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The Inequalities for Polynomials and Integration over Fractal Arcs

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Abstract. The paper is dealing with determination of the integral $\int_\gamma f\,dz$ along the fractal arc γ on the complex plane by terms of polynomial approximations of the function f. We obtain inequalities for polynomials and conditions of integrability for functions from the Hölder, Besov and Slobodetskii spaces.

Let γ be directed Jordan arc on the complex plane $\mathbb{C}$ with beginning at the point a and endpoint b. For any polynomial $p(z) = \sum_{j=0}^m p_j z^j$ we put

$$I_\gamma p = \sum_{j=0}^{m} p_j \frac{b^{j+1} - a^{j+1}}{j+1}. \tag{1}$$

If the arc γ is rectifiable, then $I_\gamma p = \int_\gamma p\,dz$ and $|I_\gamma p| \le \lambda \|p\|_{C(\gamma)}$, where $\|p\|_{C(\gamma)} = \sup\{|p(z)| : z \in \gamma\}$ is the norm of polynomial p in the space $C(\gamma)$ of continuous on γ functions, and λ is length of γ. Vice versa, if $|I_\gamma p| \le \lambda \|p\|_X$ for some constant λ and some functional space X containing polynomials, then the functional I_γ is extensible from the set $\mathbb{P}$ of all polynomials onto closure of this set in the space X. The extended functional can be considered as integral over γ even if the arc is not rectifiable. As a result, we determine integral $\int_\gamma f\,dz$ for non-rectifiable and fractal arcs γ. Another way to determinate this integral is described in the papers [6], [4], [5], [3]; see also bibliography in [7].

Thus, we are interested in inequalities of the form

$$|P(b) - P(a)| \le \lambda \|P'\|_X, \tag{2}$$

where P is arbitrary polynomial, the constant λ does not depend on P, and $X = X(\gamma)$ is certain normed space of functions defined on γ such that $\mathbb{P} \subset X$. The best possible value of the constant $\lambda = \lambda_X(\gamma)$ (*i.e.*, norm of the functional I_γ) can be considered as a generalization of the length for non-rectifiable arcs. We call it X-*length*. The following two sections treat the Hölder space as the space X. In the last one we are dealing with certain version of the Besov and Slobodetskii spaces.

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1 Inequalities in the Hölder Space

We begin from the Hölder spaces $H_\nu(\gamma)$. This space consists of defined on γ functions $f(z)$ with finite Hölder coefficient

$$h_\nu(f,\gamma) = \sup\left\{\frac{|f(t') - f(t'')|}{|t'-t''|^\nu} : t', t'' \in \gamma, t' \neq t''\right\}.$$

A norm in that space can be defined as the sum $\|f\|_{H_\nu} = h_\nu(f,\gamma) + \|f\|_{C(\gamma)}$, or $h_\nu(f,\gamma) + |f(a)|$, or $h_\nu(f,\gamma) + |f(b)|$. All these norms are equivalent.

We shall describe integral properties of the arc γ in terms of its box dimension d and growth of the function

$$k_\gamma(z) = \frac{1}{2\pi i}\log\frac{z-b}{z-a} \tag{3}$$

at the points a and b. Let us call to our mind that the box dimension d (see, for instance, [10], [2]) of the plane arc belongs to the interval $[1,2]$. It is equal to 1 if the arc is rectifiable; for fractals this dimension is fractional. The function (3) is a single-valued branch of the infinitely valued function $\frac{1}{2\pi i}\log\frac{z-b}{z-a}$ separated in $\mathbb{C}\setminus\gamma$ so that it vanishes at ∞. In general, its growth at the points a and b can be arbitrarily fast. Obviously, it is related with metric properties of the arc γ. If the arc is rectifiable, then (see [8]) the product $(z-a)(z-b)k_\gamma(z)$ vanishes at the points a, b. If a rectifiable arc satisfies the chord-arc condition, then the growth of corresponding function (3) at these points is logarithmic (see [12]). Obviously, this conclusion is valid for smooth arcs. If γ is not rectifiable, but there exists smooth (or rectifiable and satisfying chord-arc condition) arc γ' with the same beginning and end points a, b and without another common point with γ, then k_γ also has logarithmic growth at a, b, because the difference $|k_\gamma(z) - k_{\gamma'}(z)|$ is characteristical function of domain bounded by $\gamma\cup\gamma'$. We call these arcs *attainable* (*cf.* [4]).

