INVERSION OF ASYMPTOTIC GRAVITY-MODE FREQUENCIES AND ITS APPLICATION TO THE SUN

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ABSTRACT. Solar gravity modes can be used to probe the stratification below the convection zone. An asymptotic formula for high-order gravity-mode frequencies has been obtained in terms of a stratification parameter β. The domain of validity of this formula is investigated by least-squares regression of frequencies obtained from a theoretical model. Random noise is introduced to simulate real observations. An estimate is made of the critical noise-level above which β cannot be inferred.

The frequencies σ of adiabatic gravity modes of high order n and low degree ℓ in the Cowling approximation (1) satisfy the following asymptotic relation (2,3,4):

$$\frac{L}{\sigma} = \alpha \left\{ n + \frac{1}{2}\ell - \frac{1}{4} - \frac{1}{2}\nu + \frac{1}{\pi}\tan^{-1}A + \frac{P_\ell}{2\pi}\left(\frac{\sigma}{L}\right) \right\} \tag{1}$$

where

$$L^2 = \ell(\ell+1)$$

$$A = \frac{\Xi_{\sigma\ell}\sin\nu\pi}{(L/\sigma)^{2\nu} + \Xi_{\sigma\ell}\cos\nu\pi}$$

$$P_\ell = L^2 P_1 + P_2.$$

The quantities α, ν, P_1, P_2 and $\Xi_{\sigma\ell}$ depend on the equilibrium structure. $2\pi\alpha$ is the familiar asymptotic period spacing. P_1 and P_2 are weighted integrals of the buoyancy frequency in the radiative interior. $\Xi_{\sigma\ell}$ and ν depend on conditions at the base of the convection zone. In particular, ν is related to the exponent, β, of the depth below the convection zone with which the polytropic index pulls away from its adiabatic value (see Figure 1), thus: $\nu = 1/(\beta+2)$. If the stratification in the convection zone is adiabatic throughout then

$$\Xi_{\sigma\ell} = L^2\Xi_1 + \Xi_2$$

where Ξ_1 and Ξ_2 are very weakly dependent on σ. Such a model is considered here. The case of a real solar model with a non-adiabatic

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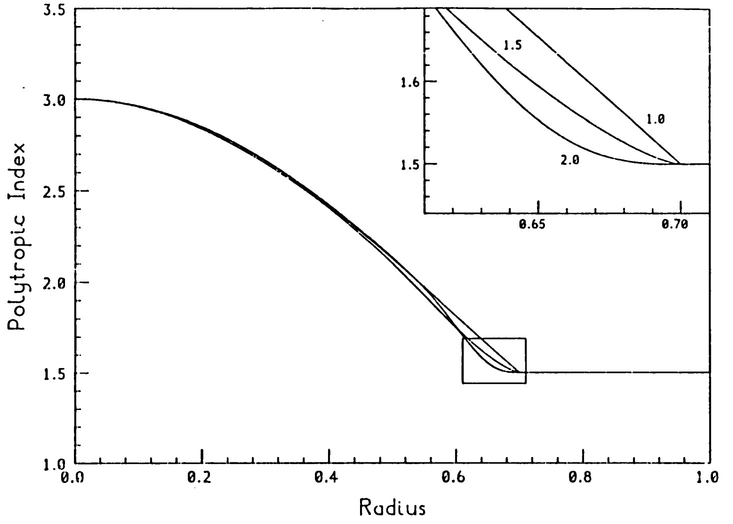

Figure 1. Polytropic indices of three solar models. The difference between the polytropic index and its adiabatic value (1.5) is proportional to the depth below the convection zone raised to some power β. Each curve in the inset is labelled by its value of β. It is thus seen that β is a measure of the smoothness of the stratification at the convective interface. Only the β=1 model is discussed in the text.

region at the top of the convection zone has been investigated by Provost and Berthomieu (4). They find $\Xi_{n\ell}$ depends significantly on frequency. It is to be hoped that this dependence can be accurately modelled.

The asymptotic domain of σ must be determined by numerical means. This involves perturbing a solar model to obtain numerical frequencies. The parameters α to Ξ_2 are calculated from the model, and the frequencies compared with those obtained by substituting these parameters into equation (1). The models used are shown in Figure 1. Their structure is very simple because it is essentially the small region just below the convection zone that is of interest. Presented here is the analysis of the β = 1 case.

Figure 2 shows the error between the numerical frequencies and the asymptotic frequencies calculated from the model. The asymptotic parameters are shown in the last row of Table I. Evidently the frequencies agree with the asymptotic formula to an absolute tolerance of 0.01 for $n+\frac{1}{2}\ell \gg 10$. The apparently good agreement in the range $5 < n+\frac{1}{2}\ell < 8$ is accidental, and arises because the error passes through a zero.

A non-linear least-squares regression was performed on the numerical frequencies over various ranges in order to estimate the asymptotic parameters. A summary is given in Table I. A study of this table will show that the parameters are best estimated when $n+\frac{1}{2}\ell > 10$. Over this range the spread in the estimate of α is a tiny 0.01%, and of ν, P_1 and Ξ_1 it is only a few percent. It should be understood that, owing to the absence of higher order correction terms, these estimates will not coincide exactly with the true values.

