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GROUPS WITH FINITE DIMENSIONAL IRREDUCIBLE MULTIPLIER REPRESENTATIONS

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1. Introduction. Let G be a locally compact group and ω a normalized multiplier on G. Denote by $V(G)$ (respectively by $V(G, \omega)$) the von Neumann algebra generated by the regular representation (respectively ω-regular representation) of G. Kaniuth [**6**] and Taylor [**14**] have characterized those G for which the maximal type I finite central projection in $V(G)$ is non-zero (respectively the identity operator in $V(G)$).

In this paper we determine necessary and sufficient conditions on G and ω such that the maximal type I finite central projection in $V(G, \omega)$ is non-zero (respectively the identity operator in $V(G, \omega)$) and construct this projection explicitly as a convolution operator on $L^2(G)$. As a consequence we prove the following statements are equivalent.

(i) $V(G, \omega)$ is type I finite,

(ii) all irreducible multiplier representations of G are finite dimensional,

(iii) G^ω (the central extension of G) is a Moore group, that is all its irreducible (ordinary) representations are finite dimensional.

A few interesting corollaries result.

Note that in the case where the multiplier ω is trivial, these results reduce to results about ordinary representations that are well known.

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2. Notation. For the basic definitions and classification of von Neumann algebras see [**12**]. Given a locally compact group G with normalized multiplier ω ([**1**]) and subgroup H, we adopt the following symbols consistently throughout the paper.

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H^- : closure of H in G
H' : commutator subgroup of H
G^ω : central extension with Weil topology (see [**8**], p. 218)
h : canonical projection $G^\omega \to G^\omega/\mathbf{T} = G$
ρ : left ω-regular representation of G

$$((\rho(g)f)(x) = \omega(g^{-1}, x)f(g^{-1}x), g, x \in G, f \in L^2(G))$$

$V(G, \omega)$: von Neumann algebra generated by ρ
$V(G)$: von Neumann algebra generated by the left regular (ordinary) representation of G
e_G : maximal type I finite central projection in $V(G)$
d_G : maximal type I finite central projection in $V(G, \omega)$
G_0 : von Neumann kernel $= \cap \{\ker \pi: \pi$ is a finite dimensional representation of $G\}$
Δ_G : topological finite class group of G = union of conjugacy classes whose closure is compact.

In the situation where it will be clear from the context which G is in question, the subscript G will be dropped from the symbols e_G, d_G and Δ_G. All isomorphisms between von Neumann algebras mentioned in this paper are spatial.

3. Preliminaries. Let G be a locally compact group. Kaniuth [**6**] and Taylor [**14**] have proved the following theorem about e_G.

THEOREM 3.1. ([**6**, **14**]). *For a locally compact group G,*
(i) *$e_G \neq 0$ implies G_0 is compact and $e_G V(G)$ is spatially isomorphic to $V(G/G_0)$*
(ii) *$e_G \neq 0$ if and only if $[G:\Delta_G] < \infty$ and $(\Delta_G)'^-$ is compact.*

Furthermore, the results of Kaniuth [**6**] and Robertson [**11**] combine to yield the following result.

THEOREM 3.2. ([**6**, **11**]). *The following statements are equivalent*:
(i) *$e_G = I$ (the identity operator)*
(ii) *$[G:\Delta_G] < \infty$, $(\Delta_G)'^-$ is compact and $G_0 = \{e\}$*
(iii) *G is a Moore group, that is all its irreducible (ordinary) representations are finite dimensional.*

For the remainder of this section we shall fix a locally compact group G with normalized multiplier ω.

LEMMA 3.3. *Let A be a subset of G^ω, then*
(i) *A^- is compact if and only if $h(A)^-$ is compact*
(ii) *if A^- is compact, then $h(A)^- = h(A^-)$.*

The proof of this result is straightforward.

COROLLARY 3.4. (i) $(\Delta_G)^\omega = \Delta_{G^\omega}$.