In what follows we restrict the growth of k_γ in terms of condition $k_\gamma \in L^q$. It means that k_γ is integrable with degree q in usual plane measure at some neighborhoods of the points a, b.

Theorem 1 *If the arc γ has the null area and $k_\gamma \in L^1$, then the inequality (2) is valid for $X = H_1(\gamma)$.*

Theorem 2 *If the box dimension of the arc γ is $d < 2$ and $k_\gamma \in L^q$, $q > 1$, then the inequality (2) is valid for $X = H_\nu(\gamma)$ under the condition*

$$1 > \nu > 1 - (1-q^{-1})(2-d). \tag{4}$$

Corollary 1 *If the box dimension of attainable arc γ is $d < 2$, then the inequality (2) is valid for $X = H_\nu(\gamma)$ under the condition*

$$\nu > d - 1. \tag{5}$$

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Proof First we obtain an integral representation for the functional I_γ.

Lemma 1 *Let $\gamma \subset D$ where D is a finite domain with rectifiable Jordan boundary ∂D. Then for any $p \in \mathbb{P}$*

$$I_\gamma p = -\int_{\partial D} p(z) k_\gamma(z)\, dz.$$

As $2\pi i k_\gamma(z) = -\sum_{j=1}^{\infty}(b^j - a^j)/jz^j$ if $|z|$ is sufficiently large, so this equality follows immediately from the residue formula.

The obtained representation does not enable us to bound the functional I_γ as we desire, because relations between the least upper bounds of the polynomial p on ∂D and on γ are rather weak. We need the Stokes formula for improvement of the situation.

Lemma 2 *The Stokes formula*

$$\int_{\partial D} u(\zeta)\, d\zeta = -\iint_D \frac{\partial u}{\partial \overline{\zeta}}\, d\zeta\, d\overline{\zeta} \tag{6}$$

remains valid if the domain D and function $u(\zeta)$ satisfy the following conditions:

- *the domain D is finite and its boundary ∂D consists of finite number of rectifiable Jordan curves;*
- *the function $u(\zeta)$ is continuous in $\overline{D}$, has bounded and continuous partial derivatives of first order in $\overline{D} \setminus \gamma$ and satisfies inequality*

$$|u(\zeta) - u(t)| \le C|\zeta - t|, \quad \zeta \in D, t \in \gamma$$

  for some $C > 0$;
- *the arc $\gamma \subset \overline{D}$ has null area.*

The proof of this proposition reduces to the standard covering of γ by a system of squares with vanishing area.

Now we consider the Whitney extension operator $\mathcal{E}_0$ for the compact γ (see, for instance, [13]). If $f \in H_\nu(\gamma)$ then the extended function $\mathcal{E}_0 f$ is defined on the whole complex plane $\mathbb{C}$, satisfies there the Hölder condition with the same exponent ν, $\sup\{|\mathcal{E}_0 f(z)| : z \in \mathbb{C}\} = \sup\{|f(z)| : z \in \gamma\}$ and $h_\nu(\mathcal{E}_0 f, \mathbb{C}) = h_\nu(f, \gamma)$. In addition, the function $\mathcal{E}_0 f(z)$ is differentiable in $\mathbb{C} \setminus \gamma$, and its gradient does not exceed the value $h_\nu(f,\gamma)\big(\operatorname{dist}(z,\gamma)\big)^{\nu-1}$. In particular, for $\nu = 1$ it is bounded by constant $h_1(f, \gamma)$.

