbb{N}$, with $s_p \to +\infty$, $\limsup_{p\to\infty}(lpn)/s_p \geq 1/q$, and such that $\gamma|_{[0,l(pn+1)]}$ is admissible of order at most s_p. Now fix $p \in \mathbb{N}$ and let $\hat{\gamma} := \gamma|_{[0,l]}(\gamma_{\rho,l})^p$. We will show that $\hat{\gamma}$ is admissible of order at most s_p. From this, it follows that the same holds for $(\gamma_{\rho,l})^p$. So, since p was arbitrary, the claim follows.

Let $\hat{\rho} = (\hat{\rho}_1, \dots, \hat{\rho}_{pn}) = (\rho_1, \dots, \rho_n, \rho_1, \dots, \rho_n, \rho_1, \dots, \dots, \rho_n)$ be the pn-tuple that is built from p repetitions of ρ. Consider the transverse paths $\hat{\gamma}_j$, $j \in \{0, \dots, pn\}$ given by $\hat{\gamma}_0 := \gamma|_{[0,l]}\gamma|_{[0,lpn]}$, $\hat{\gamma}_j := \gamma|_{[0,l]}\gamma_{\hat{\rho}_1,l} \cdots \gamma_{\hat{\rho}_j,l}\gamma|_{[lj,lpn]}$, for $j \in \{1, \dots, pn-1\}$, and $\hat{\gamma}_{pn} := \hat{\gamma}$. If $j \in \{0, \dots, np-1\}$ we will say that $\hat{\gamma}_j$ is *reducible* if it has an $\mathcal{F}$-transverse self-intersection at $\gamma|_{[lj,lpn]}(lj + t_{\hat{\rho}_{j+1}}) = \gamma|_{[lj,lpn]}(lj + t'_{\hat{\rho}_{j+1}})$. For $j \in \{0, \dots, pn\}$ consider the following statement.

$$(R_j): \text{ The path } \hat{\gamma}_j \text{ is admissible of order at most } s_p, \text{and it is reducible if } j < pn.$$

We want to show (R_{pn}), and we prove it by induction. Note that (R_0) holds. Indeed, $\hat{\gamma}_0 = \gamma|_{[0,l]}\gamma|_{[0,lpn]} = \gamma|_{[0,l(pn+1)]}$ is admissible or order at most s_p and $\gamma|_{[0,l(pn+1)]}$ has an $\mathcal{F}$-transverse self-intersection at $\gamma(l + t_{\hat{\rho}_1}) = \gamma(l + t'_{\hat{\rho}_1})$ by assumption. Assume now that (R_j) holds for some $j \in \{0, \dots, pn-1\}$. Applying Proposition 2.4 at $\gamma|_{[lj,lpn]}(lj + t_{\hat{\rho}_{j+1}}) = \gamma|_{[lj,lpn]}(lj + t'_{\hat{\rho}_{j+1}})$ yields that $\hat{\gamma}_{j+1}$ is admissible of order at most s_p. We are left to show that $\hat{\gamma}_{j+1}$ is reducible if $j \in \{0, \dots, pn-2\}$. So let $s_1 := l(j+1) + t_{\hat{\rho}_{j+2}}$ and $s_2 := l(j+1) + t'_{\rho_{j+2}}$. Take two lifts τ_1 and τ_2 of $\hat{\gamma}_{j+1}$ that intersect at a lift $\widetilde{y}$ of $y := \gamma|_{[l(j+1),lpn]}(s_1) = \gamma|_{[l(j+1),lpn]}(s_2)$.

For $l \geq 2$, consider the subpaths $\overline{\tau}_1 = \tau_1|_{[l(j+1)-1,l(j+1)+2]}$ and $\overline{\tau}_2 = \tau_2|_{[l(j+1)-1,l(j+1)+2]}$ of τ_1 and τ_2. It is enough to show that these subpaths intersect $\widetilde{\mathcal{F}}$-transversally at $\widetilde{y}$. Note that since $l \geq 2$, $\overline{\tau}_1$ and $\overline{\tau}_2$ are themselves subpaths of lifts $\widetilde{\gamma}_1 : \mathbb{R} \to \widetilde{\mathrm{dom}(\mathcal{F})}$ and $\widetilde{\gamma}_2 : \mathbb{R} \to \widetilde{\mathrm{dom}(\mathcal{F})}$ of γ that intersect $\widetilde{\mathcal{F}}$-transversally at $\widetilde{y}$. Therefore, since $|s_i - (l(j+1) - 1)| \geq 1$ and $|s_i - (l(j+1) + 2)| \geq 1$ for $i = 1, 2$, we conclude by Lemma 3.8(1) that $\overline{\tau}_1$ and $\overline{\tau}_2$ intersect $\widetilde{\mathcal{F}}$-transversally.

For $l = 1$, and $\rho_i \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$, consider the subpaths $\overline{\tau}_1 = \tau_1|_{[j+t'_1,j+1+t'_1]}$ of τ_1 and $\overline{\tau}_2 = \tau_2|_{[j+1+t_1,j+2+t_1]}$, which are also subpaths of lifts $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ of γ that intersect $\widetilde{\mathcal{F}}$-transversally at $\widetilde{y}$. It follows from Lemma 3.7 that the subpaths $\overline{\tau}_1$ and $\overline{\tau}_2$ intersect $\widetilde{\mathcal{F}}$-transversally at $\tau_1|_{[j+t'_1,j+1+t'_1]}(j+1+t_1) = \tau_2|_{[j+1+t_1,j+2+t_1]}(j+1+t'_1)$, positively if $t'_1 = \overline{t}$ and negatively if $t'_1 = \underline{t}$.

In both cases considered above, (R_{pn}) follows now by induction, that is, $\hat{\gamma}$ is admissible of order at most s_p. □

Let $z = \Gamma(0) \in \mathrm{dom}(\mathcal{F})$. Choose a lift $\widetilde{z}$ of z in $\widetilde{\mathrm{dom}(\mathcal{F})}$. Let $l \geq 1$. For $J \subset \{0, \dots, k\}$ with $0 \in J$, consider the family of paths $\gamma_{i,l}$, $i \in J$, where $\gamma_{i,l}$ is defined as above. Set $t_{\min} := \min\{t_i, i \in J \setminus \{0\}\}$ and $t'_{\max} := \max\{t'_i, i \in J \setminus \{0\}\}$. For all $i \in J$, and suitable $a_i < b_i$, let $\widetilde{\gamma}_{i,l} : [a_i, b_i] \to \widetilde{\mathrm{dom}(\mathcal{F})}$ be the lift of the path $\gamma_{i,l}\gamma_{[0,t_{\min}]}$ that starts at $\widetilde{z}$. Furthermore, define the paths $\gamma'_{i,l}$, $i \in J$, by $\gamma'_{i,l} := \gamma|_{[0,l-1+t_i]}\gamma|_{[l-1+t'_i,l]}$, for $i \neq 0$ and $\gamma'_{0,l} := \gamma|_{[0,l]}$, and consider the lifts $\widetilde{\gamma}'_{i,l} : [a'_i, b'_i] \to \widetilde{\mathrm{dom}(\mathcal{F})}$ of the paths $\gamma_{[t'_{\max},1]}\gamma'_{i,l}$, $i \in J$, that end at $\widetilde{z}$. We say that the paths $\gamma_{i,l}$, $i \in J$, *spread*

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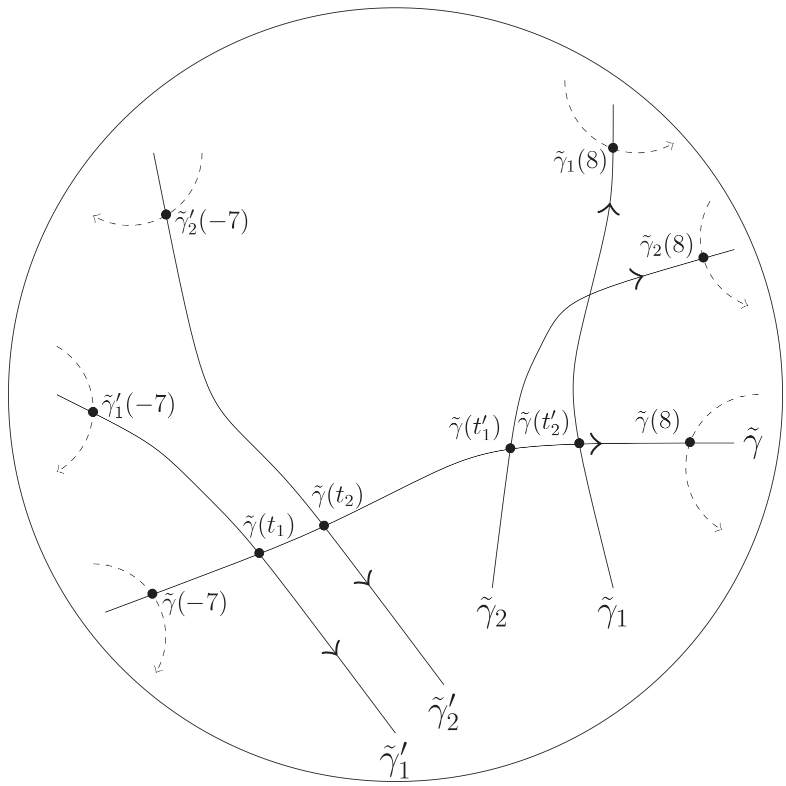

FIGURE 6. Some lifts of the loop Γ to $\widetilde{\mathrm{dom}(\mathcal{F})}$, with $t_1 < t'_1, t_2 < t'_2 \in [0, 1)$, and $\Gamma(t_1) = \Gamma(t'_1)$, $\Gamma(t_2) = \Gamma(t'_2)$ in the situation that the family $\gamma_{0,8}, \gamma_{1,8}, \gamma_{2,8}$ spreads, with $1 \prec 2 \prec 0$ and $1^* = 0$, $2^* = 1$, $0^* = 2$.

if for all $i \neq j \in J$, $\phi_{\widetilde{\gamma}_{i,l}}(b_i)$ is above or below $\phi_{\widetilde{\gamma}_{j,l}}(b_j)$ relative to $\phi_{\widetilde{z}}$, and $\phi_{\widetilde{\gamma}'_{i,l}}(a'_i)$ is above or below $\phi_{\widetilde{\gamma}'_{j,l}}(a'_j)$ relative to $\phi_{\widetilde{z}}$ (see Figure 6). We get two total orders $\prec$ and $\prec^*$ on J, by saying that $i \prec j$ if $\phi_{\widetilde{\gamma}_{i,l}}(b_i)$ is below $\phi_{\widetilde{\gamma}_{j,l}}(b_j)$ relative to $\phi_{\widetilde{z}}$, and by saying that $i \prec^* j$ if $\phi_{\widetilde{\gamma}'_{i,l}}(a'_i)$ is below $\phi_{\widetilde{\gamma}'_{j,l}}(a'_j)$ relative to $\phi_{\widetilde{z}}$. For each $i \in J$ we define $i^* \in J$ by requiring that $\#\{j \in J \mid j \prec i\} = \#\{j \in J \mid i^* \prec^* j^*\}$. Note that if $j \prec i$, then $i^* \prec^* j^*$.

A $2n$-tuple $\hat\rho = (\rho_1, \ldots, \rho_{2n})$ with $\rho_j \in J$ that satisfies $\rho_{n+j} = \rho^*_{n-j+1}$ for all $j \in \{1, \ldots, n\}$ will be called a *quasi-palindromic word of length* $2n$ *for* J. In the following, we will only consider quasi-palindromic words ρ with $\rho_1 = 0$. For such a quasi-palindromic word ρ of length $2n$ for J we consider the lift $\widetilde{\gamma}_\rho = \widetilde{\gamma}^-_\rho \widetilde{\gamma}^+_\rho$ of $\prod_{1 \le j \le 2n} \gamma_{\rho_j,l} = \gamma_{[0,l]} \prod_{2 \le j \le 2n} \gamma_{\rho_j,l} = \gamma_{[0,1]} \prod_{2 \le j \le n} \gamma'_{\rho_j,l} \gamma_{[1,l]} \prod_{n+1 \le j \le 2n} \gamma_{\rho_j,l}$, where $\widetilde{\gamma}^-_\rho$ is the lift of $\gamma_{[0,1]} \prod_{2 \le j \le n} \gamma'_{\rho_j,l} \gamma_{[1,l]}$ ending at $\widetilde{z}$ and $\widetilde{\gamma}^+_\rho$ is the lift of $\prod_{n+1 \le j \le 2n} \gamma_{\rho_j,l}$ starting at $\widetilde{z}$. Let T_ρ be the deck transformation that sends the starting point of $\widetilde{\gamma}_\rho$ to its endpoint. Let $\widetilde{\gamma}^\infty_\rho := \prod_{k \in \mathbb{Z}} T^k_\rho(\widetilde{\gamma}_\rho)$. It is a lift of $\Gamma_{\rho,l}$. We will also consider the path $\widetilde{\gamma}^2_\rho = \widetilde{\gamma}_\rho T_\rho(\widetilde{\gamma}_\rho)$.

CLAIM 2. *Let $l \ge 1$ and let $\gamma_{i,l}$, $i \in J$, be a family of paths as above that spread. If ρ and ρ' are two distinct quasi-palindromic words of the same length with $\rho_1 = \rho'_1 = 0$, then the paths $\widetilde{\gamma}_\rho$ and $\widetilde{\gamma}_{\rho'}$ intersect $\widetilde{\mathcal{F}}$-transversally at $\widetilde{z}$.*

Proof. Let $\rho \neq \rho'$ with $\rho_1 = \rho'_1 = 0$. Note that then $\rho_{2n} = \rho'_{2n}$. Hence, there is $j \in \{1, \ldots, n-1\}$ such that $\rho_{n+i} = \rho'_{n+i}$ for $0 < i < j$ and $\rho_{n+j} \neq \rho'_{n+j}$. Assume that $\rho_{n+j} \prec \rho'_{n+j}$, the other case

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being analogous. This implies that $\rho'_{n-j+1} \prec^* \rho_{n-j+1}$. There are lifts of $\prod_{n+1\le i\le n+j}\gamma_{\rho_i,l}\gamma_{[0,t_{\min}]}$ and $\prod_{n+1\le i\le n+j}\gamma_{\rho'_i,l}\gamma_{[0,t_{\min}]}$ with the same starting point $\widetilde{z}$, and the leaf through the other endpoint of the first lift is below the leaf through the other endpoint of the second lift. Since $j<n$, these lifts are subpaths of $\widetilde{\gamma}^+_\rho$ and $\widetilde{\gamma}^+_{\rho'}$. Dually, there are lifts of $\gamma_{[t'_{\max},1]}\prod_{n-j+1\le i\le n}\gamma'_{\rho_i,l}\gamma_{[1,l]}$ and $\gamma_{[t'_{\max},1]}\prod_{n-j+1\le i\le n}\gamma'_{\rho'_i,l}\gamma_{[1,l]}$ that are subpaths of $\widetilde{\gamma}^-_\rho$ and $\widetilde{\gamma}^-_{\rho'}$ with the same second endpoint $\widetilde{z}$, and the leaf through the first endpoint of the first lift is above the first endpoint of the second lift. The claim follows. $\square$

Claim 3. *Let $\gamma_{i,l}, i\in J$, be a family of paths that spread. Then there is a constant $L>0$ such that, for a given quasi-palindromic word ρ of length $2n$ with $\rho_1=0$, there are at most Ln^2 different such quasi-palindromic words ρ' such that $\Gamma_{\rho,l}$ and $\Gamma_{\rho',l}$ are equivalent.*

Proof. The proof is the same as the proof of Lemma 35 in [LeCT18]. There is a constant L' such that the number of deck transformations S such that $\widetilde{\gamma}_{i,l}$ and $S(\widetilde{\gamma}_{j,l})$ intersect is at most L' for all $i,j\in J$. Hence, there are at most $8L'n^2$ deck transformations S such that $\widetilde{\gamma}_\rho$ and $S(\widetilde{\gamma}^2_\rho)$ have an $\widetilde{\mathcal{F}}$-transverse intersection. Let ρ and ρ' be two quasi-palindromic words of length $2n$ with $\rho_1=\rho'_1=0$ such that $\Gamma_{\rho,l}$ and $\Gamma_{\rho',l}$ are equivalent. This means that there is a deck transformation $S_{\rho'}$ such that the lift $\widetilde{\gamma}^\infty_{\rho'}$ of $\Gamma_{\rho',l}$ is equivalent to the lift $S_{\rho'}\widetilde{\gamma}^\infty_\rho$ of $\Gamma_{\rho,l}$. And, if γ_1 and γ_2 are subpaths of $\widetilde{\gamma}^\infty_{\rho'}$ and $S_{\rho'}\widetilde{\gamma}^\infty_\rho$, respectively, that are equivalent, then $T_1\gamma_1$ is equivalent to $T_2\gamma_2$, where $T_1=T_{\rho'}$ and $T_2=S_{\rho'}T_\rho S^{-1}_{\rho'}$ are the shifts for those lifts. This means that $\widetilde{\gamma}_\rho$ is equivalent to a subpath γ' of $S_{\rho'}\widetilde{\gamma}^\infty_\rho$ for which T_2 sends the first endpoint to the second endpoint. Any such path γ' is contained in $T_2^kS_{\rho'}(\widetilde{\gamma}^2_\rho)$ for some $k\in\mathbb{Z}$. Hence, by replacing $S_{\rho'}$ with $S_{\rho'}T^k_\rho$, we can assume that $\widetilde{\gamma}_{\rho'}$ is equivalent to a subpath of $S_{\rho'}(\widetilde{\gamma}^2_\rho)$. Moreover, if $\rho'\neq\rho''$, it follows that $S_{\rho'}\neq S_{\rho''}$ by Claim 2. Also by Claim 2, $\widetilde{\gamma}_\rho$ and $S_{\rho'}(\widetilde{\gamma}^2_\rho)$ intersect $\widetilde{\mathcal{F}}$-transversally. This proves the claim. $\square$

Claim 4.