(ii) *If one of* $(\Delta_G)'^-$ *and* $(\Delta_{G^\omega})'^-$ *is compact, then so is the other and* $(\Delta_G)'^- = h[(\Delta_{G^\omega})'^-]$.

Proof. For (i), let A equal the conjugacy class of some $(\lambda, x) \in G^\omega$ and in (ii), let $A = (\Delta_{G^\omega})'$. Now use Lemma 3.3.

Combining 3.1 and 3.4 gives $e_G \neq 0$ if and only if $e_{G^\omega} \neq 0$. But this follows already from [**14**, Proposition 5.2].

Let E_n, $n \in Z$ be the maps on $L^2(G^\omega)$ given by

$$E_n f(\mu, x) = \mu^n \int_{\mathbf{T}} \alpha^{-n} f(\alpha, x) d\alpha,$$

for almost all $(\mu, x) \in G^\omega, f \in L^2(G^\omega)$. (Compare this with [**7**, page 563].) Clearly $E_n L^2(G^\omega)$ consists of all those functions $f \in L^2(G^\omega)$ such that

$$f(\mu, x) = \mu^n f(l, x) \text{ for almost all } (\mu, x) \in G^\omega.$$

It follows that the E_n, $n \in \mathbf{Z}$ are mutally orthogonal idempotents. For $n \in \mathbf{Z}$, E_n is just convolution by the measure obtained, if you multiply the measure on G^ω supported on $\mathbf{T}$ which restricts to Haar measure on $\mathbf{T}$, with the character χ_n of $\mathbf{T}$ given by

$$\chi_n(\lambda) = \lambda^n, \ n \in \mathbf{Z}, \lambda \in \mathbf{T}.$$

It is easy to check that E_n commutes with the right and left regular representation of G^ω, so by [**13**, Theorem 3] E_n is in the centre of $V(G^\omega)$.

THEOREM 3.5. *With the above notation, the* E_n, $n \in \mathbf{Z}$ *are mutually orthogonal central projections in* $V(G^\omega)$ *and*

(i) *The three von Neumann algebras* $E_n V(G^\omega)$, $V(G, \omega^n)$, $V(\chi_{-n} \uparrow_{\mathbf{T}}^{G^\omega})$ (*the non Neumann algebra generated by the induced representation* $\chi_{-n} \uparrow_{\mathbf{T}}^{G^\omega}$) *are spatially isomorphic.*

(ii) $\sum_{n \in \mathbf{Z}} E_n = I$ (*the identity operator*).

Proof. (i) Let τ denote the left regular representation of G^ω and ρ_n, $n \in \mathbf{Z}$, the left regular ω^n-representation of G. Fix n. Observe that the representation space of $\chi_{-n} \uparrow_{\mathbf{T}}^{G^\omega}$ is just $E_n L^2(G^\omega)$ and that

$$E_n \tau = \chi_{-n} \uparrow_{\mathbf{T}}^{G^\omega}.$$

It follows that

$$E_n V(G^\omega) = V(\chi_{-n} \uparrow_{\mathbf{T}}^{G^\omega}).$$

The map

$$\phi: E_n L^2(G^\omega) \to L^2(G)$$

defined by $\phi(f)(x) = f(1, x)$ is an isometry (see [**3**]).

The spatial monomorphism

$$E_n V(G^\omega) \to B(L^2(G)): T \to \phi \cdot T \cdot \phi^{-1}$$

is weakly continuous and maps $E_n \tau(1, x)(x \in G)$ to $\rho_n(x)$. Observe that the von Neumann algebra generated by $\{E_n \tau(1, x): x \in G\}$ is precisely $E_n V(G^\omega)$, thus the range of ϕ is contained in $V(G, \omega^n)$. Furthermore this range is weakly closed ([**12**, 1.16.2]) and contains the operators $\rho_n(x)$, $x \in G$ and thus must be equal to $V(G, \omega^n)$.

