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THE NATURAL TOPOLOGY OF MATLIS REFLEXIVE MODULES

RICHARD BELSHOFF

For a complete local ring R with maximal ideal $\mathfrak{m}$, we define a linear topology on a Matlis reflexive R-module M which coincides with the $\mathfrak{m}$-adic topology on M in case M is finitely generated. We show that a Matlis reflexive module is complete in this topology.

1. INTRODUCTION

Let $(R, \mathfrak{m})$ be a local noetherian ring and let E denote the injective envelope of $R/\mathfrak{m}$. For an R-module M the Matlis dual $\operatorname{Hom}_R(M, E)$ is denoted M^ν, and M is said to be Matlis reflexive if the canonical injection $M \to M^{\nu\nu}$ is an isomorphism. It is well known [**6**, Theorem 3.7, p.522] that there is a canonical isomorphism $R^{\nu\nu} \cong \widehat{R}$, where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R, and so any complete local ring R is a Matlis reflexive R-module. It follows that $M^{\nu\nu} \cong \widehat{M}$ if M is a finitely generated R-module, because in this case, $M^{\nu\nu} \cong M \otimes_R \operatorname{Hom}_R(E, E) \cong M \otimes_R \widehat{R} \cong \widehat{M}$.

For the rest of this article R will denote a complete local ring. In this case, finitely generated and artinian R-modules are Matlis reflexive. Also, M is an artinian (noetherian) R-module if and only if M^ν is noetherian (artinian). This is the essence of what is commonly known as "Matlis duality". See [**6**, Corollary 4.3, p.528] for more details. The main results of Matlis [**6**] are nicely presented in Matsumura's recent book [**7**].

Enochs [**5**, Proposition 1.3, p.181] has shown that an R-module M is Matlis reflexive if and only if M has a finitely generated submodule S such that M/S is artinian. This characterisation has proved to be very useful. For example, in [**2**] it was used to show that if M and N are Matlis reflexive R-modules, then so are $\operatorname{Hom}_R(M, N)$, $M \otimes_R N$, and more generally $\operatorname{Ext}^n_R(M, N)$ and $\operatorname{Tor}^R_n(M, N)$ for all $n \geqslant 1$.

In section 2 we use this characterisation to define a linear topology on a Matlis reflexive R-module M, which we call the Matlis topology, or the natural topology of the Matlis reflexive R-module M. This topology is a generalisation of the usual $\mathfrak{m}$-adic topology in the sense that these two topologies coincide when M is finitely generated. For general references on linear topologies and completions, see [**1**, **3**, or **7**]. In Section

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3 we show that for a finitely generated submodule S of a Matlis reflexive R-module M, the subspace topology that S inherits from M is the same as the m-adic topology on S. In Section 4 we show that a Matlis reflexive module is complete in its natural topology.

2. Definition of the Natural Topology

In this section we show that a Matlis reflexive module M has a natural topology defined in terms of the finitely generated submodules $S \subset M$ such that M/S is artinian. If we choose a particular finitely generated submodule $S \subset M$ with M/S artinian, then we have $S \supset \mathfrak{m}S \supset \mathfrak{m}^2 S \supset \cdots$, a decreasing filtration on M. And so there is a topology on M such that $\{\mathfrak{m}^n S\}$ form a fundamental system of neighbourhoods of 0, called *the topology on* M *defined by the filtration* $\{\mathfrak{m}^n S\}$. Let $M_n = \mathfrak{m}^n S$ for $n \geqslant 0$, and $M_n = M$ for $n < 0$. Then the filtration $(M_n)_{n \in \mathbb{Z}}$ is such that $\bigcup M_n = M$, and $\bigcap M_n = 0$ and so is exhaustive and separated (Hausdorff.)

We begin with an easy Lemma.

Lemma 1.

1. *If $S \subset M$ is a finitely generated submodule such that M/S is artinian, then $\mathfrak{m}S \subset M$ is also finitely generated with $M/\mathfrak{m}S$ artinian.*
2. *If $S_1, S_2 \subset M$ are two finitely generated submodules with $M/S_1, M/S_2$ artinian, then $S_1 \cap S_2 \subset M$ is also finitely generated and $M/S_1 \cap S_2$ is artinian.*

Proof: 1. Clearly $\mathfrak{m}S$ is finitely generated; and the kernel of the induced surjection $M/\mathfrak{m}S \longrightarrow M/S \to 0$ is $S/\mathfrak{m}S$, which is a finite dimensional vector space and so is artinian. Therefore $M/\mathfrak{m}S$ is artinian.