Let us denote $u(z) = \big(p(z) - \mathcal{E}_0 p_\gamma(z)\big) k_\gamma(z)$, where p_γ means restriction of the polynomial p on γ. Obviously, $p_\gamma \in H_1(\gamma)$. Therefore the function $u(z)$ satisfies conditions of Lemma 2 in domain $D' = D \setminus \bigcup_{j=1}^{2} D_j$, where $\gamma \subset D$, D is finite domain with rectifiable Jordan boundary ∂D, and D_1 and D_2 are disks with centers a and b and radia ε and δ respectively. Consequently,

$$\left(\int_{\partial D} - \int_{|\zeta - a| = \varepsilon} - \int_{|\zeta - b| = \delta}\right) u(\zeta)\, d\zeta = -\iint_{D'} \frac{\partial u}{\partial \overline{\zeta}}\, d\zeta\, d\overline{\zeta}.$$

If $k_\gamma \in L_1$, then the right side of the last equality has finite limits for $\varepsilon \to 0$ and for $\delta \to 0$. Hence, there exist limits $\lim_{\varepsilon\to 0}\int_{|\zeta-a|=\varepsilon} u(\zeta)\,d\zeta = c_1$ and $\lim_{\delta\to 0}\int_{|\zeta-b|=\varepsilon} u(\zeta)\,d\zeta = c_2$. The assumption $c_{1,2} \neq 0$ yields a contradiction with the condition $k_\gamma \in L_1$. Thus, the Stokes formula (6) is valid for our function u in the domain D under the additional condition $k_\gamma \in L_1$. This conclusion together with Lemma 1 implies:

Lemma 3 *Let γ be of null area and $k_\gamma \in L_1$. Then*

$$I_\gamma p = -\int_{\partial D} \big(\mathcal{E}_0 p_\gamma(z)\big)\, k_\gamma(z)\, dz - \iint_D \frac{\partial \mathcal{E}_0 p_\gamma}{\partial \bar z} k_\gamma(z)\, dz\, d\bar z \tag{7}$$

for any $p \in \mathbb{P}$ and any finite Jordan domain $D \supset \gamma$ with rectifiable boundary.

The representation (7) enables us to bound the functional I_γ in terms of the Hölder norm. Indeed, it implies inequality

$$|I_\gamma p| \le \|p\|_{C(\gamma)} \int_{\partial D} |k_\gamma(z)\, dz| + h_1(p,\gamma) \iint_D |k_\gamma(z)\, dz\, d\bar z|,$$

which proves Theorem 1.

Furthermore, $\mathbb{P} \subset H_\nu(\gamma)$ for any $\nu \in (0,1]$, and mentioned above properties of the Whitney extension yield the following bound

$$|I_\gamma p| \le \|p\|_{C(\gamma)} \int_{\partial D} |k_\gamma(z)\, dz| + h_\nu(p,\gamma) \iint_D \big(\operatorname{dist}(z,\gamma)\big)^{\nu-1} |k_\gamma(z)\, dz\, d\bar z|. \tag{8}$$

Easy calculation shows that the function $\operatorname{dist}^{-\mu}(z,\gamma)$ is integrable in finite domain $D \supset \gamma$ if $\mu < 2 - d$ (see details in [9]). This fact together with the Hölder inequality means that the last integral in (8) is finite under condition (4). Theorem 2 is proved.

Corollary 1 follows immediately from Theorem 2, because the function k_γ for attainable arc γ satisfies condition $k_\gamma \in L^q$ for arbitrarily large q.

2 The Hölder Length

The results of the preceding section enables us to bound the Hölder length in terms of the function (3). Here we estimate it in another way.

We restrict ourself by the set $\mathbb{P}_0 = \{p \in \mathbb{P} : p(a) = 0, p \not\equiv 0\}$ and define the Hölder length by equality

$$\lambda_\nu(\gamma) = \sup\left\{\frac{|I_\gamma p|}{h_\nu(p,\gamma)} : p \in \mathbb{P}_0\right\}.$$

Definition 1 An arc γ is called *q-rectifiable* if the value

$$\sigma_q(\gamma) = \sup_Z \sum_{j=1}^n |z_j - z_{j-1}|^q$$

is finite; the least upper bound is taken over all finite sequences $Z = \{z_j\}_{j=1}^n \subset \gamma$ enumerated in intrinsic order.