(ii) Since $\bigoplus_{n \in \mathbf{Z}} \chi_n$ is the regular representation of $\mathbf{T}$, we infer that

$$\bigoplus_{n \in \mathbf{Z}} (\chi_n \uparrow_{\mathbf{T}}^{G^\omega}) = \left(\bigoplus_{n \in \mathbf{Z}} \chi_n\right) \uparrow_{\mathbf{T}}^{G^\omega}$$

is the regular representation of G^ω, hence

$$V(G^\omega) = \bigoplus_{n \in \mathbf{Z}} V(\chi_n \uparrow_{\mathbf{T}}^{G^\omega}),$$

hence the result.

LEMMA 3.6. *Suppose $e_G \neq 0$ (or equivalently $e_{G^\omega} \neq 0$) and that G has a finite dimensional ω-representation π, then $K = h((G^\omega)_0)$ is compact and ω is similar to a multiplier which is lifted from G/K.*

Proof. That K is compact follows from 3.1. Suppose $(\lambda, k) \in (G^\omega)_0$, then since $(\mu, g) \to \mu\pi(g)$, $(\mu, g) \in G^\omega$ is a finite dimensional (ordinary) representation of G^ω, we have $I = \lambda\pi(k)$, that is $k \in p$-ker π. The result now follows from [**1**, Lemma 1.3].

4. The main theorems.

THEOREM 4.1. *Let G be a locally compact group with normalized Borel multiplier ω. Adopt the notation of Section 2, then the following three conditions are equivalent.*

(i) *$e_G \neq 0$ (or equivalently $e_{G^\omega} \neq 0$) and there exists a finite dimensional ω-representation π of G.*

(ii) *$d_G \neq 0$.*

(iii) *$[G:\Delta_G] < \infty$, $(\Delta_G)'^-$ is compact and G has a finite dimensional ω-representation π.*

Note if ω is a trivial multiplier then ω is of the form

$$\omega(x, y) = \gamma(x)\gamma(y)/\gamma(xy)$$

and thus there automatically exists a finite dimensional ω-representation of G. In this case the theorem reduces to 3.1 (ii). See also the comments preceding Example 5.1.

Proof. (i) and (iii) are obviously equivalent using 3.1 (ii). Suppose (ii) is true and let π' be a non-degenerate finite dimensional algebra representation of $dV(G, \omega)$, then

$$g \rightarrow \pi'(d\rho(g^{-1}))^*$$

is a finite dimensional ω-representation of G. By Theorem 3.5, $e_{G^\omega} \neq 0$. This shows (ii) implies (i); that (i) implies (ii) will be proved together with Theorem 4.3.

LEMMA 4.2. *Suppose K is a compact normal subgroup of G and ω is lifted from a multiplier ω' on G/K, then K is also a compact normal subgroup of G^ω and G^ω/K is topologically isomorphic to $(G/K)^{\omega'}$.*

The proof of this result follows easily from the definition of group extensions.

THEOREM 4.3. *Suppose G is a locally compact group and d_G is non-zero. We assume (using Lemma 3.6) that ω is lifted from a multiplier ω' of G/K, where K is the compact normal subgroup $K = h((G^\omega)_0)$. Then d_G is the operator $L^2(G) \rightarrow L^2(G)$ defined by*

$$d_G f(x) = \int_K f(k^{-1}x)d\lambda(k), \quad \textit{almost all } x \in G, f \in L^2(G),$$

where λ is Haar measure on G normalized such that $\lambda(K) = 1$. Furthermore, for each $n \in \mathbf{Z}$, d_G is the maximal type I finite central projection in $V(G, \omega^n)$ and

$$d_G V(G, \omega^n) \simeq V(G/K, (\omega')^n).$$

Proof. First we give a proof, as promised, of the statement '(i) implies (ii)' of Theorem 4.1. Let $\alpha: L^2(G) \rightarrow L^2(G)$ be defined by

$$\alpha f(x) = \int_K f(k^{-1}x)d\lambda(k), \quad \text{almost all } x \in G, f \in L^2(G).$$