The frequencies measured on the sun are subject to observational errors. I have attempted to model this error in order to investigate

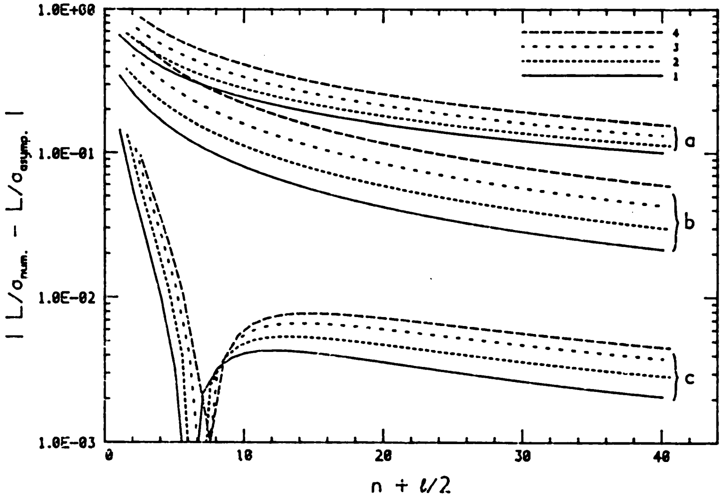

Figure 2. Deviation of frequencies from asymptotic behaviour. The degree of the mode is shown in the key at the top right-hand corner. The figure is in three parts. It shows the deviation of the numerical frequencies from formula (1) employing: (a) only the first three terms inside the curly brackets; (b) the first five terms; (c) the full expression.

the effect of noise on the estimates of the asymptotic parameters. The numerical frequencies were multiplied by a Gaussian distribution of mean 1 and standard deviation ranging optimistically from 10^{-6} to 10^{-2}. A regression was then performed for each value of the standard deviation on those frequencies with $20 \leq n+\frac{1}{2}\ell \leq 30$. This range was chosen because the agreement between estimated and expected asymptotic parameters was especially close. The results are shown in Table II. It is seen that, although the estimate of α is fairly insensitive to the amount of noise present, the higher-order parameters are faithfully estimated only below a noise level of about 10^{-4}. If we take the 160 minute oscillation as an example of a gravity mode, this must be observed over at least 3 years in order to yield its frequency to such an accuracy to provide information on the stratification just below the convection zone.

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range	size	α	ν	P_1	P_2	Ξ_1	Ξ_2	rms dev.
20 - 41	86	1.34851	0.3286	0.510	4.3	0.060	3.20	1.5 (-4)
15 - 35	82	1.34874	0.3281	0.506	4.1	0.062	3.28	1.6 (-4)
10 - 30	82	1.34871	0.3547	0.489	2.7	0.076	4.59	1.7 (-4)
5 - 25	82	1.34944	0.3940	0.470	1.7	0.105	6.27	6.2 (-4)
1 - 20	74	1.35074	0.4323	0.417	1.4	0.161	7.42	4.4 (-3)
30 - 41	46	1.34860	0.3361	0.481	4.8	0.068	3.30	1.3 (-4)
25 - 35	42	1.34858	0.3382	0.504	3.9	0.065	3.65	1.5 (-4)
20 - 30	42	1.34854	0.3336	0.497	4.1	0.066	3.43	1.4 (-4)
15 - 25	42	1.34848	0.3328	0.498	3.7	0.067	3.54	1.3 (-4)
10 - 20	42	1.34839	0.3372	0.491	3.2	0.074	3.92	1.2 (-4)
5 - 15	42	1.35103	0.4313	0.461	1.1	0.121	7.46	5.0 (-4)
1 - 10	34	1.35280	0.4564	0.398	1.3	0.199	7.49	5.7 (-3)
MODEL	-	1.34856	0.3333	0.504	3.8	0.054	3.35	-

Table I. Estimates of the asymptotic parameters by regression on the numerical frequencies over different ranges of $n+\frac{1}{2}\ell$. The expected values, as calculated directly from the model, are shown in the last row, accurate to the number of decimal places given. The second column gives the number of data points in the regression. The final column shows the minimised root-mean-square residue.

noise	α	ν	P_1	P_2	Ξ_1	Ξ_2	error
-6.0	1.34851	0.3303	0.495	4.3	0.065	3.28	-3.8
-5.5	1.34856	0.3306	0.494	4.6	0.064	3.16	-3.8
-5.0	1.34852	0.3295	0.489	4.4	0.066	3.20	-3.5
-4.5	1.34873	0.3330	0.546	5.0	0.047	3.03	-3.0
-4.0	1.34910	0.3357	0.702	5.1	0.001	2.80	-2.5
-3.5	1.34277	0.2416	0.664	-12.4	1.392	40.98	-1.9
-3.0	1.35785	0.6945	-2.188	8.9	4.960	26.75	-1.4
-2.5	1.35698	0.6616	-2.017	-16.4	5.312	50.82	-1.0
-2.0	1.34630	0.3858	-0.100	-29.0	4.165	92.71	-0.4
MODEL	1.34856	0.3333	0.504	3.8	0.054	3.35	-

Table II. Estimates of the asymptotic parameters by regression on the numerical frequencies with random noise. The frequencies were multiplied by a Gaussian distribution with mean 1 and standard deviation equal to 10^{noise}, which is given in the first column. The regression was performed over the range $20 \leqslant n+\frac{1}{2}\ell \leqslant 30$ on 39 data points. The last row of Table I is repeated here. The minimised root-mean-square residue is 10^{error}, as given in the last column.