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Let us denote the class of all q-rectifiable arcs by R_q. Obviously, R_1 consists of usual rectifiable arcs; for $s < t$ we have $R_s \subset R_t$ and this inclusion is strict.

Lemma 4 *The following propositions are valid:*

i. *the box dimension d of a q-rectifiable arc γ does not exceed q;*
ii. *an arc γ is q-rectifiable if and only if there exists a one-to-one mapping $z(x)$ of segment $I = [0,1]$ onto γ belonging to the Hölder space $H_{1/q}(I)$;*
iii. *the mapping $z(x)$ can be chosen so that $h_{1/q}(z, I) = \sigma_q^{1/q}(\gamma)$.*

Proof In order to prove the first proposition of the theorem, we fix $\delta > 0$ and divide the complex plane into grid of squares with mesh δ. Let $N(\delta)$ be the number of squares Q such that the intersection $Q \cap \gamma$ is not empty. Now we consider a special sequence of points. We put $z_0 = a$ and define z_1 as the first point on γ (let us call to mind that the arc γ is directed from a to b) satisfying condition $|z_1 - z_0| = \delta$; if that point does not exist then we put $z_1 = b$. Analogously, z_2 is the next after z_1 point of the set $\{z \in \gamma : |z_2 - z_1| = \delta\}$, and $z_2 = b$ if this set is empty, and so on. As a result we obtain a sequence $Z = \{z_j\}_{j=1}^m$ such that $|z_j - z_{j-1}| = \delta$ for $j = 1, \dots, m-1$, and $|z_m - z_{m-1}| \le \delta$. Hence, $(m-1)\delta^q < \sum_{j=1}^m |z_j - z_{j-1}|^q \le \sigma_q(\gamma)$ and $m < 1 + \sigma_q(\gamma)\delta^{-q}$. On the other hand, any subarc of γ with beginning z_{j-1} and end z_j, $j = 1, \dots, m$, is contained in a disk of radius δ which intersects no more than 12 squares. Thus, $N(\delta) \le 12m < c + c\delta^{-q}$, where the constant c does not depend on δ, and $d = \limsup \frac{\log N(\delta)}{-\log \delta} \le q$ by definition of the box dimension. The proposition (i) is proved.

If a function $z(x) \in H_{1/q}(I)$ is one-to-one mapping of the unit segment I onto γ, then any point $z_j \in \gamma$ is image of point $x_j \in I$, $0 = x_0 < x_1 < \cdots < x_n = 1$, and, consequently, $\sum_{j=1}^n |z_j - z_{j-1}|^q \le \sum_{j=1}^n h_{1/q}^q(z, I)|x_j - x_{j-1}| = h_{1/q}^q(z, I)$. Thus, $\gamma \in R_q$ and $\sigma_q(\gamma) \le h_{1/q}^q(z, I)$.

Now let $\gamma \in R_q$. We consider the subarc γ_t of the arc γ with beginning a and end $t \in \gamma$. The function $\sigma(t) = \sigma_q(\gamma_t)$ continuously increases if the point t runs from a to b. This function maps γ on the segment $[0, \sigma(b)]$ where $\sigma(b) = \sigma_q(\gamma)$. Then the function $x = \sigma_0(t) = \sigma(t)/\sigma(b)$ is a homeomorphism of γ onto I, and its inverse function $z_0(x)$ maps I onto γ. According to Definition 1 we have $|\sigma(t') - \sigma(t)| \ge |t' - t|^q$. Therefore the inverse function $z_0(x) \colon I \mapsto \Gamma$ satisfies the inequality $|z_0(x) - z_0(x')| \le \big(\sigma(b)|x - x'|\big)^{1/q}$, *i.e.*, it belongs $H_{1/q}(I)$ and $h_{1/q}(z_0, I) \le \sigma_q^{1/q}(\gamma)$. But we have proved above the inverse inequality for any Hölder mapping $z \colon I \mapsto \gamma$. Hence, $h_{1/q}(z_0, I) = \sigma_q^{1/q}(\gamma)$, and the lemma is proved.