The proof that α is a central idempotent in $V(G, \omega^n) = E_n V(G^\omega)$ (and hence in $V(G^\omega)$) and that $\alpha V(G, \omega^n)$ and $V(G/K, (\omega')^n)$ are spatially isomorphic is similar to the proof of the corresponding facts about E_n in Theorem 3.5. Since

$$V(G, \omega^n) \simeq V(G/K, (\omega')^n) \oplus V(G/K, (\omega')^n)^\perp$$

(where $\perp$ denotes orthogonal complement), we have by Theorem 3.5,

$$V(G^\omega) \simeq V((G/K)^{\omega'}) \oplus V((G/K)^{\omega'})^\perp$$

but $(G/K)^{\omega'}$ and $G^\omega/(G^\omega)_0$ are topologically isomorphic and $V(G^\omega/(G^\omega)_0)$ is isomorphic to the maximal type I finite direct summand in $V(G^\omega)$ (3.1 (i)), it follows that $V(G/K, \omega')^n)$ is isomorphic to the maximal type I direct summand of $V(G, \omega^n)$. In particular we have $d_G = \alpha \neq 0$.

Now assume $d_G \neq 0$, then by '(ii) implies (i)' of Theorem 4.1, $e_{G^\omega} \neq 0$, thus by the same argument as above, we reach the desired conclusion.

COROLLARY 4.4. *Suppose* $d_G \neq 0$ *and let* $n \in \mathbf{Z}$, *then the following equations obtain*

$$G_0 = h[(G^\omega)_0] = h[\cap\{\ker \pi: \pi \text{ is a finite dimensional representation of } G^\omega \text{ such that } \pi|_{\mathbf{T}}(t) = t^n\}]$$

$$= \{g \in G: \text{there exists } \gamma(g) \in \mathbf{T} \text{ such that } \pi(g) = \gamma(g)\, I \text{ for all finite dimensional } \omega^n\text{-representations of } G\}.$$

Proof. Let $K = h[(G^\omega)_0]$ and denote the last two sets in the above equality by H and L respectively. That $K \subseteq H \subseteq L$ is clear from the definitions and the property that an ω^n-representation π of G extends to an ordinary representation π' of G^ω such that

$$\pi'|_{\mathbf{T}}(t) = t^n.$$

Using the proof of Lemma 3.6, we assume that ω is lifted from a multiplier on G/L. The non-degenerate finite dimensional representations of $d_G V(G, \omega^n)$ separate the points of $d_G V(G, \omega^n)$, hence $\pi^0(g) = I$ if and only if

$$\pi(d_G \rho(g^{-1}))^* = \pi(d_G)^*$$

for all such representations π, where π^0 denotes the ω^n-representation

$$g \to \pi(d_G \rho(g^{-1}))^*$$

and ρ is the left regular ω^n-representation of G. This happens if and only if $\rho(g)d_G = d_G$ and using the previous Theorem, this occurs if and only if $g \in K$. This shows

$$K = \cap\ \{\ker \pi^0: \pi \text{ is a finite dimensional non-degenerate representation of } d_G V(G, \omega^n)\} \supseteq L.$$

If we let $n = 0$, we obtain the remainder of the corollary: $G_0 = L$.

THEOREM 4.5. *Let G be a locally compact group and ω a normalised Borel multiplier on G. The following conditions are equivalent.*

(a) *$V(G, \omega)$ is type I finite.*

(b) *All irreducible ω-representations of G are finite dimensional.*

(c) *The following conditions hold:*

(i) $[G:\Delta_G] < \infty$

(ii) $(\Delta_G)'^{-}$ *is compact*

(iii) *G admits a finite dimensional ω-representation*

(iv) $G_0 = \{e\}$

(d) *G^ω is a Moore group, that is all irreducible ordinary representations of G^ω are finite dimensional.*

Proof. To obtain (a) is eqivalent to (c) combine 4.1, 4.3 and 4.4. To prove (b) implies (a), apply [**2**, 4.2.1 and 5.5.2] and the proof of Moore [**10**, Lemma 4.1] to the twisted group C^*-algebra $C^*(G, \omega)$. Assume (c). If π is a finite dimensional ω-representation of G, then the n-fold tensor product $\pi \otimes \ldots \otimes \pi$ is a finite dimensional ω^n-representation of G, hence $V(G, \omega^n)$ is type I finite and by Theorem 3.5 (ii), $V(G^\omega)$ is type I finite. It follows from Theorem 3.2 that G^ω is a Moore group.