(1) *The family $\mathcal{P}_1$ of paths $\gamma_{i,8}$, $i\in\{0,1,\dots,k\}$, spread.*
(2) *The pair $\mathcal{P}_2$ of paths $\gamma_{0,1}$, $\gamma_{1,1}$ spread.*

Proof. We first show (2). Let $\widetilde{\gamma}_{0,1}=\widetilde{\gamma}_0|_{[0,1+t_1]}$ and $\widetilde{\gamma}_{1,1}=\widetilde{\gamma}_0|_{[0,t_1]}\widetilde{\gamma}_1|_{[t'_1,1+t_1]}$ be the lifts of $\gamma_{0,1}\gamma|_{[0,t_1]}$ and $\gamma_{1,1}\gamma|_{[0,t_1]}$, respectively, that start at the point $\widetilde{z}$. Here $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are suitable lifts of γ. Note that $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ have an $\widetilde{\mathcal{F}}$-transverse intersection. So, by Lemma 3.7 and since $\{t_1,t'_1\}=\{\overline{t},\underline{t}\}$, we have that $\phi_{\widetilde{\gamma}_0(t'_1)}$ does not intersect $\widetilde{\gamma}_{1,1}$ and $\phi_{\widetilde{\gamma}_1(1+t_1)}$ does not intersect $\widetilde{\gamma}_{0,1}$. It is then easy to see that $\phi_{\widetilde{\gamma}_0(1+t_1)}$ is above or below $\phi_{\widetilde{\gamma}_1(1+t_1)}$ relative to $\phi_{\widetilde{z}}$. Similarly, one can see that the leaf through the starting point of $\widetilde{\gamma}'_{0,1}$ is above or below the leaf through the starting point of $\widetilde{\gamma}'_{1,1}$ relative to $\phi_{\widetilde{z}}$, where these paths lift $\gamma|_{[t'_1,1]}\gamma'_{0,1}$ and $\gamma|_{[t'_1,1]}\gamma'_{1,1}$ with a common endpoint.

We now turn to (1). Consider first a pair $\gamma_{0,8}=\gamma_{[0,8]}$ and $\gamma_{i,8}=\gamma_{[0,t_i]}\gamma|_{[t'_i,8]}$, for some $i\in\{1,\dots,k\}$. The same argument as in the proof of (2) shows that such a pair of paths spread. Now let $i,j\in\{1,\dots,k\}$ with $i\neq j$, and consider $\gamma_{i,8}=\gamma_{[0,t_i]}\gamma|_{[t'_i,8]}$ and $\gamma_{j,8}=\gamma_{[0,t_j]}\gamma|_{[t'_j,8]}$, and their lifts $\widetilde{\gamma}_{i,8}=\widetilde{\gamma}|_{[0,t_i]}\widetilde{\gamma}_i|_{[t'_i,8]}$ and $\widetilde{\gamma}_{j,8}=\widetilde{\gamma}|_{[0,t_j]}\widetilde{\gamma}_j|_{[t'_j,8]}$ that start at $\widetilde{z}$. Here $\widetilde{\gamma}$, $\widetilde{\gamma}_i$, and $\widetilde{\gamma}_j$ are suitable lifts of γ. We will show that $\phi_{\widetilde{\gamma}_i(8)}$ is below or above $\phi_{\widetilde{\gamma}_j(8)}$ relative to $\phi_{\widetilde{z}}$. Since the lifts $\widetilde{\gamma}_{i,8}$ and $\widetilde{\gamma}_{j,8}$ start at the same point, it is sufficient to show that neither $\widetilde{\gamma}_{i,8}$ is equivalent to a subpath of $\widetilde{\gamma}_{j,8}$ nor $\widetilde{\gamma}_{j,8}$ is equivalent to a subpath of $\widetilde{\gamma}_{i,8}$. Assume that $\widetilde{\gamma}_{i,8}$ is equivalent to a subpath of $\widetilde{\gamma}_{j,8}$. Then, in particular, $\widetilde{\gamma}_i|_{[t'_i+1,8]}$ is equivalent to a subpath of $\widetilde{\gamma}_{j,8}$. Since $\widetilde{\gamma}_i$ and $\widetilde{\gamma}$ intersect $\mathcal{F}$-transversally in $\widetilde{\gamma}_i(t'_i)=\widetilde{\gamma}(t_i)$, there is, by Lemma 3.7, $t^*\in(t'_i,t'_i+1)$ such that the leaf $\phi_{\widetilde{\gamma}_i(t^*)}$ does not intersect $\widetilde{\gamma}$. Moreover, $\phi_{\widetilde{\gamma}_i(t^*)}$ separates $\widetilde{\gamma}_i|_{[t'_i+1,8]}$ and $\widetilde{\gamma}$ and therefore no subpath of $\widetilde{\gamma}_i|_{[t'_i+1,8]}$ is equivalent to a subpath of $\widetilde{\gamma}|_{[0,t_j]}$. Hence, $\widetilde{\gamma}_i|_{[t'_i+1,8]}$ is equivalent to

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a subpath of $\widetilde{\gamma}_j|_{[t'_j,8]}$. Since $\widetilde{\gamma}_i$ and $\widetilde{\gamma}_j$ are not translates of each other, and $8-(t'_i+1)>6$, we obtain a contradiction by Lemma 3.8(3). Analogously $\widetilde{\gamma}_{j,8}$ is not equivalent to a subpath of $\widetilde{\gamma}_{i,8}$. Similarly, one can show, for $i,j\in\{0,\ldots,k\}$ with $i\neq j$, that the leaf through the starting point of $\widetilde{\gamma}'_{i,8}$ is above or below the leaf through the starting point of $\widetilde{\gamma}'_{j,8}$ relative to $\phi_{\widetilde{z}}$, where these paths are lifts of $\gamma'_{i,8}$ and $\gamma'_{j,8}$, respectively, with common endpoint at $\widetilde{z}$. $\square$

By Claim 3 applied to the family $\mathcal{P}_1$ and by Claim 1(1), there is a constant $L>0$ such that for every n there are at least $(k+1)^{n-1}/Ln^2$ different equivalence classes of linearly admissible loops of order $16nq$, hence, by Proposition 2.6, there are at least $(k+1)^{n-1}/Ln^2$ many fixed points of f^{16nq}, and so

$$\mathrm{Per}^{\infty}(f)\geq\limsup_{n\to+\infty}\frac{1}{16nq}\log\frac{(k+1)^{n-1}}{Ln^2}=\log(k+1)/16q.$$

Similarly, by Claim 3 applied to the pair of paths $\mathcal{P}_2$, we obtain that $\mathrm{Per}^{\infty}(f)\geq(\log 2)/2q$.

The proof is easily adapted to the general case $m>1$. As noted above, we can assume that $\Gamma=(\Gamma')^m$ and Γ' satisfy the properties of Lemma 3.10. We can carry out the proof with Γ' instead of Γ, where we consider n that are multiples of m. With the proof of Claim 1 one obtains that the loops $\Gamma'_{\rho,l}$ are even linearly admissible of order at most lnq/m. So one obtains at least $(k+1)^{n-1}/Ln^2$ different equivalence classes of linearly admissible loops of order $16nq/m$, and also at least $2^{n-1}/Ln^2$ different equivalence classes of linearly admissible loops of order $2nq/m$. Hence, the lower bound on $\mathrm{Per}^{\infty}(f)$ claimed in Proposition 4.1 follows as above. $\square$

Proof of Proposition 4.2. As before, we first assume that Γ is primitive in $\mathrm{dom}(I)$ (i.e. $m=1$). The proof in [LeCT18] carries over with slight technical changes. Hence, we will indicate these changes and refer for the full proof to [LeCT18]. The outline of the proof in [LeCT18] can be described as follows. One considers a pair of paths that spread with some suitable $l\geq 1$ and such that for every quasi-palindromic word ρ of length $2n$ (in the situation in [LeCT18] it is also a palindromic word), there is a linearly admissible loop Γ_ρ of order $2lnq$. Each such loop gives rise to a fixed point of f^{2lnq}. Furthermore, there are at least $2^n/Ln^2$ equivalence classes of such loops Γ_ρ. Consider the one-point compactification $\mathrm{dom}(I)\cup\{\infty\}$ of $\mathrm{dom}(I)$ and the extension $\hat{f}$ of $f|_{\mathrm{dom}(I)}$ that fixes $\{\infty\}$. For every $p\in\mathbb{N}$ a family of suitable coverings $\mathcal{V}^p$ of $\mathrm{dom}(I)\cup\{\infty\}$ is constructed (the coverings differ by the size of their neighborhoods at ∞) such that for each element $V=\bigcap_{0\leq k\leq 2lnq}\hat{f}^{-k}(V^k)$, $V^k\in\mathcal{V}^p$, of the covering $\bigvee_{0\leq k\leq 2lnq}\hat{f}^{-k}(\mathcal{V}^p)$ there are at most $M^{lnq/p}$ equivalence classes of those loops Γ_ρ that are associated to some fixed point of $\hat{f}^{2lnq}$ that lies in V. Here M is a suitable constant independent of p. Hence, the minimal cardinality $N_{2lnq}(\hat{f},\mathcal{V}^p)$ of a subcover of $\bigvee_{0\leq k\leq 2lnq}f^{-k}(\mathcal{V}^p)$ is at least $2^n/(Ln^2M^{lnq/p})$. A lower bound on the entropy can then be directly obtained, using the classical definition of topological entropy. The main idea in the proof is to show that orbits that stay close to ∞ for 'many' iterates only contribute to rather 'few' non-equivalent paths. See [LeCT18] for the proof.

One can adapt the proof to other families of paths that spread. We consider the two families $\mathcal{P}_1=\{\gamma_{0,8},\ldots,\gamma_{k,8}\}$ and $\mathcal{P}_2=\{\gamma_{0,1},\gamma_{1,1}\}$. We use the same notation as in the proof of Proposition 4.1 and for a given $n\in\mathbb{N}$ let $\Gamma_{\rho,l}=\prod_{i=1}^{2n}\gamma_{\rho_i,l}$, where $\rho=(\rho_1,\ldots,\rho_{2n})$ with $\rho_i\in\{0,\ldots,l\}$, $l=8$ in the case of the family $\mathcal{P}_1$, and $l=1$ in the case of the family $\mathcal{P}_2$. In order to find the suitable coverings that allow us to make the conclusions above, it is clear from the proof in [LeCT18] that it is sufficient to check the following condition: there exist a finite family of paths $\tau_1,\ldots,\tau_N$, and a transverse loop Γ^* with natural lift γ^* such that

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- τ_i, for any $i \in \{1, \dots, N\}$, has a leaf on its right and a leaf on its left, and
- if a path σ in $\mathrm{dom}(I)$ is equivalent to a subpath of $\prod_{i=1}^{2n} \gamma_{\rho_i,l}$, for some $n \in \mathbb{N}$, and there is no subpath of σ that is equivalent to τ_i, for some $i \in \{1, \dots, N\}$, then σ is equivalent to a subpath of γ^*.

For $\mathcal{P}_1$ this condition holds, as one can easily check for the choices $\tau_i := \gamma_{i,8}$, $i \in \{0, \dots, k\}$, and $\Gamma^* := \prod_{1\le i\le k} \prod_{1\le j\le k} \tau_i\tau_j$. One then can conclude from Proposition 4.1, considering quasi-palindromic words ρ with $\rho_1 = 0$ and arguing as in the proof of [LeCT18, Proposition 38], that there is a constant $M_1 > 0$ and for every $p \in \mathbb{N}$ a covering $\mathcal{V}_1^p$ such that the cardinality of $\bigvee_{0\le k\le 16nq} f^{-k}(\mathcal{V}_1^p)$ is at least $(k+1)^{n-1}/Ln^2M_1^{8nq/p}$. We conclude that

$$\begin{aligned} h_{\mathrm{top}}(f) &\ge \sup_{p\in\mathbb{N}} h_{\mathrm{top}}(\hat{f}, \mathcal{V}_1^p) \ge \sup_{p\in\mathbb{N}} \lim_{n\to\infty} \frac{1}{16nq} \log \frac{(k+1)^{n-1}}{Ln^2 M_1^{8nq/p}} \\ &= \sup_{p\in\mathbb{N}} \left\{ \frac{\log(k+1)}{16q} - \frac{\log M_1}{2p} \right\} = \frac{\log(k+1)}{16q}, \end{aligned} \tag{7}$$

where $h_{\mathrm{top}}(\hat{f}, \mathcal{U}) := \limsup_{n\to\infty} \log(N_n(\hat{f}, \mathcal{U}))/n$ for a covering $\mathcal{U}$ of $\mathrm{dom}(I) \cup \{\infty\}$.

In the case of $\mathcal{P}_2$ we choose some large $u \in \mathbb{N}$ and consider only quasi-palindromic words ρ of length $2n$ such that $\rho_{\nu u} = 0$ for all $\nu \in \mathbb{N}$. Let $\tau_1 := \gamma_{0,1}^2$, $\tau_2 := \gamma_{0,1}\gamma_{1,1}$, $\tau_3 := \gamma_{1,1}\gamma_{0,1}$, and $\Gamma^* = \gamma_{0,1}^3\gamma_{1,1}^u$. The paths τ_1, τ_2, and τ_3 have all a leaf on their right and a leaf on their left, as can easily be checked. Any path σ that is equivalent to a subpath of $\prod_{1\le j\le 2n} \gamma_{\rho_j,1}$ and does not have a subpath equivalent to some $\tau_i, 1 \le i \le 3$, must be equivalent to a subpath of $\gamma_{0,1}$, $\gamma_{0,1}^2$, $\gamma_{0,1}^3$, $\gamma_{0,1}^2\gamma_{1,1}$, $\gamma_{1,1}\gamma_{0,1}^2$, $\gamma_{0,1}\gamma_{1,1}^s$, $\gamma_{1,1}^s\gamma_{0,1}$, $\gamma_{1,1}^s$, $1 \le s \le u-1$, hence is a subpath of the natural lift γ^* of Γ^*. One can conclude that there is a constant $M_2 > 0$ and for every $p \in \mathbb{N}$ a covering $\mathcal{V}_2^p$ such that the cardinality of $\bigvee_{0\le k\le 2nq} f^{-k}(\mathcal{V}_2^p)$ is at least $2^{((u-1)/u)(n-1)}/Ln^2M_2^{nq/p}$. Hence,

$$\begin{aligned} h_{\mathrm{top}}(f) &\ge \sup_{p\in\mathbb{N}} h_{\mathrm{top}}(\hat{f}, \mathcal{V}_2^p) \ge \sup_{p\in\mathbb{N}} \lim_{n\to\infty} \frac{1}{2nq} \log \frac{2^{((u-1)/u)(n-1)}}{Ln^2 M_2^{nq/p}} \\ &= \sup_{p\in\mathbb{N}} \left\{ \frac{u-1}{u}\frac{\log 2}{2q} - \frac{\log M_2}{2p} \right\} = \frac{u-1}{u}\frac{\log 2}{2q}. \end{aligned} \tag{8}$$

But since u can be chosen arbitrarily large, we get that $h_{\mathrm{top}}(f) \ge \log 2/2q$.

The proof for the case $m > 1$ is again almost identical. From the proof of Proposition 4.1 for the families $\mathcal{P}_1$ or $\mathcal{P}_2$ we get, for every quasi-palindromic word ρ of order $2n$ with $\rho_1 = 0$ and $n \in \mathbb{N}$ that is a multiple of m, transverse loops $\Gamma_{\rho,l}$ that are linearly admissible of order $2lnq/m$. One then argues as above to get the lower bounds claimed in Proposition 4.2. □

5. Turaev's cobracket and orbit growth

Goldman's bracket [Gol86] and Turaev's cobracket [Tur91] define a Lie bialgebra structure on the free $\mathbb{Z}$-module of non-trivial free homotopy classes of loops $\widehat{\pi}(M)^* := \widehat{\pi}(M) \setminus \{[*]\}$ on a surface M.[4] Turaev's cobracket $v : \mathbb{Z}[\widehat{\pi}(M)^*] \to \mathbb{Z}[\widehat{\pi}(M)^*] \otimes \mathbb{Z}[\widehat{\pi}(M)^*]$ applied to a free homotopy class α gives some information about the free homotopy classes of the loops that split at intersection points of any representative Γ of α. We adopt the construction of v and refine it to keep additional information about how these loops relate to Γ in the fundamental group $\pi_1(M, \Gamma(0))$. Via this invariant, we then define an exponential growth rate $\mathrm{T}^\infty(\alpha)$ of α, and prove Theorem 1.3.

[4] Similar mappings were investigated earlier in [Tur78].