Theorem 3 *If the arc γ is q-rectifiable, $1 < q < 2$, then the inequality (2) is valid for $X = H_\nu(\gamma)$ under condition*

$$\nu > q - 1, \tag{9}$$

and its Hölder length satisfies the following inequality

$$\lambda_\nu(\gamma) \le \left(1 + \zeta\left(\frac{\nu+1}{q}\right)\right) \sigma_q^{(\nu+1)/q}(\gamma),$$

where $\zeta(x)$ is the Riemann ζ-function.

Proof If a function f is defined on the segment I then the value $\nu_q(f) = \sup_\chi \sum_{j=1}^n |f(x_{j-1}) - f(x_j)|^q$ is its q-variation (the least upper bound is taken over all finite sequences $\chi = \{x_j\}_{j=0}^n$ such that $0 = x_0 < x_1 < \cdots < x_n = 1$). The class $V_q(I)$ consists of all functions with finite q-variations. The Stieltjes integral $\int_0^1 f\,dg$ exists if $g \in V_q(I)$, $f \in V_r(I)$, $q^{-1} + r^{-1} > 1$, and the functions f, g have no common singularities (L. C. Young [14]). If, in addition, $f(0) = 0$, then $|\int_0^1 f\,dg| \le \bigl(1 + \zeta(q^{-1} + r^{-1})\bigr)\,\nu_r^{1/r}(f; I)\nu_q^{1/q}(g; I)$. Obviously, any continuous one-to-one mapping $z(x)$ of the segment I onto q-rectifiable arc γ belongs to the class $V_q(I)$ and $\nu_q(z) = \sigma_q(\gamma)$. For any function $F \in H_\nu(\gamma)$ we have $\sum_{j=1}^n \bigl|F\bigl(z(x_{j-1})\bigr) - F\bigl(z(x_j)\bigr)\bigr|^{q/\nu} \le h_\nu^{q/\nu}(F, \gamma) \sum_{j=1}^n |z(x_{j-1}) - z(x_j)|^q \le h_\nu^{q/\nu}(F, \gamma)\sigma_q(\gamma)$, *i.e.*, $F\bigl(z(x)\bigr) \in V_{q/\nu}(I)$ and $\nu_{q/\nu}(F \circ z) \le h_\nu^{q/\nu}(F, \gamma)\sigma_q(\gamma)$. Hence, the integral $\int_0^1 F\bigl(z(x)\bigr)\,dz(x)$ exists under condition (9) and $\bigl|\int_0^1 F\bigl(z(x)\bigr)\,dz(x)\bigr| \le 1 + \zeta(\frac{1+\nu}{q})h_\nu(F, \gamma)\sigma_q^{(1+\nu)/q}(\gamma)$ if $F(a) = 0$. Hence, the Stieltjes integral $\int_0^1 p\bigl(z(x)\bigr)\,dz(x)$ exists for any $p \in \mathbb{P}$ and $\bigl|\int_0^1 p\bigl(z(x)\bigr)\,dz(x)\bigr| \le 1 + \zeta(\frac{1+\nu}{q})h_\nu(p, \gamma)\sigma_q^{(1+\nu)/q}(\gamma)$ for any $p \in \mathbb{P}_0$ and any $\nu > q - 1$. It remains to show that $\int_0^1 p\bigl(z(x)\bigr)\,dz(x) = I_\gamma p$. Let us put $z_j = z(j/n)$, $j = 0, 1, \ldots, n$, and $S_n(p) = \sum_{j=1}^n p(z_{j-1})(z_j - z_{j-1})$. Obviously, $I_\gamma p = \sum_{j=1}^n \int_{l(j)} p(z)\,dz$, where $l(j)$ stands for the linear segment $[z_{j-1}, z_j]$, and $|I_\gamma p - S_n(p)| \le \sum_{j=1}^n |\int_{l(j)} p(z) - p(z_{j-1})| \le h_1(p, U) \sum_{j=1}^n |z_j - z_{j-1}|^2 \le h_1(p, U)\delta_n^{2-q}\sigma_q(\gamma)$, where U is closed convex hull of γ and δ_n is maximal of differences $|z_j - z_{j-1}|$, $j = 1, 2, \ldots, n$. Consequently, $\lim_{n\to\infty} S_n(p) = I_\gamma p$. But $S_n(p)$ is a integral sum for the Stieltjes integral $\int_0^1 p\bigl(z(x)\bigr)\,dz(x)$. As this integral exists,