For 2, it's clear that $S_1 \cap S_2$ is finitely generated. And $M/(S_1 \cap S_2)$ is isomorphic to a submodule of the artinian module $M/S_1 \times M/S_2$, and so is artinian. □

If we had chosen another finitely generated submodule $T \subset M$ with M/T artinian to define the topology, then by part (3) of the next Lemma, both S and T give rise to the same topology on M. We will call this topology the *natural topology of the Matlis reflexive module* M. So a basic neighbourhood of 0 will be any finitely generated submodule $S \subset M$ such that M/S is artinian.

Lemma 2. *Suppose that an R-module M has a finitely generated submodule $S \subset M$ such that M/S is artinian. Let $T \subset M$ be any submodule. Then the following are equivalent:*

(1) *T is finitely generated and M/T is artinian.*
(2) *Both $(S+T)/S$ and $(S+T)/T$ have finite length.*
(3) *For some integer n, $\mathfrak{m}^n S \subset T$ and $\mathfrak{m}^n T \subset S$.*

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PROOF: (1) $\Longrightarrow$ (2). Since T is finitely generated, the image $T/(S \cap T)$ is too. But $T/(S \cap T) \cong (S+T)/S$ is also artinian, since it's a submodule of the artinian module M/S, and similarly for $(S+T)/T$.

(2) $\Longrightarrow$ (3). If $(S+T)/T$ has finite length, then since

$$(S+T)/T \supset (\mathfrak{m}S+T)/T \supset (\mathfrak{m}^2S+T)/T \supset \cdots,$$

we must have $(\mathfrak{m}^nS+T)/T = 0$ for some integer n. Hence $\mathfrak{m}^nS + T = T$ for some n and so we have $\mathfrak{m}^nS \subset T$. Similarly, if $(S+T)/S$ is of finite length, then use the descending chain

$$(S+T)/S \supset (S+\mathfrak{m}T)/S \supset (S+\mathfrak{m}^2T)/S \supset \cdots$$

to get that $\mathfrak{m}^nT \subset S$.

(3) $\Longrightarrow$ (1). Since $\mathfrak{m}^nT \subset S$ and S is finitely generated, then $\mathfrak{m}^nT$ is finitely generated. And $M/\mathfrak{m}^nT$ is artinian. Then $T/\mathfrak{m}^nT \subset M/\mathfrak{m}^nT$ is also artinian. And $T/\mathfrak{m}^nT$ is a module over $R/\mathfrak{m}^n$, an artinian ring. So $T/\mathfrak{m}^nT$ is finitely generated. Then it follows that T is finitely generated. The induced onto homomorphism $M/\mathfrak{m}^nS \longrightarrow M/T \longrightarrow 0$ and the fact that $M/\mathfrak{m}^nS$ is artinian show that M/T is artinian. $\square$

The next result is a generalisation of "Chevalley's theorem". The theorem states that if $(R,\mathfrak{m})$ is a complete local ring, M is a finitely generated R-module and $(S_t)_{t=1}^{\infty}$ is a nonincreasing sequence of submodules such that $\bigcap_{t=1}^{\infty} S_t = 0$, then for every n there exists $t(n)$ such that $S_{t(n)} \subset \mathfrak{m}^nM$.

In other words given a finitely generated module over a complete local ring, if a nonincreasing sequence of submodules has intersection 0 then the terms are eventually contained in a large power of the maximal ideal times the module. (See [**4**, **7**, Exercise 8.7, p.63, or **8**].)

We show next that Chevalley's theorem is true not only for finitely generated modules but also for Matlis reflexive modules. So Chevalley's theorem is valid in the special case when M is artinian.

PROPOSITION 1. *Let M be an R-module with a finitely generated submodule T such that M/T is artinian. If*

$$S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} \supset \cdots$$

is any chain of submodules of M such that $\bigcap_{n=1}^{\infty} S_n = 0$, then for every n there exists an integer $s(n)$ such that $S_{s(n)} \subset \mathfrak{m}^nT$.