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Recall that we denote by $\mathcal{S}(\Gamma) = \{y \in M \mid y = \Gamma(t) = \Gamma(t'), t \neq t'\}$ the set of self-intersection points of a loop $\Gamma : S^1 \to M$. We say that a smooth loop Γ is in *general position* if it is an immersion, has only double intersection points which, moreover, are transverse, and $\Gamma(0) \notin \mathcal{S}(\Gamma)$. We first recall the construction of Turaev's cobracket. Let $\alpha \in \widehat{\pi}(M)^*$ be a free homotopy class of loops in M. Choose a smooth representative $\Gamma : [0,1] \to M$, $\Gamma(0) = \Gamma(1)$ of α that is in general position. For $y \in \mathcal{S}(\Gamma)$, let $\vec{v}_1$ and $\vec{v}_2$ be the two tangent vectors of Γ at y, labeled such that the pair $\vec{v}_1, \vec{v}_2$ is positively oriented (see Figure 7). Consider two loops u_1^y, u_2^y that start both at y in the direction $\vec{v}_1$ and $\vec{v}_2$, respectively, follow along Γ until their first return to y (see Figures 8 and 9). Set

$$v(\alpha) := \sum_{y \in \mathcal{S}(\Gamma)} [u_1^y]^* \otimes [u_2^y]^* - [u_2^y]^* \otimes [u_1^y]^* \in \mathbb{Z}[\widehat{\pi}(M)^*] \otimes \mathbb{Z}[\widehat{\pi}(M)^*],$$

where $[u]^*$ denotes the free homotopy class of u if u is non-contractible and $[u]^* := 0$ otherwise. One can show that the right-hand side is invariant under free homotopies and hence v is well defined (see [Tur91]). Extending v linearly to $\mathbb{Z}[\widehat{\pi}(M)^*]$, sending the trivial class to 0, defines Turaev's cobracket.

To refine this construction, we make the following definitions. Choose a basepoint $x_0 \in M$. For $g \in \pi_1(M, x_0)$, we say that two elements a and a' in $\pi_1(M, x_0)$ are *g-equivalent* if there is $k \in \mathbb{Z}$ with $g^k a g^{-k} = a'$. Denote by $\pi_1(M, x_0)^* = \pi_1(M, x_0) \setminus \{1\}$ the set of non-trivial elements of $\pi_1(M, x_0)$, and by $\pi_1(M, x_0)_g^*$ the set of g-equivalence classes in $\pi_1(M, x_0)^*$. Let $\mathbb{Z}[\pi_1(M, x_0)_g^*]$ be the free $\mathbb{Z}$-module over $\pi_1(M, x_0)_g^*$. We view the disjoint union $\mathcal{H} := \bigcup_{g \in \pi_1(M, x_0)} \mathbb{Z}[\pi_1(M, x_0)_g^*] \otimes \mathbb{Z}[\pi_1(M, x_0)_g^*]$ as a bundle $p : \mathcal{H} \to \pi_1(M, x_0)$ over $\pi_1(M, x_0)$ with fiber $p^{-1}(g) = \mathbb{Z}[\pi_1(M, x_0)_g^*] \otimes \mathbb{Z}[\pi_1(M, x_0)_g^*]$. We define a section $\mu : \pi_1(M, x_0) \to \mathcal{H}$ as follows. Let Γ with $\Gamma(0) = \Gamma(1) = x_0$ be any smooth loop in general position that represents g. For each $y \in \mathcal{S}(\Gamma)$ let v_1, v_2, u_1^y, u_2^y be as above, and additionally consider two paths q_1^y, q_2^y that both start at $\Gamma(0)$, follow Γ in positive direction, and end at y such that the following holds: q_1^y ends at y as soon as it reaches y the first time for which its tangent vector coincides with v_2, and q_2^y ends at y as soon as it reaches y the first time for which its tangent vector coincides with v_1 (see Figures 10 and 11). Let $a_i^y = \langle q_i^y u_i^y \overline{q_i^y} \rangle \in \pi_1(M, x_0)$, $i = 1, 2$, be the element represented by the loop $q_i^y u_i^y \overline{q_i^y}$, where $\overline{q_i^y}$ is the reverse path of q_i^y (see Figures 12 and 13). One checks that

$$[a_2^y]_g = [(a_1^y)^{-1} g]_g, \quad [a_1^y]_g = [g (a_2^y)^{-1}]_g. \tag{9}$$

Define

$$\mu(g) := \sum_{y \in \mathcal{S}(\Gamma)} [a_1^y]_g^* \otimes [a_2^y]_g^* - [a_2^y]_g^* \otimes [a_1^y]_g^* \in \mathcal{H}, \tag{10}$$

where $[a]_g^*$ denotes the g-equivalence class of a if $a \neq 1$ and $[1]_g^* := 0$.

We now show that μ is well defined, and respects conjugation in the following sense. Let $g \in \pi_1(M, x_0)$. Conjugation by $h \in \pi_1(M, x_0)$ defines a map $\phi_h^g : \pi_1(M, x_0)_g^* \to \pi_1(M, x_0)_{hgh^{-1}}^*$, $[a]_g \mapsto [hah^{-1}]_{hgh^{-1}}$. By extending linearly to $\mathbb{Z}[\pi_1(M, x_0)_g]^*$ and taking tensor products, we obtain mappings $\overline{\phi_h} = \bigcup_{g \in \pi_1(M, x_0)} \phi_h^g \otimes \phi_h^g : \mathcal{H} \to \mathcal{H}$.

LEMMA 5.1. *The map μ is well defined and $\mu(hgh^{-1}) = \overline{\phi_h}(\mu(g))$ for all $h, g \in \pi_1(M, x_0)$.*

Proof. In the space of C^∞ loops $C^\infty(S^1, M)$, we denote the set of loops in general position by Ω and denote by Ω^* the space of loops that satisfy the properties of being in general position except at one exceptional point $y = \Gamma(t_0)$, where exactly one of the following occurs: (1) the differential

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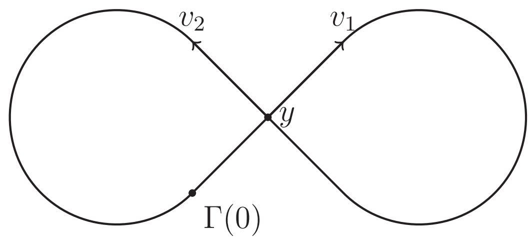

Figure 7. A self-intersecting loop Γ.

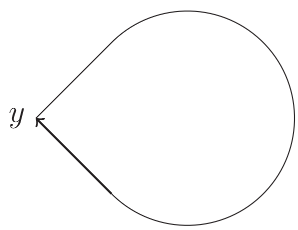

Figure 8. u_1^y.

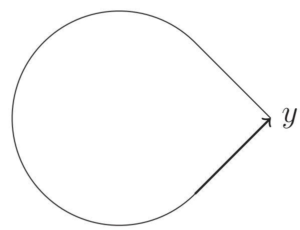

Figure 9. u_2^y.

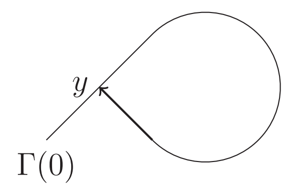

Figure 10. q_1^y.

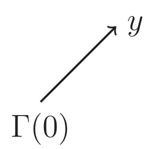

Figure 11. q_2^y.

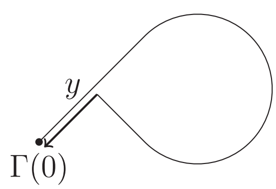

Figure 12. A loop representing a_1^y.

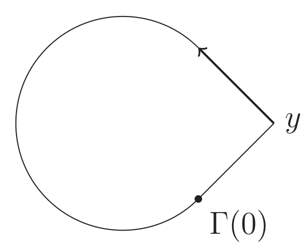

Figure 13. A loop representing a_2^y.

$(d/dt)\Gamma(t_0)$ vanishes and $(d^2/dt^2)\Gamma(t_0) \neq 0$; (2) there is a self-tangency at y; (3) there is a triple intersection at y; (4) $y = \Gamma(0) \in \mathcal{S}(\Gamma)$.

Let g and g' be two elements in $\pi_1(M, x_0)$, and Γ, Γ' be generic free loops in Ω with $\Gamma(0) = \Gamma'(0) = x_0$, representing g and g', respectively. A generic path of loops $\Gamma_s, s \in [0,1]$, from Γ to Γ' lies in $\Omega \cup \Omega^*$ and intersects Ω^* transversally for finitely many parameter values $A \subset (0,1)$ (see [Gol86, pp. 291–294]). According to which of the cases (1)–(4) occurs for some $s_* \in A$, we have that near s_* the homotopy Γ_s behaves like one of the following elementary moves: (1) birth–death of a monogon; (2) birth–death of a bigon; (3) jumping over an intersection; (4) jumping over $\Gamma_s(0)$. (For (1)–(3) see [Gol86], (4) is illustrated in Figure 14). Write A as a disjoint union $A = A_1 \cup A_2 \cup A_3 \cup A_4$ such that $s_* \in A_i$ if and only if situation (i), for $i \in \{1,2,3,4\}$, occurs near s_*.

Consider the s-family of paths $b_s(t)$, $s \in [0,1]$, with $b_0(t) \equiv x_0$ and $b_s(t) = \Gamma_{ts}(0)$, which all start at x_0 and end at $\Gamma_s(0)$. For each $s \in [0,1]$, and each $y_s \in \mathcal{S}(\Gamma_s)$ that is not an exceptional point, consider paths $q_1^{y_s}$ and $q_2^{y_s}$ as in the definition of μ for Γ_s, from $\Gamma_s(0)$ to y_s, and parametrized proportional to Γ_s. Similarly define $u_i^{y_s}$, $i = 1,2$. Consider for any $s \in [0,1] \setminus A$ the loops $p_i^{y_s} = b_s q_i^{y_s} u_i^{y_s} \overline{q_i^{y_s}}\, \overline{b_s}$ and the following element in $\mathbb{Z}[\pi_1(M,x_0)_g^*] \otimes \mathbb{Z}[\pi_1(M,x_0)_g^*]$:

$$\xi_s := \sum_{y_s \in \mathcal{S}(\Gamma_s)} [\langle p_1^{y_s} \rangle]_g \otimes [\langle p_2^{y_s} \rangle]_g - [\langle p_2^{y_s} \rangle]_g \otimes [\langle p_1^{y_s} \rangle]_g.$$

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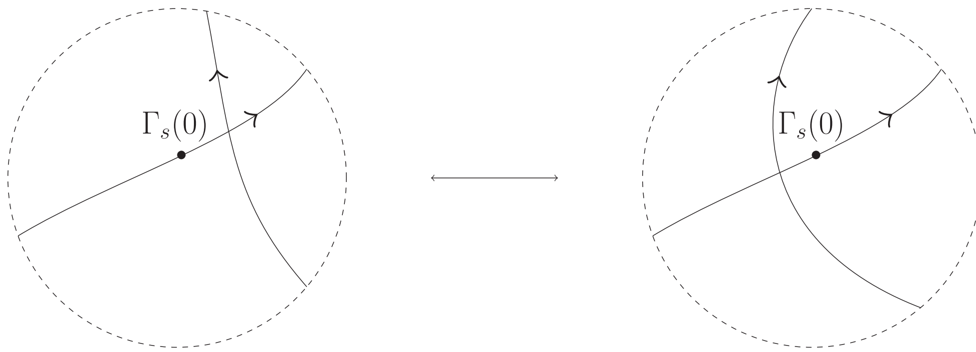

FIGURE 14. Jumping over $\Gamma_s(0)$.

We claim that ξ_s is constant along the homotopy Γ_s. To see this, one has to check how ξ_s is affected when s crosses some $s_* \in A$, since outside of A one can choose all the data continuously in s. The argument that ξ_s does not change near some $s_* \in A_1 \cup A_2 \cup A_3$ is similar to the argument of invariance of Turaev's cobracket [Tur91], this holds even without taking g-equivalence classes. We check invariance near some $s_* \in A_4$. For $s \in [s_* - \epsilon, s_* + \epsilon]$, $\epsilon > 0$ sufficiently small, let y_s be that intersection point of Γ_s such that $y_{s_*} = \Gamma_{s_*}(0)$ and y_s varies continuously in $s \in [s_* - \epsilon, s_* + \epsilon]$. Let $q_i^{y_s}, u_i^{y_s}$, $i = 1, 2$, be defined as above. Note that either for $i = 1$ or $i = 2$ we have that $\lim_{s \searrow s_*} q_i^{y_s} \equiv \Gamma_{s_*}(0)$ is constant and $\lim_{s \nearrow s_*} q_i^{y_s}(t) = \Gamma_{s_*}(t)$, or vice versa, interchanging $s \searrow s_*$ with $s \nearrow s_*$. Assume that $\lim_{s \searrow s_*} q_1^{y_s} \equiv \Gamma_{s_*}(0)$; the argument for the other cases is analogous. Clearly $\langle b_s q_2^{y_s} u_2^{y_s} \overline{q_2^{y_s}}\, \overline{b_s} \rangle \in \pi_1(M, x_0)$ is constant for all $s \in [s_* - \epsilon, s_* + \epsilon]$, since all paths vary continuously in s. Also, $\lim_{s \searrow s_*} \langle b_s q_1^{y_s} u_1^{y_s} \overline{q_1^{y_{s*}}}\, \overline{b_s} \rangle = \langle b_{s_*} u_1^{y_{s*}} \overline{b_{s_*}} \rangle =: f \in \pi_1(M, x_0)$, and $\lim_{s \nearrow s_*} \langle b_s q_1^{y_s} u_1^{y_s} \overline{q_1^{y_{s*}}}\, \overline{b_s} \rangle = \langle b_{s_*} \Gamma_{s_*} u_1^{y_{s*}} \overline{\Gamma_{s_*}}\, \overline{b_{s_*}} \rangle = gfg^{-1}$, since $g = \langle b_{s_*} \Gamma_{s_*} \overline{b_{s_*}} \rangle$. All other terms of ξ_s for $s \in [s_* - \epsilon, s_* + \epsilon]$ correspond to non-exceptional intersection points if ϵ is sufficiently small and clearly do not change along $s \in [s_* - \epsilon, s_* + \epsilon]$.

If in the above discussion we take $g' = g$ and a homotopy Γ_s such that $\Gamma_s(0)$ stays close to x_0, then the loop $b_1(t)$ is contractible, and using $\xi_1 = \xi_2$ we see that the definition of $\mu(g)$ does not depend on choosing the loop Γ or Γ'.

If we take $g' = hgh^{-1}$ for some $h \in \pi_1(M, x_0)$, then

$$
\begin{aligned}
\mu(g') &= \sum_{y \in \mathcal{S}(\Gamma_1)} [a_1^y]_{g'} \otimes [a_2^y]_{g'} - [a_2^y]_{g'} \otimes [a_1^y]_{g'} \\
&= \sum_{y \in \mathcal{S}(\Gamma_1)} [hh^{-1} a_1^y hh^{-1}]_{g'} \otimes [hh^{-1} a_2^y hh^{-1}]_{g'} - [hh^{-1} a_2^y hh^{-1}]_{g'} \otimes [hh^{-1} a_1^y hh^{-1}]_{g'} \\
&= \overline{\phi_h}(\xi_1) \\
&= \overline{\phi_h}(\mu(g)),
\end{aligned}
$$

where in the third equation we use that the loop b_1 represents h^{-1} as well as the definition of ξ_1, and in the last equation we use that ξ_s is constant along the homotopy Γ_s. $\square$

Note that $\widehat{\pi}(M)$ can be identified with the set of conjugacy classes of $\pi_1(M, x_0)$. By Lemma 5.1, μ induces a section $\widehat{\mu} : \widehat{\pi}(M) \to \widehat{\mathcal{H}}$, where $\widehat{\mathcal{H}}$ is obtained from $\mathcal{H}$ if we mod out by the action $\pi_1(M, x_0) \times \mathcal{H} \to \mathcal{H}$, $(h, \xi) \mapsto \overline{\phi_h}(\xi)$, and hence $\widehat{\mathcal{H}}$ is actually a bundle over $\widehat{\pi}(M)$. Turaev's cobracket can be recovered from $\widehat{\mu}$ by sending (tensors of) g-equivalence classes in $\widehat{\mathcal{H}}$ to (tensors of) conjugacy classes.

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We will now define the growth rate $\mathrm{T}^\infty(\alpha)$ of α in terms of $\widehat{\mu}(\alpha)$. Roughly speaking, up to a modification due to some parametrization issues, $\mathrm{T}^\infty(\alpha)$ is the exponential growth rate in k of the number of free homotopy classes of loops that can be obtained by following a loop representing α a total of k times, each time allowing a shortcut at some $\widehat{\mu}(\alpha)$-relevant self-intersection point and imposing that the turning at these points is always to the right or always to the left.

Let $S = \{s_1, \dots, s_m\} \subset \pi_1(M, x_0)$. Denote by $B(n, S)$ the set of elements in $\pi_1(M, x_0)$ that can be written as a product of up to n factors that are elements in S. Let $\widehat{B}(n, S)$ be the set of conjugacy classes of $B(n, S)$, $\widehat{N}(n, S) := \#\widehat{B}(n, S)$ the cardinality of $\widehat{B}(n, S)$, and $\Gamma(S) := \limsup_{n\to\infty} \log(\widehat{N}(n, S))/n < \infty$. Given $g \in \pi_1(M, x_0)$ and a set $\mathfrak{S}$ of g-equivalence classes of elements in $\pi_1(M, x_0)$, we define $\Gamma(\mathfrak{S}, g) = \inf\{\Gamma(S) \,|\, S \subset \pi_1(M, x_0),\ [S]_g = \mathfrak{S}\}$, where by $[S]_g$ we denote the set of g-equivalence classes of elements of $S \subset \pi_1(M, x_0)$.