$$I_\gamma p = \int_0^1 p\bigl(z(x)\bigr)\,dz(x), \tag{10}$$

and the theorem is proved.

Note 1 We can construct arcs of box dimension $d < 2$ which are q-rectifiable only for $q > 2$ or for no one q at all. Hence, the q-rectifiability is a sufficient but not necessary condition for finiteness of the Hölder length.

Note 2 L. C. Young [15] obtained an existence theorem for the Stieltjes integral which generalized the result from [14] applied above. By means of this generalization we are able to prove a version of Theorem 3 for the spaces of functions with prescribed modulus of continuity.

The conditions of Theorems 1, 2 and 3 ensure extendability of the functional I_γ onto closures of $\mathbb{P}$ in corresponding Hölder spaces. In other words, we have obtained sufficient conditions for integrability of functions from these closures along the arc γ. But the Hölder spaces are not separable ones. Consequently, the closures of $\mathbb{P}$ in these spaces cannot coincide with the whole spaces, and these conditions are weaker than analogous integrability conditions in the papers [6], [4], [5], [3], [7]. But we have to note that unlike our theorems the results of these papers (excluding [3] and [7]) concern only closed curves and attainable arcs. In addition, the representation (7) can be considered as explicit formula for extension of I_γ onto the whole Hölder space: we must only replace there $p \in \mathbb{P}$ by $f \in H_\nu(\gamma)$. The Young-Stieltjes integral representation (10) is applicable to this end, too.

3 Inequalities in the Besov Spaces

The Besov space $B^\alpha_{r,1}$ on the segment I consists of functions $f \in L^r$ satisfying condition

$$\int_0^1 s^{-\alpha-1}\omega_r(f;s)\,ds = N_{r,\alpha}(f;I) < \infty,$$

where

$$\omega_r(f;s) = \left(\int_0^{1-s} |f(x+s) - f(x)|^r\,dx\right)^{1/r}, \quad 0 < s \leq 1.$$

The value $N_{r,\alpha}(f;I)$ is seminorm of the space. An equivalent seminorm is defined by equalities

$$\tilde{\omega}_r(f;s) = \sup\{\omega_r(f;t) : 0 < t \leq s\}, \quad \tilde{N}_{r,\alpha}(f;I) = \int_0^1 s^{-\alpha-1}\tilde{\omega}_r(f;s)\,ds.$$

The sum $\tilde{N}_{r,\alpha}(f;I) + \|f\|_{L^r}$ is a norm in the space $B^\alpha_{r,1}$. The class $B^\alpha_{1,1}$ coincides with the Slobodetskii space W^α_1 on the segment I.

Let γ be q-rectifiable arc. We consider functions $\sigma(t)$, $\sigma_0(t)$ and $z_0(x)$ which are constructed for that arc in the proof of Lemma 4. The function $\sigma(t)$ increases if the point t runs along γ from a to b. Hence, it determines the measure $d\sigma$ on γ. Furthermore, the equality $\sigma_0(\tau_s t) = \sigma_0(t) + s$ determines the translation $\tau_s t$ along γ. It is defined for $t \in \gamma^{1-s}$, where subarc $\gamma^{1-s} \subset \gamma$ begins at the point a and ends at $z_0(1-s)$.