PROOF: If the conclusion is not true, then there exists an integer n_0 so that for every s, $S_s \not\subset \mathfrak{m}^nT$ for all $n \geqslant n_0$. Since $M/\mathfrak{m}^nT$ is artinian for every n, the chain

$$M/\mathfrak{m}^nT \supset (S_1+\mathfrak{m}^nT)/\mathfrak{m}^nT \supset (S_2+\mathfrak{m}^nT)/\mathfrak{m}^nT \supset \cdots$$

stabilises, so for each n there exists an integer $s(n)$ such that

$$S_{s(n)} + \mathfrak{m}^n T = S_{s(n)+1} + \mathfrak{m}^n T = S_{s(n)+2} + \mathfrak{m}^n T = \cdots$$

Note that we can take $s(0) < s(1) < s(2) < \cdots$. Also note that since $S_s \not\subset \mathfrak{m}^n T$ for all s and for all $n \geqslant n_0$, we have $S_s + \mathfrak{m}^n T \neq \mathfrak{m}^n T$ for all s and for all $n \geqslant n_0$.

Next we construct a sequence. There exists an element $x_{n_0} \in S_{s(n_0)}$ with $x_{n_0} \notin \mathfrak{m}^{n_0} T$. Since

$$S_{s(n_0)} + \mathfrak{m}^{n_0} T = S_{s(n_0)+1} + \mathfrak{m}^{n_0} T = \cdots = S_{s(n_0+1)} + \mathfrak{m}^{n_0} T = \cdots,$$

there exists $x_{n_0+1} \in S_{s(n_0+1)}$ such that $x_{n_0} = x_{n_0+1} + y_{n_0}$ for some $y_{n_0} \in \mathfrak{m}^{n_0} T$. And since

$$S_{s(n_0+1)} + \mathfrak{m}^{n_0+1} T = S_{s(n_0+1)+1} + \mathfrak{m}^{n_0+1} T = \cdots = S_{s(n_0+2)} + \mathfrak{m}^{n_0+1} T = \cdots,$$

there exists $x_{n_0+2} \in S_{s(n_0+2)}$ such that $x_{n_0+1} = x_{n_0+2} + y_{n_0+1}$ for some $y_{n_0+1} \in \mathfrak{m}^{n_0+1} T$. Continuing in this way we get a sequence $x_{n_0}, x_{n_0+1}, x_{n_0+2}, \cdots$ of points of T such that $x_{n_0+k} - x_{n_0+k+1} \in \mathfrak{m}^{n_0+k} T$ for all integers $k \geqslant 0$ and so this sequence is a Cauchy sequence.

Since T is finitely generated over a complete local ring R, then T is complete and so there exists an element $x = \lim_{i \to \infty} x_{n_0+i}$. Note that $x \in S_{s(n_0+k)}$ for all $k \geqslant 0$, and so $x \in \bigcap_{i=0}^{\infty} S_{s(n_0+i)}$. Therefore $x = 0$.

And by definition of limit, there exists an N so that for $i \geqslant N$ we have $x_{n_0+i} \in \mathfrak{m}^{n_0} T$. But we have

$$x_{n_0} + \mathfrak{m}^{n_0} T = x_{n_0+1} + \mathfrak{m}^{n_0} T = x_{n_0+2} + \mathfrak{m}^{n_0} T = \cdots = x_{n_0+N} + \mathfrak{m}^{n_0} T = \cdots =$$

and so, since $x_{n_0+N} \in \mathfrak{m}^{n_0} T$ we see that $x_{n_0} \in \mathfrak{m}^{n_0} T$, a contradiction. $\square$

3. Comparison of Topologies

Given a Matlis reflexive module M, choose a finitely generated submodule $T \subset M$ with M/T artinian so that $\{\mathfrak{m}^n T\}$ is a fundamental system of neighbourhoods of 0. Let $S \subset M$ be any finitely generated submodule. Then S has two topologies, $\mathcal{T}$ and $\mathcal{T}'$, both given by systems of neighbourhoods of 0. One is the neighbourhood base $\{\mathfrak{m}^n S\}$ of 0 giving us the topology $\mathcal{T}$, the usual $\mathfrak{m}$-adic topology on S. Another is the induced neighbourhood base $\{\mathfrak{m}^n T \cap S\}$ giving an induced topology $\mathcal{T}'$, when S is thought of as a subspace of M with its natural topology. We first show that these two topologies on S, $\mathcal{T}$ and $\mathcal{T}'$, are the same.