For a set $\mathfrak{S} = \{[a_1]_g, \dots, [a_k]_g\}$ of g-equivalence classes and for $h \in \pi_1(M, x_0)$, denote $h \cdot \mathfrak{S} \cdot h^{-1} := \{[ha_1h^{-1}]_{hgh^{-1}}, \dots, [ha_kh^{-1}]_{hgh^{-1}}\}$. Any h induces in this way a bijection from the class of finite sets of g-equivalence classes to the class of finite sets of hgh^{-1}-equivalence classes. Moreover, one can check directly that

$$\Gamma(\mathfrak{S}, g) = \Gamma(h \cdot \mathfrak{S} \cdot h^{-1}, hgh^{-1}), \tag{11}$$

for all $g, h \in \pi_1(M, x_0)$.

Now let $\alpha \in \widehat{\pi}(M)$. Choose $g \in \pi_1(M, x_0)$ representing the class $[g] = \alpha$. The element $\mu(g) \in \mathcal{H}$ can be written as

$$\mu(g) = \sum_{\mathfrak{a},\mathfrak{b}\in\pi_1(M,x_0)^*_g} k_{\mathfrak{a},\mathfrak{b}}\big(\mathfrak{a} \otimes \mathfrak{b}\big), \tag{12}$$

where $k_{\mathfrak{a},\mathfrak{b}} \in \mathbb{Z}$, $\mathfrak{a}, \mathfrak{b}$ are g-equivalence classes, and $k_{\mathfrak{a},\mathfrak{b}} = -k_{\mathfrak{b},\mathfrak{a}}$. Let $\mathrm{Comp}(\mu(g))$ be the collection of terms $\mathfrak{a} \otimes \mathfrak{b}$ with $k_{\mathfrak{a},\mathfrak{b}} > 0$. By (9), $\mathfrak{a} \otimes \mathfrak{b} \neq \mathfrak{a}' \otimes \mathfrak{b}' \in \mathrm{Comp}(\mu(g))$ implies $\mathfrak{a} \neq \mathfrak{a}'$ and $\mathfrak{b} \neq \mathfrak{b}'$. Note that by Lemma 5.1, for any $h \in \pi_1(M, x_0)$, $\phi^g_h : \pi_1(M, x_0)_g \to \pi_1(M, x_0)_{hgh^{-1}}$ defines a bijection

$$\mathrm{Comp}(\mu(g)) \cong \mathrm{Comp}(\mu(hgh^{-1})). \tag{13}$$

Let $\mathrm{Comp}_+(\mu(g)) = \{\mathfrak{a} \,|\, \exists \mathfrak{b} : \mathfrak{a} \otimes \mathfrak{b} \in \mathrm{Comp}(\mu(g))\}$, and $\mathrm{Comp}_-(\mu(g)) = \{\mathfrak{b} \,|\, \exists \mathfrak{a} : \mathfrak{a} \otimes \mathfrak{b} \in \mathrm{Comp}(\mu(g))\}$. Note that $\#\mathrm{Comp}(\mu(g)) = \#\mathrm{Comp}_+(\mu(g)) = \#\mathrm{Comp}_-(\mu(g))$. Consider

$$\Gamma^g := \min_{\pm} \min_{\mathfrak{S}} \big\{\Gamma(\mathfrak{S} \cup [g]_g, g) \,\big|\, \mathfrak{S} \subset \mathrm{Comp}_\pm(\mu(g)),\ \#\mathfrak{S} = \big\lceil \tfrac{1}{2}\#\mathrm{Comp}(\mu(g))\big\rceil\big\}.$$

By (11), $\Gamma^g = \Gamma^{hgh^{-1}}$, for any $h \in \pi_1(M, x_0)$. Hence, this expression is independent of the choice of g with $[g] = \alpha$ and we denote it by

$$\mathrm{T}^\infty(\alpha) := \Gamma^g.$$

We now give a proof of Theorem 1.3.

Proof of Theorem 1.3. We may assume $T^\infty(\alpha) \neq 0$, and in particular $\mathrm{si}_M(\alpha) \neq 0$. We consider a maximal identity isotopy $\widehat{I}$ of f, and a foliation $\mathcal{F}$ transverse to $\widehat{I}$, which is possible by Theorem 2.2. Let Γ be a q-admissible $\mathcal{F}$-transverse path associated to x. Since $[\Gamma]_{\widehat{\pi}(M)} = \alpha$ is assumed to be primitive, $[\Gamma]_{\widehat{\pi}(\mathrm{dom}(\widehat{I}))}$ is also primitive. Let $x_0 = \Gamma(0)$ and let $g = \langle\Gamma\rangle \in \pi_1(M, x_0)$ be the element that is represented by Γ. Let γ be the natural lift of Γ.

Claim 5. *After possibly modifying Γ in its equivalence class, there exist, for any $\mathfrak{a} \otimes \mathfrak{b} \in \mathrm{Comp}(\mu(g))$, an intersection point $y = \Gamma(t) = \Gamma(t')$ with $t < t'$, and $a \in \pi_1(M, x_0)$ with $[a]_g = \mathfrak{a}$, such that $\gamma|_{[0,t']}\gamma|_{[t,t']}\overline{\gamma|_{[0,t']}}$ is a representative of a and $\gamma|_{[0,t]}\gamma|_{[t',1]}\gamma|_{[0,t]}\overline{\gamma|_{[0,t]}} = \gamma|_{[0,t]}\gamma|_{[t',1]}$ is a representative of b, or vice versa. Furthermore, one can choose t and t' above such that any two lifts $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ of Γ that intersect in $\widetilde{\gamma}(t) = \widetilde{\gamma}'(t')$ intersect $\widetilde{\mathcal{F}}$-transversally.*

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We will prove the claim below. It follows from Claim 5 that for $m = \lceil \frac{1}{2}\#\mathrm{Comp}(\mu(g)) \rceil$ there is $\mathfrak{S} = \{\mathfrak{a}_1, \dots, \mathfrak{a}_m\} \subset \mathrm{Comp}_+(\mu(g))$ or $\mathfrak{S} = \{\mathfrak{a}_1, \dots, \mathfrak{a}_m\} \subset \mathrm{Comp}_-(\mu(g))$ such that, with the above choices of corresponding parameters $t_1, \dots, t_m$, $t'_1, \dots, t'_m \in [0, 1)$, $t_i < t'_i$, and a set $S = \{a_1, \dots, a_m\} \subset \pi_1(M, x_0)$ with $[S]_g = \mathfrak{S}$, we have that $\gamma_i := \gamma|_{[0,t_i]}\gamma|_{[t'_i,1]}$ represents the element a_i, and any two lifts $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ of Γ that intersect in $\widetilde{\gamma}(t_i) = \widetilde{\gamma}'(t'_i)$ intersect $\mathcal{F}$-transversally, for all $i = 1, \dots, m$.

Set $\gamma_0 := \gamma|_{[0,1]}$. With notation similar to that in the proof of Proposition 4.1 for $\rho = (\rho_1, \dots, \rho_n)$ with $\rho_i \in \{0, 1, \dots, k\}$, consider the path $\gamma_\rho := \gamma_{\rho_1}\gamma_{\rho_2}\cdots\gamma_{\rho_n}$. The path defines a transverse loop Γ_ρ. As in the proof of Claim 1 of the proof of Proposition 4.1, one shows that for any $\rho = (\rho_1, \dots, \rho_n)$ with $\rho_i \in \{0, \dots, k\}$, $i = 1, \dots, n$, Γ_ρ is linearly admissible of order nq: the fact that, for any $p \in \mathbb{N}$, $\widehat{\gamma} := \gamma_{[0,1]}\gamma_\rho^p$ is admissible of order pnq follows from Proposition 2.5, since by assumption $\mathfrak{S} \subset \mathrm{Comp}_+(\mu(g))$ or $\mathfrak{S} \subset \mathrm{Comp}_-(\mu(g))$, and so all relevant $\mathcal{F}$-transverse self-intersections have the same sign.

Consider the set of free homotopy classes that are defined by Γ_ρ for all $\rho = (\rho_1, \dots, \rho_n)$ with $\rho_1 = \rho_2 = \rho_3 = 0$. With the notation above, this set coincides with $\widehat{N}(n-3, S)$. It follows from Lemma 3.8(1) that Γ_ρ has an $\mathcal{F}$-transverse self-intersection and hence, by Proposition 2.6, there is for each ρ of the above form a periodic point $z \in \mathrm{dom}(\mathcal{F})$ of period nq such that Γ_ρ is associated to z, so there are at least $\widehat{N}(n-3, S)$ periodic points of period nq that are of pairwise different free homotopy classes. Hence,

$$\mathrm{H}^\infty(f) = \limsup_{n\to\infty} \frac{\log(N_{\mathrm{h}}(f, nq))}{nq} \geq \limsup_{n\to\infty} \frac{\log(\widehat{N}(n-3, S))}{nq} \geq \frac{1}{q}\Gamma^g = \frac{1}{q}\mathrm{T}^\infty(\alpha). \qquad \square$$

Proof of Claim 5. Note that by the definition of $\mu(g)$, the first part of the claim is immediate if the loop Γ was smooth. We explain how to construct a smooth, not necessarily $\mathcal{F}$-transverse, loop Θ with natural lift θ such that, after modifying Γ in its equivalence class,

(1) $\Theta(0) = \Gamma(0)$,
(2) Θ is homotopic to Γ in $\mathrm{dom}(\mathcal{F})$ relative to $\Gamma(0)$, and
(3) for any intersection point $\Theta(s) = \Theta(s')$ of Θ, the loops $\theta|_{[0,s']}\theta|_{[s,s']}\overline{\theta|_{[0,s']}}$ (respectively, $\theta|_{[0,s]}\theta|_{[s',1]}$) are homotopic relative to $\Gamma(0)$ to the loops $\gamma|_{[0,t']}\gamma|_{[t,t']}\overline{\gamma|_{[0,t']}}$ (respectively, $\gamma|_{[0,t]}\gamma|_{[t',1]}$) for some intersection point $\Gamma(t) = \Gamma(t')$ of Γ.

It is easy to see that we can assume that Γ has, after possibly modifying it in its equivalence class, only finitely many self-intersection points $y_1 = \Gamma(t_1) = \Gamma(t'_1), \dots, y_n = \Gamma(t_n) = \Gamma(t'_n)$, and that all its self-intersection points are double intersections points and do not coincide with $\Gamma(0)$. We still write $x_0 = \Gamma(0)$. Equip $\mathrm{dom}(\mathcal{F})$ with any Riemannian metric. Choose pairwise disjoint open balls $B_i \subset \mathrm{dom}(\mathcal{F})$, $i = 0, 1, \dots, n$, such that

- $x_0 \in B_0$ and $y_i \in B_i$, for all $i = 1, \dots, n$, and
- all B_i are strongly convex, which means that any two points in $\overline{B_i}$ are connected by a unique geodesic segment in $\overline{B}_i$ with its interior contained in B_i.

Let $J_1, \dots, J_k$, $k \in \mathbb{N}$, be the connected components of $[0, 1] \setminus \{t_1, \dots, t_n, t'_1, \dots, t'_n\}$. Using continuous charts in which the foliation is vertical, we can easily choose an open neighborhood U of $\{x_0\} \cup \{y_1, \dots, y_n\}$ with $\overline{U} \subset \bigcup_{i=0}^n B_i$ and such that the sets $\widehat{J}_i$, $i = 1, \dots, k$, defined by $\widehat{J}_i := \Gamma^{-1}(\mathrm{dom}(\mathcal{F}) \setminus U) \cap J_i$, are precisely the connected components of $\Gamma^{-1}(\mathrm{dom}(\mathcal{F}) \setminus U)$. Choose, for $i = 1, \dots, n$, sufficiently small open neighborhoods V_i of $\Gamma(\widehat{J}_i) \subset \mathrm{dom}(\mathcal{F})$ such that $V_i \cap (V_j \cup \Gamma(J_j)) = \emptyset$ if $i \neq j$. Choose a loop Λ in $\mathrm{dom}(\mathcal{F})$ that agrees with Γ on $[0, 1] \setminus \bigcup_{i=1}^k \widehat{J}_i$, and such that

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- each $\Gamma|_{\widehat{J}_i}$ is replaced by a piecewise geodesic path $\Lambda|_{\widehat{J}_i}$ in V_i that is homotopic to $\Gamma|_{\widehat{J}_i}$ relative to its endpoints, and
- no path $\Lambda|_{\widehat{J}_i}$ has self-intersection points.

That we can find a loop Λ satisfying the first property is clear. It is then not hard to see that we can modify Λ in the neighborhoods V_i such that both properties are satisfied. Note that $\Lambda|_{J_i}$ still might have self-intersection points inside U.

For each $i = 0, \dots, n$, let $(I_i^j)_{j\in\mathcal{J}_i}$ be the (finitely many) connected components of $\Lambda^{-1}(\overline{B_i})$, except for the two components of the form $[0, a]$ and $[b, 1]$, for which we consider their union $[b, 1] \cup [0, a]$ as an element in $\{(I_0^j)_{j\in\mathcal{J}_0}\}$. Choose a smooth loop $\Theta : [0, 1] \to \mathrm{dom}(\mathcal{F})$ where

- $\Theta(0) \in B_0$,
- each $\Lambda|_{I_i^j}$ is replaced by the (suitably reparametrized) geodesic segment in $\overline{B}_i$ with the same endpoints (in ∂B_i) as $\Lambda|_{I_i^j}$,
- there are no self-intersection points of Θ in $\mathrm{dom}(\mathcal{F}) \setminus \bigcup_{i=1}^n B_i$, and
- the subpaths of Θ and Γ defined on the connected components of $\Gamma^{-1}(\mathrm{dom}(\mathcal{F}) \setminus \bigcup_{i=0}^n \overline{B_i}) \subset [0, 1]$ are homotopic relative to their endpoints.

Note that there is at most one self-intersection point of Θ in each B_i, $i = 1, \dots, n$. In the construction we can choose B_0 arbitrarily small and hence, by again modifying Γ in its $\mathcal{F}$-equivalence class in a neighborhood of x_0 and by reparametrizing Θ a little near $\Theta(0)$, we can achieve that $\Gamma(0) = \Theta(0)$. It follows directly from the construction that for the smooth loop Θ properties (1)–(3) above are satisfied. This implies the first part of the claim.

By Lemma 3.8, any two lifts $\widetilde{\gamma}$ and $\widetilde{\gamma}' = S\widetilde{\gamma}$, where $S \neq T^k$ and T is the shift of $\widetilde{\gamma}$, can only be $\widetilde{\mathcal{F}}$-equivalent along a finite interval. In particular, if in addition they do not intersect $\widetilde{\mathcal{F}}$-transversally, then $\widetilde{\gamma}|_{(B,\infty)}$, and $\widetilde{\gamma}|_{(-\infty,-B)}$ will be on the same side of $\widetilde{\gamma}'$ for B sufficiently large, say on the left of $\widetilde{\gamma}'$. It is now sufficient to show that for any such lifts and $a, b, a', b' \in \mathbb{R}$, $a < b$, $a' < b'$, such that $\widetilde{\gamma}(a) = \widetilde{\gamma}'(a')$, $\widetilde{\gamma}(b) = \widetilde{\gamma}'(b')$, and $\widetilde{\gamma}|_{(a,b)}$ lies on the right of $\widetilde{\gamma}'$, the contributions to $\mu(\alpha)$ at the intersection points $y = \Gamma(a)$ and $y' = \Gamma(b)$ cancel each other out. So take such a, b, a', b' and $\widetilde{\gamma}, \widetilde{\gamma}'$. We may, by interchanging the lifts and translating if necessary, assume that $0 < a < a' < 1$. We note that $b < b' < b + 1$. Indeed, if $b \geq b'$, then $\widetilde{\gamma}|_{[a,b]}$ is equivalent to a subpath of $S\widetilde{\gamma}|_{[a,b]}$, and by iterating this argument and applying the deck transformations S^{-k}, $k \in \mathbb{N}$, we see that in fact $S^{-k}\widetilde{\gamma}|_{[a,b]}$ is equivalent to a subpath of $\widetilde{\gamma}_{[a,b]}$ for all $k \in \mathbb{N}$. On the other hand, the pairwise different lifts $S^{-k}\widetilde{\gamma}$ all intersect $\widetilde{\gamma}$, and since S is not a multiple of T and Γ has only finitely many double self-intersection points, there is $k \in \mathbb{N}$ such that a subpath of $S^{-k}\widetilde{\gamma}$ is equivalent to $\widetilde{\gamma}|_J$ for an interval $J \supset [a, b]$ of length at most 6. This contradicts Lemma 3.8(3). If $b' \geq b + 1$, then the path $T\widetilde{\gamma}|_{[a,b]} = \widetilde{\gamma}|_{[a+1,b+1]}$ is a subpath of $\widetilde{\gamma}|_{[a',b']}$ and hence equivalent to a subpath of $S^{-1}\widetilde{\gamma}|_{[a,b]}$. Therefore the lifts $(ST)^k\widetilde{\gamma}|_{[a,b]}$, $k \in \mathbb{N}$, are equivalent to a subpath of $\widetilde{\gamma}|_{[a,b]}$. This leads to a contradiction as above.