Now we can define the Besov space $B^\alpha_{r,1}(\gamma)$ as the set of all functions $f \in L^r(d\sigma;\gamma)$ with finite seminorm

$$\tilde{N}_{r,\alpha}(f;\gamma) = \int_0^1 s^{-\alpha-1}\tilde{\omega}_{r,\gamma}(f;s)\,ds,$$

where

$$\tilde{\omega}_{r,\gamma}(f;s) = \sup\{\omega_{r,\gamma}(f;t) : 0 < t \leq s\}, \;\; \omega_{r,\gamma}(f;s) = \left(\int_{\gamma^{1-s}} |f(\tau_s t) - f(t)|^r\,d\sigma_t\right)^{1/r}.$$

The value $\|f\|_{B^\alpha_{r,1}(\gamma)} = \tilde{N}_{r,\alpha}(f;\gamma) + \|f\|_{L^r(d\sigma)}$ is a norm in the space $B^\alpha_{r,1}(\gamma)$.

Obviously, $f \in B^\alpha_{r,1}(\gamma)$ if and only if $f\bigl(z_0(x)\bigr) \in B^\alpha_{r,1}(I)$. By virtue of Lemma 4 $z_0(x) \in H_{1/q}(I)$. Therefore $|\tau_s t - t| \leq \bigl(s\sigma_q(\gamma)\bigr)^{1/q}$ and, consequently, $\mathbb{P} \subset B^\alpha_{r,1}(\gamma)$ for $q\alpha < 1$.

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V. I. Matsaev and M. Z. Solomyak [11] proved that the integral $\int_0^1 f\,dg$ exists for $f \in B^{\alpha}_{1,1}$, $g \in H_{1-\alpha}(I)$, $0 < \alpha < 1$ and for $f \in B^{\alpha}_{1/\alpha,1}$, $g \in V_{(1-\alpha)^{-1}}$ as a limit of sums

$$\sum_{j=1}^{n} \frac{g(x_j) - g(x_{j-1})}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} f(x)\,dx$$

(a special case of so-called *Hellinger integral*). Thus, the integral $\int_0^1 p\big(z_0(x)\big)\,dz_0(x)$ exists in the Hellinger sense if $1 - \alpha = q^{-1}$ and $q\alpha < 1$. These conditions are compatible for $q < 2$. If a function f is integrable on the segment I in the Riemann sense and the Hellinger integral $\int_0^1 f\,dg$ exists, then this integral exists in the Stieltjes sense and its Hellinger and Stieltjes values are equal (see [1]). We have proved in the previous section that the Stieltjes value of the integral $\int_0^1 p\big(z_0(x)\big)\,dz_0(x)$ is equal to $I_\gamma p$. Thus, the representation (10) is valid for $q < 2$ in the Hellinger sense and we can apply here the bounds for the Hellinger integral from [11]. These bounds yield the following result:

Theorem 4 *If the arc γ is q-rectifiable, $1 < q < 2$ and $\alpha = 1 - q^{-1}$, then the inequality (2) is valid for $X = B^{\alpha}_{1,1}(\gamma)$ and for $X = B^{\alpha}_{1/\alpha,1}(\gamma)$, and in both these cases the Besov length of γ (i.e., the constant λ in the inequality (2)) does not exceed $\frac{\alpha}{1-2^{-\alpha}}\sigma_q^{1/q}(\gamma)$.*

This proposition establishes existence of the integral $\int_\gamma f\,dz$ along q-rectifiable arc γ for $f \in B^{1/q'}_{1,1}(\gamma)$ and for $f \in B^{1/q'}_{q',1}(\gamma)$ (as usual, q' stands here for $(1-q^{-1})^{-1}$). The first case is of special interest because the space $B^{\alpha}_{1,1}$ contains discontinuous and unbounded functions, and in general the integral $\int_\gamma f\,dz$ cannot exist in the Riemann-Stieltjes sense. As shown above, its evaluation is possible by terms of a polynomial approximation.

For example, if γ is an arc of the von Koch snowflake, then its box dimension d is $\log_3 4$. One can easily verify that it is q-rectifiable for $q = \log_3 4$, too. Therefore the integral $\int_\gamma f\,dz$ exists for any function $f \in B^{\alpha}_{1,1}(\gamma)$, $\alpha = \log_4 \frac{4}{3}$, as the limit of integrals of polynomials approximating f in this space.

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