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By Proposition 1, given any n there exists an integer $s(n)$ so that $\mathfrak{m}^{s(n)}S \subset \mathfrak{m}^n T$. Therefore $\mathfrak{m}^{s(n)}S \subset \mathfrak{m}^n T \cap S$. This shows that any set which is open in $\mathcal{T}'$, the induced Matlis topology on S, is also open in $\mathcal{T}$, the $\mathfrak{m}$-adic topology on S. That is $\mathcal{T}' \subset \mathcal{T}$, or $\mathcal{T}$ is finer than $\mathcal{T}'$.

To show that $\mathcal{T}'$ is finer than $\mathcal{T}$ it would be enough to prove the Artin-Rees Lemma, that is, for some n_0,

$$\mathfrak{m}^n(\mathfrak{m}^{n_0}T \cap S) = \mathfrak{m}^{n_0+n}T \cap S \tag{1}$$

for all n. Since given any neighbourhood $\mathfrak{m}^k S$, we certainly have

$$\mathfrak{m}^k(\mathfrak{m}^{n_0}T \cap S) \subset \mathfrak{m}^k S;$$

but then by (1) we would get

$$\mathfrak{m}^{n_0+k}T \cap S \subset \mathfrak{m}^k S.$$

We now prove (1). Since R is noetherian, the graded ring $R^* = \oplus_{n=0}^{\infty}\mathfrak{m}^n$ is also noetherian. And the graded module $T^* = \oplus_{n=0}^{\infty}\mathfrak{m}^n T$ is a finitely generated R^*-module. Now we look at the filtration on S,

$$S \supset T \cap S \supset \mathfrak{m}T \cap S \supset \mathfrak{m}^2 T \cap S \supset \cdots$$

where we set $S_n := \mathfrak{m}^n T \cap S$ for $n \geqslant 0$. Then $\mathfrak{m}S_n \subset S_{n+1}$, and so $S^* := \oplus_{n=0}^{\infty} S_n$ is an R^*-module. In fact, S^* is an R^*-submodule of T^*, and so is a finitely generated R^*-module. Hence [1, Lemma 10.8, p.107] for some n_0 we have $\mathfrak{m}S_n = S_{n+1}$ for all $n \geqslant n_0$. So we have

$$\mathfrak{m}(\mathfrak{m}^{n_0}T \cap S) = \mathfrak{m}^{n_0+1}T \cap S$$

which proves (1). So the two topologies $\mathcal{T}$ and $\mathcal{T}'$ on the finitely generated submodule S are the same.

4. A Reflexive Module is Complete

Suppose now that (x_n) is a Cauchy sequence in M, a Matlis reflexive R-module, where M is given its natural Matlis topology. This means that given any basic neighbourhood S of 0, there exists an n_0 so that $x_n - x_m \in S$ for n, $m \geqslant n_0$. In particular $x_{n_0} - x_m \in S$ for all $m \geqslant n_0$. Hence x_m is in the finitely generated module $S + Rx_{n_0}$ for all $m \geqslant n_0$. So the entire Cauchy sequence (x_n) is in the finitely generated submodule $\widetilde{S} = S + Rx_{n_0} + \cdots + Rx_2 + Rx_1$. We know that this finitely generated module $\widetilde{S}$ is complete with the $\mathfrak{m}$-adic topology, and so by the above it is complete with the induced subspace topology it inherits from M. Since $\widetilde{S}$ is complete the sequence converges. Hence M is complete.

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References

[1] M.F. Atiyah and I.G. Macdonald, *Introduction to commutative algebra* (Addison-Wesley, Reading, MA, 1969).

[2] Belshoff, 'Matlis reflexive modules', *Comm. Algebra* **19** (1991), 1099–1118.

[3] N. Bourbaki, *Commutative algebra* (Addison-Wesley, Reading, MA, 1972).

[4] C. Chevalley, 'On the theory of local rings', *Ann. Math.* **44** (1943), 690–708.

[5] E. Enochs, 'Flat covers and flat cotorsion modules', *Proc. Amer. Math. Soc.* **92** (1984), 179–184..

[6] E. Matlis, 'Injective modules over noetherian rings', *Pacific J. Math.* **8** (1958), 511–528..

[7] H. Matsumura, *Commutative ring theory* (Cambridge University Press, Cambridge, 1986).

[8] M. Nagata, *Local rings* (Interscience, New York, 1962).

Department of Mathematics
Southwest Missouri State University
Springfield MO 64804-0094
United States of America