Let $k, k' \in \mathbb{Z}$ with $k < b < k + 1$ and $k' < b' < k' + 1$. We have that $\widetilde{\gamma}|_{[0,a]}\widetilde{\gamma}'|_{[a',1]}\widetilde{\gamma}'|_{[1,k'+1]}$ is homotopic relative to its endpoints to $\widetilde{\gamma}|_{[0,k]}\widetilde{\gamma}|_{[k,b]}\widetilde{\gamma}'|_{[b',k'+1]}$. By the construction of $\mu(g)$ and by (9) it is enough to show that $[f]_g = [f']_g$, where $f \in \pi_1(M, x_0)$ is represented by the loop in M that we obtain by projecting $\widetilde{\gamma}|_{[0,a]}\widetilde{\gamma}'|_{[a',1]}$ to $\mathrm{dom}(\widehat{I}) \subset M$, and f' is, if (i) $b < b' < k + 1 = k' + 1$, represented by the loop in M that we obtain by projecting $\widetilde{\gamma}|_{[k,b]}\widetilde{\gamma}'|_{[b',k'+1]}$ to $\mathrm{dom}(\widehat{I}) \subset M$, and (ii) $b < k' = k + 1 < b' < b + 1$, represented by the loop in M that we obtain by projecting

$\widetilde{\gamma}|_{[k,b]}\overline{\widetilde{\gamma}'|_{[k',b']}}$ to $\mathrm{dom}(\widehat{I}) \subset M$. If (i) we get $fg^{k'} = g^k f$, and if (ii) we get $fg^{k'} = g^k f' g$. Hence, in both cases $[f]_g = [f']_g$. $\square$

6. Floer theory and persistence modules prerequisites

6.1 A bit of Floer homology

Let (M^{2n}, ω) be a symplectic manifold, and let $H : S^1 \times M \to \mathbb{R}$ be a smooth function, called a Hamiltonian function. The function H induces a time-dependent vector field $X_H : S^1 \times M \to TM$ on M that satisfies

$$\forall t \in S^1 : \omega(\cdot, X_{H_t}) = dH_t(\cdot),$$

where $X_{H_t} : M \to TM$ is the vector field at time $t \in S^1$ and $H_t : M \to \mathbb{R}$ is $H_t(\cdot) = H(t, \cdot)$. The flow induced by H, or by X_H, is the family of maps $\varphi_H^t : [0,1] \times M \to M$ satisfying

$$\forall t \in [0,1], x \in M : \frac{d}{dt}\varphi_H^t(x) = X_H(t,x).$$

The time-1 map induced by H is the map $\varphi_H : M \to M$ defined by $\varphi_H = \varphi_H^1$. We say also that H generates φ_H. The set of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\omega) := \{\varphi_H \,|\, H : S^1 \times M \to \mathbb{R} \text{ smooth}\}$ is given a group structure by composition.

In [Hof90], Hofer defined a remarkable metric $d_{\mathrm{Hofer}} : \mathrm{Ham}(M,\omega) \times \mathrm{Ham}(M,\omega) \to \mathbb{R}$. This metric is induced by the Hofer norm $||\cdot||_{\mathrm{Hofer}} : \mathrm{Ham}(M,\omega) \to \mathbb{R}$, which is defined as follows:

$$||\psi||_{\mathrm{Hofer}} = \inf\left\{ \int_0^1 \max_M H_t - \min_M H_t \; dt \,\middle|\, H \text{ smooth}, \, \varphi_H = \psi \right\},$$

where the infimum is taken over all Hamiltonians H which have ψ as the time-1 map of their Hamiltonian flow.

Recall the setting of Morse theory: a manifold X and a Morse function $f : X \to \mathbb{R}$ are given. Morse theory reveals a connection between the homology of sublevel sets $\{x \in X \,|\, f(x) < t\}_{t\in\mathbb{R}}$ and critical points of f. Hamiltonian Floer theory is an analogous theory, set in an infinite-dimensional manifold of loops on a manifold, which is equipped with the action functional (see [Sal99]).

Let (M^{2n}, ω) be a closed, symplectically aspherical symplectic manifold and let $\alpha \in \widehat{\pi}(M)$ be a free homotopy class of loops in M. Denote by $\mathcal{L}_\alpha(M) = \{x : S^1 \to M \text{ smooth } \,|\, [x]_{\widehat{\pi}(M)} = \alpha\}$ the set of all smooth loops in M which represent the class α. Assume that α is symplectically atoroidal; that is, that for any loop ρ in $\mathcal{L}_\alpha(M)$, which is to be thought of as a function $\rho : \mathbb{T}^2 \to M$,

$$\int_{\mathbb{T}^2} \rho^*\omega = \int_{\mathbb{T}^2} \rho^* c_1 = 0,$$

where c_1 is the first Chern class of (M,ω) and $\mathbb{T}^2$ is the 2-torus.

Note for future reference that if M is a surface of genus $g \geq 2$ this condition holds trivially, since $[\mathbb{T}^2, M] = 0$. Fix a reference loop $\eta_\alpha \in \mathcal{L}_\alpha(M)$ and let $x \in \mathcal{L}_\alpha(M)$. The above condition means that the quantity $\int_{\bar{x}} \omega := \int_{S^1\times[0,1]} \bar{x}^*\omega$ is well defined and independent of $\bar{x}$, for a map $\bar{x} : S^1 \times [0,1] \to M$ with $\bar{x}|_{S^1\times\{0\}} = \eta_\alpha$ and $\bar{x}|_{S^1\times\{1\}} = x$.

Let $H : S^1 \times M \to \mathbb{R}$ be a Hamiltonian function on M. The action functional associated to H, $\mathcal{A}_H : \mathcal{L}_\alpha(M) \to \mathbb{R}$, is defined as follows:

$$\mathcal{A}_H(x) = \int_0^1 H(t, x(t))\, dt - \int_{\bar{x}} \omega.$$

A fixed point z of φ_H shall be called *non-degenerate* if the differential $(\varphi_H)_* : T_zM \to T_zM$ does not have 1 as an eigenvalue. A Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ and its time-1 map φ_H shall be called *non-degenerate* if all fixed points z of φ_H are non-degenerate. A non-degenerate Hamiltonian diffeomorphism ϕ has isolated fixed points. The property of non-degeneracy corresponds, in the analogy between Morse and Floer theories, to the function $f : X \to \mathbb{R}$ being a Morse function.

Let $x \in \mathcal{L}_\alpha(M)$. The tangent space to $\mathcal{L}_\alpha(M)$ at the point x, $T_x\mathcal{L}_\alpha(M)$, is identified with the space of vector fields ξ along x, that is, with the space of maps $\xi : S^1 \to TM$ which are compositions of x with a section of $TM \to M$. One shows that the differential of $\mathcal{A}_H$ is

$$(d\mathcal{A}_H)_x(\xi) = \int_0^1 \omega(\xi, X_H - \dot{x}(t))\, dt.$$

This formula for the differential of the action implies the following characterization of critical points of the action functional.

Proposition 6.1 (The least action principle). *The critical points of $\mathcal{A}_H$ are exactly the 1-periodic orbits of the flow generated by the Hamiltonian H.*

Endow $\mathcal{L}_\alpha(M)$ with an auxiliary Riemannian metric as follows. Choose a loop of ω-compatible almost complex structures $J(t)$, that is, choose smoothly for each $t \in S^1$ a map $J(t) : TM \to TM$ with $J(t)^2 = -\operatorname{id}_{TM}$, such that $\omega(\cdot, J\cdot)$ defines a Riemannian metric on M. The inner product at the point $x \in \mathcal{L}_\alpha(M)$, $\langle \cdot, \cdot \rangle_x : T_x\mathcal{L}_\alpha(M) \times T_x\mathcal{L}_\alpha(M) \to \mathbb{R}$, is defined to be

$$\langle \xi, \zeta \rangle_x = \int_0^1 \omega(\xi(t), J(t)\zeta(t))\, dt.$$

Denote

$$P(M,H)_\alpha = \{x \in \mathcal{L}_\alpha(M) \,|\, x \text{ is a 1-periodic orbit of } \varphi_H^t\}.$$

We wish to grade $P(M,H)_\alpha$; this is done using the Conley–Zehnder index as follows (for a definition of the Conley–Zehnder index, see [Gut12]). First, if Φ is a path of symplectic matrices starting with the identity matrix and such that $\Phi(1)$ does not have 1 as an eigenvalue, denote by $\mu_{\mathrm{CZ}}(\Phi)$ the Conley–Zehnder index of the path Φ. Fix a trivialization $\eta_\alpha^*TM \simeq S^1 \times (\mathbb{R}^{2n}, \omega_0)$ of the symplectic vector bundle η_α^*TM. Let $x \in P(M,H)_\alpha$. For any annulus $w : [0,1] \times S^1 \to M$ connecting η_α to x, w defines a trivialization $x^*TM \simeq S^1 \times (\mathbb{R}^{2n}, \omega_0)$. Note that the symplectic atoroidality condition implies that this trivialization does not depend on w. Using the trivialization of x^*TM, the differential $d(\varphi_H^t)_{x(0)}$ for $t \in [0,1]$ is a symplectic map of $(\mathbb{R}^{2n}, \omega_0)$. Denote $\operatorname{Ind}(x) = n - \mu_{\mathrm{CZ}}(\{t \mapsto d(\varphi_H^t)_{x(0)}\})$. Note that by non-degeneracy of H, $d(\varphi_H^1)_{x(0)}$ does indeed not have 1 as an eigenvalue.

Let $x, y \in P(M,H)_\alpha$ with $\operatorname{Ind}(x) = \operatorname{Ind}(y) + 1$; denote by $\tilde{\mathcal{M}}(x,y)$ the space of solutions $u(s,t) : \mathbb{R} \times S^1 \to M$ to the Floer equation,

$$\bar{\partial}_{H,J}(u) = \partial_s u + J(t)\big(\partial_t u - X_{H_t}\big) = 0,$$

which have boundary conditions $\lim_{s\to-\infty} u(s,t) = x(t)$, $\lim_{s\to\infty} u(s,t) = y(t)$. Note that the loops $\{u(s,\cdot) \,|\, s \in \mathbb{R}\}$ for $u \in \tilde{\mathcal{M}}(x,y)$ are gradient descent trajectories on the space $\mathcal{L}_\alpha(M)$ with respect to the auxiliary metric defined above and the action functional $\mathcal{A}_H$. The space $\tilde{\mathcal{M}}(x,y)$ has an obvious $\mathbb{R}$-action, and one can show that for a generic choice of J, $\tilde{\mathcal{M}}(x,y)/\mathbb{R}$ is a compact zero-dimensional manifold (i.e. a finite set of points). Denote in that case $n(x,y) = \#(\tilde{\mathcal{M}}/\mathbb{R}) \bmod 2$.

Consider the Floer complex over $\mathbb{Z}_2$, filtered by action less than $r \in \mathbb{R}$:

$$CF_k^r(M,H)_\alpha = \mathrm{Span}_{\mathbb{Z}_2}\{x \in P(M,H)_\alpha \mid \mathrm{Ind}(x) = k,\, \mathcal{A}_H(x) < r\},$$

and consider the linear map $\partial_k : CF_k^r(M,H)_\alpha \to CF_{k-1}^r(M,H)_\alpha$ which linearly extends the following map on $\{x \in P(M,H)_\alpha \mid \mathrm{Ind}(x) = k, \mathcal{A}_H(x) < r\}$:

$$x \mapsto \sum_{\substack{y \in P(M,H)_\alpha \\ \mathrm{Ind}(y)=k-1}} n(x,y)y.$$

It can be shown that $\partial : CF_*^r(M,H)_\alpha \to CF_*^r(M,H)_\alpha$ is a differential, that is, that $\partial^2 = 0$. Here, $CF_*^r(M,H)_\alpha = \bigoplus_k CF_k^r(M,H)_\alpha$ denotes the Floer complex in all degrees. Since $(CF_*^r(M,H)_\alpha, \partial)$ defines a chain complex, it has a well-defined homology, the filtered Hamiltonian Floer homology of action below r in class α, which is denoted by $HF_*^r(M,H)_\alpha$. One shows that this homology does not depend on the choice of almost complex structure $J(t)$ and, in addition, since M is symplectically aspherical, that it does not depend on the Hamiltonian H, but only on the time-1 map of its flow. Thus for $\phi \in \mathrm{Ham}(M,\omega)$, denote its Hamiltonian Floer homology in free homotopy class α, filtered with action less than r and over $\mathbb{Z}_2$, by $HF_*^r(\phi)_\alpha$.

6.2 Persistence modules and barcodes

We begin this subsection by defining a persistence module. For more background on the definitions and theorems appearing in this subsection, see [PRSZ20].

DEFINITION 6.2. Let $\mathbb{F}$ be a field. A *persistence module* is a pair (V,π) where $V = (V_t)_{t\in\mathbb{R}}$ is a family of finite-dimensional $\mathbb{F}$-vector spaces and $\pi = (\pi_{s,r})_{s,r\in\mathbb{R},s\le r}$ is a family of linear maps $\pi_{s,r} : V_s \to V_r$ such that the following conditions hold.

- For all $r \le s \le t$, $\pi_{s,t} \circ \pi_{r,s} = \pi_{r,t}$.
- There exists a finite set $\mathrm{Spec}(V,\pi) \subset \mathbb{R}$, such that for any $t \notin \mathrm{Spec}(V,\pi)$ there exists a neighborhood U of t such that for all $r, s \in U$, with $r \le s$, $\pi_{r,s}$ is an isomorphism.
- For any $t \in \mathbb{R}$ there exists ϵ such that for all $t - \epsilon < s \le t$, $\pi_{s,t}$ is an isomorphism.
- There exists $s_- \in \mathbb{R}$ such that, for all $s < s_-$, $V_s = 0$.

We present a few examples of persistence modules.

- Let $\mathbb{F}$ be a field, and let $I \subset \mathbb{R}$ be an interval of the form $(a,b]$ with $a \in \mathbb{R}$, $a < b \in \mathbb{R} \cup \{\infty\}$, where the interval $(a,\infty] \subset \mathbb{R}$ is to be interpreted as equal to $(a,\infty) \subset \mathbb{R}$. The persistence module $\mathbb{F}(I)$ consists of the following data: the vector spaces $\mathbb{F}(I)_t$, for $t \in \mathbb{R}$, are

$$\mathbb{F}(I)_t = \begin{cases} \mathbb{F} & t \in I \\ 0 & t \notin I, \end{cases}$$

  and the linear maps $\pi_{s,r} : \mathbb{F}(I)_s \to \mathbb{F}(I)_r$ are

$$\pi_{s,r} = \begin{cases} \mathrm{id}_{\mathbb{F}} & s, r \in I \\ 0 & \text{otherwise.} \end{cases}$$

- Let $\mathbb{F}$ be a field, let X be a closed manifold and let $f : X \to \mathbb{R}$ be a Morse function. The Morse homology $H_*(\{x \in X | f(x) < t\}; \mathbb{F})$ for $t \in \mathbb{R}$ induces an $\mathbb{F}$-persistence module: the vector spaces are $V_t = H_*(\{x \in X \mid f(x) < t\}; \mathbb{F})$, and the linear maps $\pi_{r,s} : V_r \to V_s$ are induced by the inclusion maps $i_{r,s} : \{x \in X \mid f(x) < r\} \hookrightarrow \{x \in X \mid f(x) < s\}$.

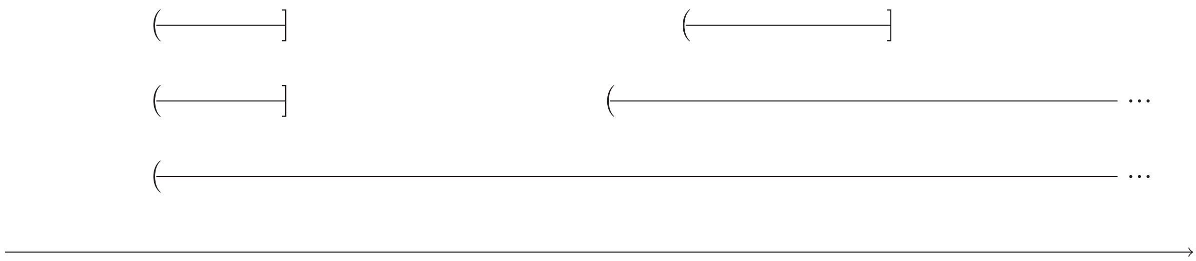

Figure 15. An example barcode.

- Let (M,ω) be a symplectic manifold, let $\alpha \in \widehat{\pi}(M)$, and let $\phi \in \mathrm{Ham}(M,\omega)$ be a non-degenerate Hamiltonian diffeomorphism. Denote by $HF_*^{\bullet}(\phi)_\alpha = (HF_*^r(\phi)_\alpha)_{r\in\mathbb{R}}$ the persistence module whose vector spaces are the filtered Floer homology vector spaces in class α, $HF_*^r(\phi)_\alpha$ for $r \in \mathbb{R}$, and whose linear maps $\pi_{r,s} : HF_*^r(\phi)_\alpha \to HF_*^s(\phi)_\alpha$ are induced by the inclusion maps $CF_*^r(M,H)_\alpha \to CF_*^s(M,H)_\alpha$, where H is a Hamiltonian that generates ϕ.

The direct sum of two persistence modules is defined as follows.

Definition 6.3. Let (V,π) and (W,τ) be two persistence modules over the same field. Their direct sum, denoted by $V \oplus W$, is the persistence module whose vector spaces $(V\oplus W)_t$ are $(V\oplus W)_t = V_t \oplus W_t$ and whose linear maps $(\pi\oplus\tau)_{r,s} : (V\oplus W)_r \to (V\oplus W)_s$ are $(\pi\oplus\tau)_{r,s} = \pi_{r,s} \oplus \tau_{r,s}$.

Definition 6.4. A *barcode* is a finite multiset of intervals. Explicitly, a barcode is a finite set of pairs of intervals of $\mathbb{R}$ and their multiplicities, $\{(I_i,m_i)\}_{i=1}^N$ for some $N\in\mathbb{N}$, where $I_i \subset \mathbb{R}$ is an interval of the form $(a,b]$ for some $a\in\mathbb{R}, a<b\in\mathbb{R}\cup\{\infty\}$, and $m_i\in\mathbb{N}$ is the multiplicity of I_i. The intervals which make up a barcode are called its *bars* (see Figure 15).

Persistence modules and barcodes are matched in a one-to-one manner, as stated in Theorem 6.5.

Theorem 6.5 (Normal form theorem). *Let (V,π) be a persistence module over a field $\mathbb{F}$. Then there exists a unique barcode $\mathcal{B}(V) = \{(I_i,m_i)\}_{i=1}^N$ such that*

$$V = \bigoplus_{i=1}^{N} \mathbb{F}(I_i)^{m_i},$$

where equality is to be understood as persistence module isomorphism, and uniqueness of the barcode is up to permutation of the order in which its bars appear.

We next recall a metric on the space of all barcodes, called the bottleneck distance, and denoted by d_{bot}.

Definition 6.6. Let $I=(a,b]$ be an interval with $a\in\mathbb{R}, a<b\in\mathbb{R}\cup\{\infty\}$, and let $\delta>0$. Denote by $I^{-\delta}$ the interval $(a-\delta, b+\delta]$. Let $\mathcal{B}$ be a barcode. Denote $\mathcal{B}_\delta = \{(I,m)\in\mathcal{B} \,|\, I = (a,b] \text{ with } b-a>\delta\}$, that is, $\mathcal{B}_\delta$ is the set of bars of $\mathcal{B}$ which have length greater than δ.

Let X, Y be multisets. A matching $\mu : X \to Y$ is a bijection $\mu : X' \to Y'$, where *coim* $\mu = X' \subseteq X$, *im* $\mu = Y' \subset Y$ are submultisets.

Let $\mathcal{A},\mathcal{B}$ be barcodes. A δ-*matching* between $\mathcal{A},\mathcal{B}$ is a matching $\mu : \mathcal{A}\to\mathcal{B}$ such that $\mathcal{A}_{2\delta} \subseteq$ *coim* μ, $\mathcal{B}_{2\delta} \subseteq$ *im* μ, and, for any $I \in$ *coim* μ, $I \subset (\mu(I))^{-\delta}$ and $\mu(I) \subset I^{-\delta}$.

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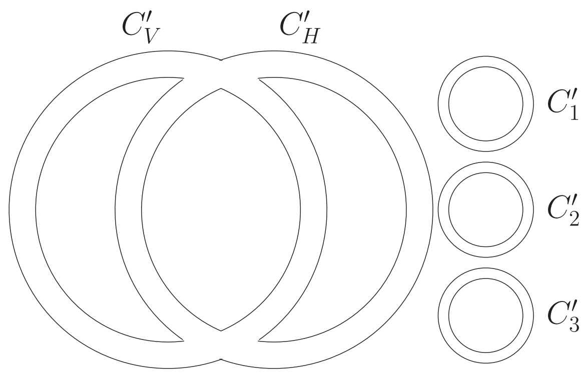

FIGURE 16. The eggbeater surface C_3.

The *bottleneck distance* between two barcodes $\mathcal{A}, \mathcal{B}$ is

$$d_{\mathrm{bot}}(\mathcal{A}, \mathcal{B}) = \inf\{\delta \mid \exists \mu : \mathcal{A} \to \mathcal{B} \ \delta\text{-matching}\}.$$

It can be easily shown that this is a genuine metric and not a pseudo-metric, that is, that if $\mathcal{A}, \mathcal{B}$ are two distinct barcodes, then $d_{\mathrm{bot}}(\mathcal{A}, \mathcal{B}) > 0$. The following theorem relates the Hofer distance between two Hamiltonian diffeomorphisms and the bottleneck distance of the barcodes associated to their Floer homology persistence modules (see [PRSZ20]).

THEOREM 6.7 (Dynamical stability theorem). *Let (M, ω) be a symplectic manifold with $\pi_2(M) = 0$, $\alpha \in \widehat{\pi}(M)$, and $\phi, \psi \in \mathrm{Ham}(M, \omega)$ non-degenerate. Then*

$$d_{\mathrm{bot}}(\mathcal{B}(HF_*^\bullet(\phi)_\alpha), \mathcal{B}(HF_*^\bullet(\psi)_\alpha)) \leq d_{\mathrm{Hofer}}(\phi, \psi).$$

7. Eggbeater maps

We first recall in § 7.1 the definitions of eggbeater surfaces (C_g, ω_0) and eggbeater maps on C_g. The eggbeater maps on surfaces (Σ_g, σ_g) of genus g are the images under the pushforward $(i_{\Sigma_g})_*$ of eggbeaters on C_g for specific embeddings $i_{\Sigma_g} : (C_g, \omega_0) \to (\Sigma_g, \omega)$. For some eggbeater maps ϕ and well-chosen free homotopy classes α, one obtains lower bounds on the length of some bars in $HF_*^\bullet(\phi)_\alpha$ (see § 7.2), as well as computations for $\mathrm{T}^\infty(\alpha)$ (respectively, $\mathrm{si}(\alpha)$) (see § 7.3). We then prove Theorems 1.4 and 1.6, using tools from § 6.

7.1 Definition of the eggbeater surfaces and maps

The eggbeater surface C_g is constructed as follows (see Figure 16). Fix $L \geq 4$ and denote by C' the cylinder of width 2 and length L, $[-1, 1] \times \mathbb{R}/L\mathbb{Z}$, equipped with the standard symplectic form $dx \wedge dy$. Let $g \geq 2$ be an integer, which will later be the genus of the surface of interest. Denote by $C'_V, C'_H, C'_1, C'_2, C'_3$ five copies of C', with $c_V : C' \to C'_V$, $c_H : C' \to C'_H$, $c_1 : C' \to C'_1, c_2 : C' \to C'_2, c_3 : C' \to C'_3$ identity maps, and consider the squares $S_0 = [-1, 1] \times [-1, 1]/L\mathbb{Z} \subset C'$, $S_1 = [-1, 1] \times [L/2 - 1, L/2 + 1]/L\mathbb{Z} \subset C'$. Define the symplectomorphism $VH : c_V(S_0) \bigsqcup c_V(S_1) \to c_H(S_0) \bigsqcup c_H(S_1)$ by $VH = VH_0 \bigsqcup VH_1$, where $VH_0 : c_V(S_0) \to c_H(S_0)$ and $VH_1 : c_V(S_1) \to c_H(S_1)$ are defined as follows:

$$VH_0(x, [y]) = (-y, [x]),$$
$$VH_1(x, [y]) = \left(y - \frac{L}{2}, \left[-x + \frac{L}{2}\right]\right).$$

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The eggbeater surface C_g, which depends on the genus g of the surface of interest, is the following disjoint union of a surface $C'_V \bigcup_{VH} C'_H$ and possibly other annuli:

$$C_2 = C'_V \bigcup_{VH} C'_H \bigsqcup C'_1 \bigsqcup C'_2,$$

$$C_3 = C'_V \bigcup_{VH} C'_H \bigsqcup C'_1 \bigsqcup C'_2 \bigsqcup C'_3,$$

$$C_g = C'_V \bigcup_{VH} C'_H, \quad g \geq 4.$$

The eggbeater surface C_3, for example, is depicted in Figure 16.

Eggbeater surfaces are symplectic manifolds with the standard symplectic form $\omega_0 = dx \wedge dy$ on each copy of C' (see Figure 16). Denote the natural injections of individual cylinders into C_g by $i_V : C'_V \hookrightarrow C_g, i_H : C'_H \hookrightarrow C_g$, and in the cases $g = 2, 3$, also consider the natural injections $i_1 : C'_1 \hookrightarrow C_g, i_2 : C'_2 \hookrightarrow C_g$. If $g = 3$, denote $i_3 : C'_3 \hookrightarrow C_g$. We will identify $C'_V, C'_H, C'_1, C'_2, C'_3$ with their images in C_g $(g \geq 2)$; this will be clear from the context.

These eggbeater surfaces C_g will, later in this section, be embedded in a closed surface of genus $g \geq 2$. A Hamiltonian function will be defined on C_g which will induce some dynamics on it. We want to push forward this dynamical system to the closed surface of genus g, by extending it by the identity map. In order for this map to be a Hamiltonian diffeomorphism, some condition on the embedding and on the Hamiltonian function on C_g must be satisfied. This condition is given in the following definition.

Definition 7.1. Let X, Y be compact topological spaces, and $i : X \hookrightarrow Y$ a continuous embedding. Let $f : X \to \mathbb{R}$ be a continuous map on X, and assume the following condition holds.

- For any path component C of $Y \setminus i(X)$, $f \restriction_{i^{-1}(\partial C)}$ is constant.

Let C_y be the path component of Y that contains $y \in Y$, and denote $D_i = \bigcup_{y \in \mathrm{Im}(i)} C_y \subseteq Y$. For all $y \in D_i$, denote by $\beta_{i,y} : [0, 1] \to C_y$ a continuous path with $\beta_{i,y}(0) = y$, $\beta_{i,y}(1) \in \mathrm{Im}(i)$, and such that if $\beta_{i,y}(t) \in \mathrm{Im}(i)$ for some $t \in [0, 1]$, then $\beta_{i,y} \restriction_{[t,1]}$ is constant. Note that if $y \in \mathrm{Im}(i)$, then $\beta_{i,y} \equiv y$.

Denote the following, not necessarily continuous, map:

$$b_i : D_i \to \mathrm{Im}(i),$$
$$y \mapsto \beta_{i,y}(1).$$

Consider the following map, the *pushforward* of f through i:

$$i_* f : D_i \to \mathbb{R},$$
$$y \mapsto f \circ i^{-1} \circ b_i(y).$$

By the conditions on f and i, this is a continuous map $D_i \to \mathbb{R}$ that does not depend on the choice of the $\beta_{i,y}$.

We now turn to describe the dynamics on the eggbeater surface. Consider the function $u_0 : [-1, 1] \to \mathbb{R}$ given by $u_0(s) = 1 - |s|$. Take an even, non-negative, sufficiently C^0-close smoothing u to u_0 such that u is supported away from $\{\pm 1\}$, both $u - u_0$ and $\int_{-1}^{r} (u(s) - u_0(s))\, ds$ are supported in a sufficiently small neighborhood of $\{\pm 1, 0\}$, and $\int_{-1}^{1} (u(s) - u_0(s))\, ds = 0$.

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For $k \in \mathbb{N}$, define the autonomous Hamiltonian function

$$h_k : C' \to \mathbb{R},$$
$$h_k(x, [y]) = -\frac{1}{2}k + k \int_{-1}^{x} u(s)\, ds.$$

The Hamiltonian h_k induces the following five autonomous Hamiltonian functions:

$$(i_V \circ c_V)_* h_k,\ (i_H \circ c_H)_* h_k : C'_V \bigcup_{VH} C'_H \to \mathbb{R},$$
$$(i_1 \circ c_1)_* h_k : C'_1 \to \mathbb{R},$$
$$(i_2 \circ c_2)_* h_k : C'_2 \to \mathbb{R},$$
$$(i_3 \circ c_3)_* h_k : C'_3 \to \mathbb{R}.$$

Consider the following two autonomous Hamiltonian functions $C_g \to \mathbb{R}$. In an abuse of notation, we will denote the two Hamiltonian functions in the same way, for all three cases $g = 2, g = 3$, $g \geq 4$, even though the definitions differ. For $g = 2$, set

$$h_{V,k} = (i_V \circ c_V)_* h_k \bigsqcup -(i_1 \circ c_1)_* h_k \bigsqcup (i_2 \circ c_2)_* h_k,$$
$$h_{H,k} = (i_H \circ c_H)_* h_k \bigsqcup (i_1 \circ c_1)_* h_k \bigsqcup (i_2 \circ c_2)_* h_k.$$

For $g = 3$, set

$$h_{V,k} = (i_V \circ c_V)_* h_k \bigsqcup (i_1 \circ c_1)_* h_k \bigsqcup (i_2 \circ c_2)_* h_k \bigsqcup 0 \restriction_{C'_3},$$
$$h_{H,k} = (i_H \circ c_H)_* h_k \bigsqcup 0 \restriction_{C'_1} \bigsqcup (i_2 \circ c_2)_* h_k \bigsqcup (i_3 \circ c_3)_* h_k.$$

For $g \geq 4$, set

$$h_{V,k} = (i_V \circ c_V)_* h_k,$$
$$h_{H,k} = (i_H \circ c_H)_* h_k.$$

These Hamiltonian functions generate Hamiltonian diffeomorphisms $f_{V,k}, f_{H,k} \in \mathrm{Ham}_c(C_g, \omega_0)$, respectively.

Define a homomorphism

$$\Psi_k : F_2 \to \mathrm{Ham}_c(C_g, \omega_0),$$
$$V \mapsto f_{V,k}, H \mapsto f_{H,k}.$$

Note that the image of a word $w = V^{N_1} H^{M_1} \cdots V^{N_r} H^{M_r} \in F_2$ is $f_{H,k}^{M_r} \circ f_{V,k}^{N_r} \circ \cdots \circ f_{H,k}^{M_1} \circ f_{V,k}^{N_1}$. We call these images *eggbeater maps* in C_g.

Denote by $S_0, S_1 \subset C_g$ the identification of the squares $c_V(S_0), c_H(S_0)$ and $c_V(S_1), c_H(S_1)$. In fact, $S_0 \cup S_1 = C'_V \cap C'_H$ (recall that C'_V and C'_H are identified with their images in C_g). Fix two points $s_0 \in S_0, s_1 \in S_1$. Define four paths: two paths q_1, q_3 from s_0 to s_1, and two paths q_2, q_4 from s_1 to s_0 as shown in Figure 17; q_1, q_2 are paths on C'_V, and q_3, q_4 are paths on C'_H.

Note that $\pi_1(C_g, s_0) \simeq F_3$, the free group on three generators. The three generators a, b, c of $\pi_1(C_g, s_0)$ are taken to be

$$a = \langle q_1 q_2 \rangle_{\pi_1(C_g, s_0)},$$
$$b = \langle q_3 q_4 \rangle_{\pi_1(C_g, s_0)},$$
$$c = \langle q_3 q_2 \rangle_{\pi_1(C_g, s_0)}.$$

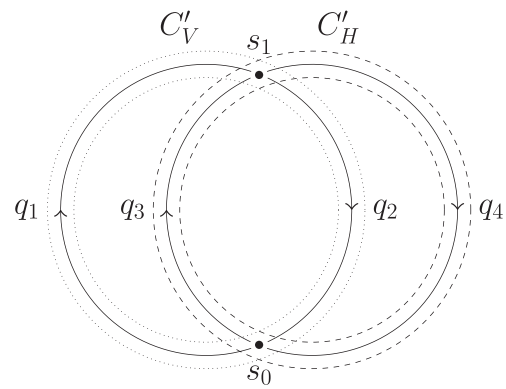

Figure 17. The paths q_1, q_2, q_3, q_4 in C_g.

That is, a is the class of a loop going around C'_V, positively oriented (i.e. $a = [t \mapsto (0, Lt) \in C'_V]$ as an element in $\pi_1(C_g)$), and b is the class of a loop going around C'_H, again positively oriented.

7.2 Action gaps in eggbeater maps

The next result is a consequence of the proof of Proposition 5.1 in [PS16]. Fix $L > 4$.

Proposition 7.2. *Let $w = VH \in F_2$. There exist certain $\nu, \mu \in (0, 1)$, an unbounded subset $K \subset \mathbb{N}$ and a family of primitive free homotopy classes $\alpha_k = [a^{\nu k/L} b^{\mu k/L}] \in \widehat{\pi}(C_g)$, for $k \in K$, such that for large enough $k \in K$, there are exactly four non-degenerate fixed points of $\Psi_k(w)$ that are in class α_k, and different such fixed points have action gaps that grow linearly with k: that is, for such fixed points y, z,*

$$|\mathcal{A}(y) - \mathcal{A}(z)| \geq c \cdot k + O(1),$$

as $k \to \infty$, for some global constant $c > 0$.

In order to get a similar result for eggbeater-like maps on surfaces of genus $g \geq 2$, let (Σ_g, σ_g) be a surface of genus g, equipped with symplectic form σ_g. In [PS16] ($g \geq 4$) and [Cho22] ($g = 2, 3$), symplectic embeddings $i_{\Sigma_g} : (C_g, \omega_0) \hookrightarrow (\Sigma_g, \sigma_g)$ are constructed, with the property that the pairs $i_{\Sigma_g}, h_{V,k}$ and $i_{\Sigma_g}, h_{H,k}$ both satisfy the condition of Definition 7.1.[5] Thus the Hamiltonian functions $(i_{\Sigma_g})_* h_{V,k}, (i_{\Sigma_g})_* h_{H,k} : \Sigma_g \to \mathbb{R}$ generate Hamiltonian diffeomorphisms on Σ_g denoted by $(i_{\Sigma_g})_* f_{V,k}, (i_{\Sigma_g})_* f_{H,k}$. Similarly to the construction of Ψ_k in C_g, define a homomorphism

$$(i_{\Sigma_g})_* \Psi_k : F_2 \to \mathrm{Ham}(\Sigma_g, \sigma_g),$$
$$V \mapsto (i_{\Sigma_g})_* f_{V,k}, \quad H \mapsto (i_{\Sigma_g})_* f_{H,k}.$$

We call the diffeomorphisms in $\mathrm{Ham}(\Sigma_g, \sigma_g)$ that are elements in the images of $(i_{\Sigma_g})_* \Psi_k$ *eggbeater maps in* Σ_g. The set of eggbeaters on Σ_g is denoted by $\mathcal{E}_g$.

We will use the following properties of the embeddings i_{Σ_g} given in [Cho22].

- $(i_{\Sigma_g})_* : \pi_1(C_g, s_0) \to \pi_1(\Sigma_g, i_{\Sigma_g}(s_0))$, $g \geq 2$, are injective.
- In the case $g = 2$, writing $\pi_1(C_2, s_0) = \langle a, b, c \rangle$, and, for a suitable choice of generators, $\pi_1(\Sigma_2, s_0) = \langle g_1, \ldots, g_4 \,|\, [g_1, g_2][g_3, g_4] \rangle$, the pushforward $(i_{\Sigma_2})_* : \pi_1(C_2, s_0) \to \pi_1(\Sigma_2, i_{\Sigma_g}(s_0))$ is given by

$$(i_{\Sigma_2})_*(a) = g_1 g_3, \quad (i_{\Sigma_2})_*(b) = g_3 g_2 g_1^{-1} g_2^{-1}, \quad (i_{\Sigma_2})_*(c) = g_3. \tag{14}$$

[5] To find such an embedding one generally has to multiply ω_0 and the Hamiltonians in the construction with a sufficiently small $\delta > 0$, and we implicitly assume this, since everything below will not depend on this rescaling.

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- The induced maps $(i_{\Sigma_g})_* : \widehat{\pi}(C_g, s_0) \to \widehat{\pi}(\Sigma_g, i_{\Sigma_g}(s_0))$ on the set of conjugacy classes of the fundamental groups, which can be identified with free homotopy classes of loops in $C'_V \bigcup_{VH} C'_H$ (respectively, Σ_g), are injective for $g \geq 3$, and $(i_{\Sigma_2})_*$ is injective up to the relations

$$\begin{aligned}(i_{\Sigma_2})_*[c^j] &= (i_{\Sigma_2})_*[(ac^{-1}b)^j], \quad j \in \mathbb{Z} \setminus \{0\},\\ (i_{\Sigma_2})_*[(ac^{-1})^j] &= (i_{\Sigma_2})_*[(b^{-1}c)^j], \quad j \in \mathbb{Z} \setminus \{0\}.\end{aligned} \tag{15}$$

In [PS16, Cho22] results analogous to Proposition 7.2 for surfaces of genus $g \geq 2$ are shown. The case of genus $g \geq 4$ is a consequence of Proposition 5.1 in [PS16], and the cases $g = 2, 3$ are consequences of Proposition 3.9 in [Cho22]. The results are stated as follows.

PROPOSITION 7.3. *Let Σ_g be a surface of genus $g \geq 2$, and let $w = VH \in F_2$. There exist certain $\nu, \mu \in (0, 1)$, an unbounded subset $K \subset \mathbb{N}$, and a family of primitive free homotopy classes $\alpha_k = (i_{\Sigma_g})_*[a^{\nu k/L} b^{\mu k/L}] \in \widehat{\pi}(\Sigma_g)$, for $k \in K$, such that for large enough $k \in K$, there are exactly four non-degenerate fixed points of $(i_{\Sigma_g})_*\Psi_k(w)$ of class α_k, and different such fixed points have action gaps that grow linearly with k: that is, for such fixed points y, z,*

$$|\mathcal{A}(y) - \mathcal{A}(z)| \geq c \cdot k + O(1),$$

as $k \to \infty$, for some global constant $c > 0$.

Note that the K, α_k, ν, μ, c given by this proposition may not be the same as those of Proposition 7.2. Also, for future reference, note the following key observation: if a Hamiltonian diffeomorphism ϕ has an action gap of $A > 0$ for fixed points of class α (in the sense of Proposition 7.3), then all the bars in the barcode of its Floer persistence module, $\mathcal{B}(HF_*^\bullet(\phi)_\alpha)$, are of length greater than or equal to A.

Remark 7.4. The topological entropy of eggbeater maps $\phi = (i_{\Sigma_g})_*\Psi_k(a)$, $a \in F_2$, is positive as long as a is not of the form H^n or V^n. This can, for example, be seen directly by showing the existence of a horseshoe, as in [Dev78], or with the results in the present paper. Moreover, we note that there is a constant $c_0 > 0$ such that $h_{\text{top}}(\phi) \leq \log((c_0 k)^n)$, where n is the length of $a \in F_2$ and k is as in the definition of ϕ.

To see this, endow Σ_g with a Riemannian metric that, when restricted to $i_{\Sigma_g}(C_g)$ and pulled back by i_{Σ_g}, agrees with the standard Riemannian metric on C_g. Observe that both $\max_{\Sigma_g} \|d(i_{\Sigma_g})_* f_{V,k}\|$ and $\max_{\Sigma_g} \|d(i_{\Sigma_g})_* f_{H,k}\|$ are bounded from above by $c_0 k$ for a constant $c_0 > 0$, and therefore $(i_{\Sigma_g})_*\Psi_k(a)$ is $(c_0 k)^n$-Lipschitz. Hence, for any given rectifiable path γ in Σ_g,

$$\limsup_{m \to \infty} \frac{\log \operatorname{length} \phi^m(\gamma)}{m} \leq \log((c_0 k)^n),$$

where length(γ) denotes the length of the path γ. By [New88, Theorem 1] this yields the desired upper bound on the topological entropy of ϕ.

7.3 Self-intersection number and T^∞ for a family of free homotopy classes

We now focus our attention on the family of free homotopy classes given by Proposition 7.3, namely $(i_{\Sigma_g})_*[a^m b^n] \in \widehat{\pi}(\Sigma_g)$, for $m, n \in \mathbb{N}$. We first compute their geometric self-intersection number and then compute a lower bound on their T^∞-growth rate, defined in § 5.

Recall from the introduction that for a compact surface M, and a loop $\Gamma : S^1 \to M$ in general position, we denote by $\operatorname{si}(\Gamma)$ the total number of self-intersections of Γ, and that the (geometric) self-intersection number $\operatorname{si}_M(\alpha)$ of a free homotopy class $\alpha \in \widehat{\pi}(M)$ is defined to be $\min \operatorname{si}(\Gamma)$, where the minimum is taken over all self-transverse loops $\Gamma : S^1 \to M$ that represent α. This is well defined since loops in general position in M have finitely many self-intersections.

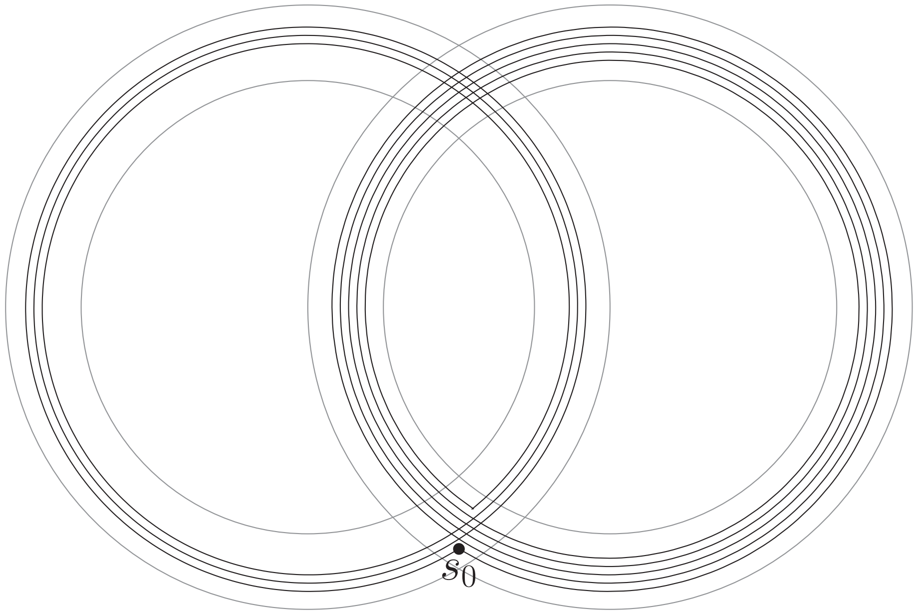

FIGURE 18. The loop $\Gamma_{3,5} : S^1 \to C_g$. This figure only shows the annuli C'_V, C'_H.

DEFINITION 7.5. Let M be a closed surface, and let $\Gamma : S^1 \to M$ be a loop. The loop Γ is said to be in *minimal position* if $\mathrm{si}(\Gamma) = \mathrm{si}_M([\Gamma]_{\widehat{\pi}(M)})$.

DEFINITION 7.6. Let M be a surface, and $\Gamma : S^1 \to M$ be a loop in general position. The loop Γ is said to form *a bigon* with itself if there are subarcs A, B of S^1 such that Γ identifies the endpoints of A with those of B and the loop $\Gamma \restriction_{A\cup B}$ is null-homotopic on M. Similarly, Γ is said to form *a monogon* with itself if there exists a subarc $A \subset S^1$ such that Γ identifies its endpoints and $\Gamma \restriction_A$ is null-homotopic on M.

In order to check whether a given loop $\Gamma : S^1 \to M$ is in minimal position, and to calculate the self-intersection number of $[\Gamma]$, one may use the following fact (Theorem 3.5 from [HS85]).

FACT 7.7. Let M be a surface, and $\Gamma : S^1 \to M$ a loop in general position. If Γ does not form any bigons or monogons then Γ is in minimal position.

Construct a loop $\Gamma = \Gamma_{m,n} : S^1 \to C_g$, whose image is a subset of $i_V(C'_V) \bigcup_{VH} i_H(C'_H)$ and whose free homotopy class is the primitive free homotopy class $[a^m b^n]$, as follows. $\Gamma_{m,n}$ starts at s_0, and performs m rounds of the annulus C'_V, while regularly spiralling inwards. After finishing m rounds of C'_V, Γ starts making n rounds of the annulus C'_H, while regularly spiralling outwards. As Γ finishes these rounds, it reaches the square $c_V(S_0)$ again, and connects back to s_0, without any further self-intersections. An example loop $\Gamma_{m,n}$ can be seen in Figure 18.

By invoking Fact 7.7, one deduces that $\Gamma_{m,n}$ is in minimal position. Counting self-intersections in $\Gamma_{m,n}$, one sees that $\mathrm{si}_{C_g}([a^m b^n]) = mn + (m-1)(n-1)$, where the $(m-1)(n-1)$ term comes from the self-intersections in the square $c_V(S_0)$, and the mn term comes from the self-intersections in the other square $c_V(S_1)$.

However, we also want to calculate the self-intersection number $\mathrm{si}_{\Sigma_g}((i_{\Sigma_g})_*[a^m b^n])$ of the pushforward of the free homotopy class considered up to this point under i_{Σ_g}, for $g \geq 2$. This is performed in the following lemma.

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Lemma 7.8. *Let Σ_g be a closed surface of genus $g \geq 2$, and consider the embedding $i_{\Sigma_g} : C_g \to \Sigma_g$. Then*

$$\mathrm{si}_{\Sigma_g}((i_{\Sigma_g})_* [a^m b^n]) = mn + (m-1)(n-1).$$

Proof. Recall that in [PS16, Cho22] the embeddings i_{Σ_g} are shown to induce injections $(i_{\Sigma_g})_* : \pi_1(C_g, s_0) \to \pi_1(\Sigma_g, i_{\Sigma_g}(s_0))$. Assume by contradiction that $i_{\Sigma_g} \circ \Gamma_{m,n} : S^1 \to \Sigma_g$ is not in minimal position. By Fact 7.7, $i_{\Sigma_g} \circ \Gamma_{m,n}$ forms a monogon or a bigon with itself.

Let $\delta : S^1 \to \Sigma_g$ be the boundary of this monogon or bigon; δ is null-homotopic. Since $\operatorname{Im} \delta \subset \operatorname{Im} i_{\Sigma_g} \circ \Gamma_{m,n}$ and i_{Σ_g} is injective, $i_{\Sigma_g}^{-1} \circ \delta : S^1 \to C_g$ is a well-defined loop in C_g, which is made up of one or two arcs of $\Gamma_{m,n}$. Since $(i_{\Sigma_g})_*$ is injective, $i_{\Sigma_g}^{-1} \circ \delta$ is also null-homotopic. This implies that $\Gamma_{m,n}$ forms a monogon or a bigon with itself, in contradiction to the above discussion. □

We now give a lower bound on the growth rate $\mathrm{T}^\infty(\alpha)$, where $\alpha = {i_{\Sigma_g}}_*[a^m b^n] \in \widehat{\pi}(\Sigma_g)$.

Lemma 7.9. *The growth rate of α is bounded below by $\mathrm{T}^\infty(\alpha) \geq \log(\lceil mn/2 \rceil + 1)$.*

Proof. We use the notation from § 5, and consider

$$\mu : \pi_1(\Sigma_g, s_0) \to \mathcal{H} = \bigcup_{y \in \pi_1(\Sigma_g, s_0)} \mathbb{Z}[\pi_1(\Sigma_g, s_0)^*_y] \otimes \mathbb{Z}[\pi_1(\Sigma_g, s_0)^*_y].$$

Here we write s_0 instead of $i_{\Sigma_g}(s_0)$, also we will write $a, b, c, \Gamma_{m,n}$ instead of writing $(i_{\Sigma_g})_* a, (i_{\Sigma_g})_* b, (i_{\Sigma_g})_* c \in \pi_1(\Sigma, s_0)$, $i_{\Sigma_g} \circ \Gamma_{m,n} : [0,1] \to \Sigma_g$, whenever the context is clear. We can read from the loop $\Gamma_{m,n}$ that $\mu(a^m b^n)$ has the form

$$\begin{aligned} \mu(a^m b^n) = &\sum_{i=1}^{m-1} \sum_{k=1}^{n-1} [a^i b^k]_{a^m b^n} \otimes [a^m b^{n-k} a^{-i}]_{a^m b^n} - [a^m b^{n-k} a^{-i}]_{a^m b^n} \otimes [a^i b^k]_{a^m b^n} \\ &+ \sum_{i=1}^{m} \sum_{k=0}^{n-1} [a^m b^k c a^{-i}]_{a^m b^n} \otimes [a^i c^{-1} b^{n-k}]_{a^m b^n} - [a^i c^{-1} b^{n-k}]_{a^m b^n} \otimes [a^m b^k c a^{-i}]_{a^m b^n}. \end{aligned} \tag{16}$$

In particular, if $m = 1$ and $n = 1$,

$$\mu(ab) = [aca^{-1}]_{ab} \otimes [ac^{-1}b]_{ab} - [ac^{-1}b]_{ab} \otimes [aca^{-1}]_{ab}. \tag{17}$$

We first claim that no terms cancel out in (16). Note first that this is the case if the expression in (16) is considered as one in terms of the free group in a, b, c. Therefore, if the genus is $g \geq 3$, no cancellation of terms in (16) follows from the fact that $(i_{\Sigma_g})_* : \widehat{\pi}(C_g, s_0) \to \widehat{\pi}(\Sigma_g, s_0)$ is injective. $(i_{\Sigma_2})_* : \widehat{\pi}(C_2, s_0) \to \widehat{\pi}(\Sigma_2, s_0)$ is injective up to (15), so clearly one only has to check whether a cancellation occurs for the terms in the second row of (16) with ($i = 1$, $k = n-1$) and ($i = m$, $k = 0$). If $m \geq 2$ or $n \geq 2$ one can readily check that this holds on the level of conjugacy classes. Consider now the remaining case $m = n = 1$. By (14),

$$(i_{\Sigma_2})_*(ab) = g_1 g_3^2 g_2 g_1^{-1} g_2^{-1}, \quad (i_{\Sigma_2})_*(aca^{-1}) = g_1 g_3 g_1^{-1}, \quad (i_{\Sigma_2})_*(ac^{-1}b) = g_1 g_3 g_2 g_1^{-1} g_2^{-1}.$$

So $\mu(ab) = 0$ if and only if there is $k \in \mathbb{Z}$ such that $f_k = 1$, where

$$f_k := (g_1 g_3^2 g_2 g_1^{-1} g_2^{-1})^k g_1 g_3 g_1^{-1} (g_1 g_3^2 g_2 g_1^{-1} g_2^{-1})^{-k} g_2 g_1 g_2^{-1} g_3^{-1} g_1^{-1}. \tag{18}$$

If $k = 0$ or $k = -1$, one checks that $f_k \neq 1$. If $k > 0$, after freely and cyclically reducing the word on the right-hand side of (18) we obtain

$$(g_1 g_3^2 g_2 g_1^{-1} g_2^{-1})^{k-1} g_1 g_3 g_1^{-1} (g_1 g_3^2 g_2 g_1^{-1} g_2^{-1})^{-k} g_2 g_1 g_2^{-1} g_3 g_2 g_1^{-1} g_2^{-1},$$

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and if $k < -1$, after freely and cyclically reducing this word we obtain

$$(g_1 g_3^2 g_2 g_1^{-1} g_2^{-1})^{k+1} g_2 g_1 g_2^{-1} g_3 g_2 g_1^{-1} g_2^{-1} (g_1 g_3^2 g_2 g_1^{-1} g_2^{-1})^{-(k+2)} g_1 g_3 g_1^{-1}.$$

Since in neither case do the reduced words contain a subword of more than four elements which is also a subword of a cyclic permutation of a relator $[g_1, g_2][g_3, g_4]$ or its inverse, Dehn's algorithm shows that $f_k \neq 1$ for all $k \in \mathbb{Z}$. Hence, the terms in (16) do not cancel out.

To estimate $\mathrm{T}^\infty(\alpha)$, we use that $\mathrm{Comp}(\mu(a^m b^n)) \supset \mathcal{T}$, where

$$\mathcal{T} := \big\{[a^m b^k c a^{-i}]_{a^m b^n} \otimes [a^i c^{-1} b^{n-k}]_{a^m b^n} \mid i = 1, \dots, m;\ k = 0, \dots, n-1\big\}.$$

Hence,

$$\mathrm{Comp}_+(\mu(a^m b^n)) \supset \mathcal{T}_+ := \{[a^m b^k c a^{-i}]_{a^m b^n} \,|\, i = 1, \dots, m;\ k = 0, \dots, n-1\}$$

and

$$\mathrm{Comp}_-(\mu(a^m b^n)) \supset \mathcal{T}_- := \{[a^i c^{-1} b^{n-k}]_{a^m b^n} \,|\, i = 1, \dots, m;\ k = 0, \dots, n-1\}.$$

We show that for any $\mathfrak{S} \subset \mathcal{T}_+$ or $\mathfrak{S} \subset \mathcal{T}_-$, and any set $S \subset \pi_1(\Sigma_g, s_0)$ with $[S]_{a^m b^n} = \mathfrak{S}$, we have that $\Gamma(S \cup \{a^m b^n\}) \geq \log(\#S + 1)$. This yields that

$$\begin{aligned}\mathrm{T}^\infty(\alpha) &\geq \min_{\pm} \min_{\mathfrak{S}} \big\{\Gamma(\mathfrak{S} \cup [g]_g, g) \ \mid \mathfrak{S} \subset \mathcal{T}_\pm, \#\mathfrak{S} = \big\lceil \tfrac{1}{2}\#\mathcal{T}\big\rceil\big\} \\ &\geq \log\left(\left\lceil \# \frac{\mathcal{T}}{2}\right\rceil + 1\right) = \log\left(\left\lceil \# \frac{mn}{2}\right\rceil + 1\right).\end{aligned}$$

We argue for $\mathfrak{S} \subset \mathcal{T}_+$; the other case is analogous.

Let $P = q^{s_0} p_1 q^{s_1} p_2 \cdots p_{l'} q^{s_{l'}}$, where $p_1, \dots, p_{l'} \in S$, $l' \in \mathbb{N}$, $q = a^m b^n$, and $s_0, \dots, s_{l'} \in \mathbb{N}$, with $l' + s_0 + \cdots + s_{l'} =: l$. Consider P as a word w in the letters a, b, and c, and consider the corresponding reduced word w'. It is clear, since $[S]_{a^m b^n} \subset \mathcal{T}_+$ and from the type of elements in $\mathcal{T}_+$, that the word w' can be written as

$$w' = w_0 c w_1 c w_2 \cdots c w_{l'},$$

for some reduced, possibly empty, words $w_0, \dots, w_{l'}$ in a and b. Moreover, if $[p_j] = [a^m b^k c a^{-i}]_{a^m b^n}$, then w_{j-1} ends with $b^{k'}$ such that $k' \sim k \bmod n$, and w_j starts with $a^{-i'}$ such that $i' \sim i \bmod m$. Therefore, such a word w' determines the original elements $p_1, \dots, p_{l'} \in S$. And so, w' also determines the original choice of $s_0, \dots, s_{l'}$. Indeed, if $p_1 = ucv$, then $w_0 u^{-1} = (a^m b^n)^{s_0}$, etc. Moreover, if two such words w_1' and w_2', coming from products P_1 and P_2, are equal up to cyclic permutation and reduction as elements in $F_3 = \langle a, b, c\rangle$, then so are P_1, P_2 as products of symbols in $S \cup \{q\}$. If the genus g is greater than or equal to 3, then $(i_{\Sigma_g})_* : \widehat{\pi}(C_g, s_0) = \langle a, b, c\rangle/\mathrm{conj} \to \widehat{\pi}(\Sigma_g)$ is injective and we obtain that $\widehat{N}(l, S \cup \{a^m b^n\}) \geq \sum_{i=1}^{l} ((\#S + 1)^l / l)$, so $\Gamma(S \cup \{a^m b^n\}) \geq \log(\#S + 1)$. $(i_{\Sigma_2})_*$ is injective up to relation (15), but for each l, only four products could appear in those equations, and hence also in this case $\Gamma(S \cup \{a^m b^n\}) \geq \log(\#S + 1)$. Note that in the case $m = n = 1$ clearly $\mathrm{T}^\infty([ab]) \leq \log(2)$, and hence in fact $\mathrm{T}^\infty([ab]) = \log(2)$. ☐

7.4 Proofs of Theorems 1.4 and 1.6, and Corollary 1.5

This subsection contains, as an application of Theorem 1.3, the proofs of Theorems 1.4 and 1.6, using Proposition 7.3 and the tools from § 6. It also contains a proof of Corollary 1.5.

In the following, for $R > 0$ and $\phi \in \mathrm{Ham}(\Sigma_g, \sigma_g)$, we denote by $B_{d_{\mathrm{Hofer}}}(\phi, R) \subset \mathrm{Ham}(\Sigma_g, \sigma_g)$ the set of $\psi \in \mathrm{Ham}(\Sigma_g, \sigma_g)$ with $d_{\mathrm{Hofer}}(\psi, \phi) < R$.

Proof of Theorem 1.4. Let $\nu, \mu \in (0, 1)$, $K \subset \mathbb{N}$, be the numbers from Proposition 7.3, and for $k \in K$ denote $m_k = \nu k / L$, $n_k = \mu k / L \in \mathbb{N}$, and $\alpha_k = (i_{\Sigma_g})_* \big[a^{m_k} b^{n_k}\big] \in \widehat{\pi}(\Sigma_g)$. Recall from

Proposition 7.3 that α_k are primitive classes. There is $c > 0$ such that the action gap for $(i_{\Sigma_g})_*\Psi_k(VH)$ guaranteed by Proposition 7.3 is greater than or equal to ck. Note also that $m_k n_k \geq (\mu\nu/L^2)k^2$. Write the elements in K as a sequence $(k_l)_{l\in\mathbb{N}}$, and let $\phi_l := (i_{\Sigma_g})_*\Psi_{k_l}(VH)$. The Hofer norm $M_l := \|\phi_l\|_{\mathrm{Hofer}}$ grows linearly in k_l, since by construction of ϕ_l, M_l grows at most linearly, and by the statement on the action gaps and the dynamical stability theorem at least linearly in k_l. Hence there are constants $\delta > 0$ and $C > 0$ such that the action gaps for ϕ_l are greater than or equal to $2\delta M_l + 1$, and $m_{k_l} n_{k_l} \geq 2CM_l^2$.

By Lemma 7.9, $\mathrm{T}^\infty(\alpha_{k_l}) \geq \log(CM_l^2)$.

Let $\psi \in B_{d_{\mathrm{Hofer}}}(\phi_l, \delta M_l)$. We show that there is a fixed point $z \in \Sigma_g$ of ψ in class α_{k_l}. To simplify notation write $\phi = \phi_l, \alpha = \alpha_{k_l}, M = M_l$.

Recall that Proposition 7.3 guarantees that ϕ has an action gap greater than or equal to $2\delta M + 1$, and that ϕ has at least one fixed point in class α. Thus, by definition, $\mathcal{B}(HF_*^\bullet(\phi)_\alpha)$ has a bar of length at least $2\delta M + 1$.

Let $H : S^1 \times \Sigma_g \to \mathbb{R}$ be a Hamiltonian function that generates ψ. Recall that the set $\{F : S^1 \times \Sigma_g \to \mathbb{R} \,|\, F \text{ is non-degenerate}\}$ is C^∞-dense in the space of all functions $S^1 \times \Sigma_g \to \mathbb{R}$, with the C^∞ topology. Thus one can take a sequence $H_j \xrightarrow{C^\infty} H$ of non-degenerate Hamiltonian functions. Denote by ψ_j the Hamiltonian diffeomorphism generated by H_j. Note that $d_{\mathrm{Hofer}}(\psi_j, \psi) \to 0$. Since $\psi \in B_{d_{\mathrm{Hofer}}}(\phi, \delta M)$, there exists an integer $j_0 \in \mathbb{N}$ such that $\psi_j \in B_{d_{\mathrm{Hofer}}}(\phi, \delta M)$ for $j > j_0$. By Theorem 6.7, $d_{\mathrm{bot}}(\mathcal{B}(HF_*^\bullet(\phi)_\alpha), \mathcal{B}(HF_*^\bullet(\psi_j)_\alpha)) \leq d_{\mathrm{Hofer}}(\phi, \psi_j) < \delta M$. Thus $\mathcal{B}(HF_*^\bullet(\psi_j)_\alpha)$ has a bar of length at least 1. By definition of the barcode, this means that ψ_j has a fixed point $z_j \in \Sigma_g$ in class α for $j > j_0$. The manifold Σ_g is compact; take a convergent subsequence of z_j, $(z_j)_{j\in J}$ for some infinite $J \subset \mathbb{N}$, which converges to $z \in \Sigma_g$. Since $(H_j)_{j\in J} \xrightarrow{C^\infty} H$ and by the Arzelà–Ascoli theorem, z is a fixed point of ψ in class α.

Finally, by Theorem 1.3, $\mathrm{H}^\infty(\alpha) \geq \mathrm{T}^\infty(\alpha) \geq \log(CM_l^2)$. By a version of Ivanov's inequality (see [Jia96, Theorem 2.7]), we obtain that $h_{\mathrm{top}}(\psi) \geq \mathrm{H}^\infty(\alpha) \geq \log(CM_l^2)$. □

Proof of Corollary 1.5. Let $\delta > 0$, $C_1 = C > 0$, and $\phi_l \in \mathcal{E}_g$, $l \in \mathbb{N}$, be given by applying Theorem 1.4, and denote $M_l = \|\phi_l\|_{\mathrm{Hofer}}$. Let $k_l \in \mathbb{N}$ be the integers given in the proof of Theorem 1.4. In the proof, it is specified that M_l is at least linear in k_l, that is, there exists some $C_2 > 0$ such that $M_l \geq C_2 k_l$ for all $l \in \mathbb{N}$. Recall that by construction, $\phi_l = (i_{\Sigma_g})_*\Psi_{k_l}(VH)$ and specifically that the word $VH \in F_2$ generating ϕ_l is of length 2.

Let $\psi \in \mathrm{Ham}(\Sigma_g, \sigma_g)$ with $d_{\mathrm{Hofer}}(\psi, \phi_l) < \delta M_l$. By Remark 7.4, there is a constant $C_3 > \sqrt{C_1}C_2$ such that $h_{\mathrm{top}}(\phi_l) \leq \log((C_3 k_l)^2)$. Therefore, setting $K = \log(C_3^2) - \log(C_1 C_2^2)$, by Theorem 1.4,

$$h_{\mathrm{top}}(\psi) \geq \log(C_1 M_l^2) \geq \log(C_1 C_2^2 k_l^2) = \log(C_3^2 k_l^2) - K \geq h_{\mathrm{top}}(\phi_l) - K. \qquad \square$$

Proof of Theorem 1.6. Let $\psi \in \mathrm{Ham}(\Sigma_g, \sigma_g)$, $\varepsilon > 0$. We must find $\hat{\chi} \in \mathrm{Ham}(\Sigma_g, \sigma_g)$ with $d_{\mathrm{Hofer}}(\hat{\chi}, \psi) < \varepsilon$ and which has some open neighborhood V in $(\mathrm{Ham}(\Sigma_g, \sigma_g), d_{\mathrm{Hofer}})$ on which $h_{\mathrm{top}} > M$; note that the union of these neighborhoods V is an open and dense set with respect to d_{Hofer}.

Choose k so large that $\log(\mathrm{si}(\alpha_k) + 1)/16 > M$ and that all fixed points of $\Psi_k(VH)$ in class α_k are non-degenerate. Let δ be so small that there is an embedding $i_{\Sigma_g} : (C_g, \delta\omega_0) \to (\Sigma_g, \sigma_g)$ as discussed in § 7.2. Recall that when multiplying by δ we implicitly multiply the Hamiltonians $\Psi_k(VH)$ by δ in order not to change the dynamics. Hence, we can additionally choose δ so small that $\phi := (i_{\Sigma_g})_*\Psi_k(VH)$ satisfies $\|\phi\|_{\mathrm{Hofer}} < \varepsilon/3$. Choose a generating Hamiltonian path ϕ_t of ϕ, $\phi_0 = \mathrm{id}$ and $\phi_1 = \phi$. Let $x \in \Sigma_g$ be a fixed point of ϕ that lies in class α_k.

Choose a generating Hamiltonian path ψ_t of ψ and choose a path of Hamiltonian diffeomorphisms τ_t, $\tau := \tau_1$, with $\tau \circ \psi \restriction_W = \mathrm{id}_W$ for a small neighborhood W of x, $\|\tau\|_{\mathrm{Hofer}} < \varepsilon/3$, and such that $\tau_t \# \psi_t(x)$ is contractible, where $\#$ denotes Hamiltonian path concatenation. Indeed, we can construct such a path τ_t as follows. Let $G : S^1 \times \Sigma_g \to \mathbb{R}$ be the Hamiltonian generating ψ_t. The Hamiltonian $Q : S^1 \times \Sigma_g \to \mathbb{R}$ defined by $Q(t,y) = -G(1-t,y)$ generates the path $\psi_{1-t} \circ \psi^{-1}$. Take two small enough tubular neighborhoods N_1 and N_2 of $\big\{(t, \psi_{1-t}(x))\big\} \subset [0,1] \times \Sigma_g$ with $\overline{N_1} \subset N_2$, such that $Q_t\big(\{p \in \Sigma_g \mid | (t,p) \in N_2\}\big) \subset \big(Q_t(\psi_{1-t}(x)) - \varepsilon/12, Q_t(\psi_{1-t}(x)) + \varepsilon/12\big)$ for all $t \in [0,1]$. For $t \in [0,1]$, set $C_t = Q_t(\psi_{1-t}(x))$, and consider the function

$$N_1 \to \mathbb{R},$$
$$(t,y) \mapsto Q_t(y) - C_t.$$

Using a suitable cutoff function, we can extend this function to a smooth function $F : [0,1] \times \Sigma_g \to \mathbb{R}$, such that $F \restriction_{[0,1]\times\Sigma_g \setminus N_2} \equiv 0$ and $F(\{t\} \times \Sigma_g) \subset \big(-\varepsilon/6, \varepsilon/6\big)$ for all $t \in [0,1]$. In fact, such a function F can be chosen to be periodic in t. Note that the Hamiltonian path τ_t generated by F satisfies the above conditions, in particular $\|\tau\|_{\mathrm{Hofer}} < \varepsilon/3$.

We define $\chi_t = \phi_t \# \tau_t \# \psi_t$, $\chi = \phi \circ \tau \circ \psi$. Let $H : S^1 \times \Sigma_g \to \mathbb{R}$ be a Hamiltonian that generates χ_t. Then x is a non-degenerate fixed point of χ in class α_k. Let $\xi(t) = \chi_t(x)$. Choose tubular neighborhoods U_0, U ($\overline{U_0} \subset U$) of $\{(t, \xi(t)) \mid t \in S^1\} \subset S^1 \times \Sigma_g$ and $\kappa > 0$ so small that $d_0(\chi(\chi_{-t}(z)), \chi_{-t}(z)) \geq \kappa$, for all $(t,z) \in U \setminus U_0$, where d_0 is some fixed metric on Σ_g. Such neighborhoods exist, since x is an isolated fixed point. A standard perturbation argument shows (see, for example, [FHS95]), that for any $\epsilon' > 0$ there are perturbations $h : S^1 \times M \to \mathbb{R}$ with $\|h\|_{C^2} < \epsilon'$ that are supported outside of U_0 and such that all periodic orbits $\eta(t)$ of the Hamiltonian diffeomorphism $\hat{\chi}$ generated by $H + h$ for which $\{(t, \eta(t)) \mid t \in S^1\} \cap U = \emptyset$, are non-degenerate. If we choose ϵ' so small that for any such $\hat{\chi}$ also $d_0(\hat{\chi}(\hat{\chi}_{-t}(z)), \hat{\chi}_{-t}(z)) \geq \kappa/2$, then actually all orbits of $\hat{\chi}$ are non-degenerate. If also $\epsilon' < \varepsilon/3$, one has $d_{\mathrm{Hofer}}(\hat{\chi}, \psi) \leq d_{\mathrm{Hofer}}(\hat{\chi}, \chi) + d_{\mathrm{Hofer}}(\chi, \tau \circ \psi) + d_{\mathrm{Hofer}}(\tau \circ \psi, \psi) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

Since $\mathrm{HF}_*^\bullet(\hat{\chi})_{\alpha_k}$ is non-zero, its barcode has a non-zero bar, say of length $\sigma > 0$. By the dynamical stability theorem, non-degenerate diffeomorphisms in $B_{d_{\mathrm{Hofer}}}(\hat{\chi}, \sigma')$, for $\sigma' < \sigma/2$, have non-zero filtered Floer homology in class α_k, and hence have a fixed point in this class. This conclusion holds also for degenerate diffeomorphisms in $B_{d_{\mathrm{Hofer}}}(\hat{\chi}, \sigma')$ with an analogous argument as in the proof of Theorem 1.4. By Theorem 1.1, this means that $h_{\mathrm{top}}|_{B_{d_{\mathrm{Hofer}}}(\hat{\chi}, \sigma')} > M$, as desired. □

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