5. Examples. The condition in 4.1 (iii) that G possess a finite dimensional ω-representation cannot be replaced by the weaker property that ω^n is trivial for some integer n (even in the case where G is discrete) as the following example shows.

Example. 5.1. Let $G = H \times H'$ with the discrete topology, where

$$H = \prod_{h=1}^{\infty} \mathbf{Z}_2 \quad \text{and} \quad H' = \bigoplus_{h=1}^{\infty} \mathbf{Z}_2$$

and define

$$\omega(\,(a_j, b_j), (a_j', b_j')\,) = \exp\left[\frac{\pi i}{2}\sum_{j=1}^{\infty} (a_jb_j' - a_j'b_j)\right],$$

$(a_j, b_j)(a_j', b_j') \in G$. Since G is abelian, Theorem 1.1 and Theorem 4.3 of [**4**] show that $V(G, \omega)$ is either type I or type II_1. But we know from [**4**, Example 4.4] that it is not type I, thus $V(G, \omega)$ is type II_1. This occurs despite the fact that (G, ω) satisfies $[G:\Delta_G] < \infty$, $(\Delta_G)'^{-}$ is compact and $\omega^2 = 1$. We observe in passing that G has no finite dimensional ω-representation (use Theorem 4.1).

Example 5.2. For each $\lambda = e^{2\pi i\alpha} \in \mathbf{T}$, ($\alpha \in [0, 2\pi[$), we obtain a multiplier ω_λ on $\mathbf{Z} \times \mathbf{Z}$ defined by

$$\omega_\lambda(\,(m, n), (m', n')\,) = \lambda^{mn'},$$

$(m, n), (m', n') \in \mathbf{Z} \times \mathbf{Z}$. Mackey [**9**, Theorem 8.6] shows that (up to similarity) all multipliers on $\mathbf{Z} \times \mathbf{Z}$ are of this form. We say that λ is rational (respectively irrational) if α is rational (respectively irrational). It follows from [**4**] that $V(\mathbf{Z} \times \mathbf{Z}, \omega_\lambda)$ is type II, if λ is irrational and if λ is rational, say $\lambda = \exp 2\pi ip/q$, where p and q are relatively prime integers, then $V(\mathbf{Z} \times \mathbf{Z}, \omega_\lambda)$ is type I (finite) and according to Hannabuss [**5**, Theorem 4.1], the irreducible ω_λ representations of $\mathbf{Z} \times \mathbf{Z}$ are all dimension q.

Let Δ_p (p-prime) be the group of p-adic integers and G be the group $\mathbf{Z} \times \mathbf{Z} \times \Delta_p$ with multiplication

$$(a, b, x)(a', b', x') = (a + a', b + b', x + x' + \theta(ab')\,),$$
$$(a, b, x)(a', b', x') \in G,$$

where $\theta:\mathbf{Z} \to \Delta_p$ is the canonical injection of $\mathbf{Z}$ into a dense subgroup of Δ_p. We topologize G so that Δ_p becomes a compact open subgroup; with this topology, G becomes a locally compact separable topological group. For each $\lambda \in \mathbf{T}$, we define a multiplier σ_λ on G as follows,

$$\sigma_\lambda = \omega_\lambda \circ k,$$

where $k:G \to \mathbf{Z} \times \mathbf{Z}$ is the canonical homomorphism

$$k(m, n, x) = (m, n).$$

Given an irreducible ω_λ-representation π of $\mathbf{Z} \times \mathbf{Z}$, denote by π' the σ_λ representation of G obtained by composing π with k.

Since Δ_p is a compact normal subgroup of G, we can apply the Mackey analysis [9] to construct all σ_λ-representations of G.

Identify the abelian group dual $\Delta_p^\wedge$ of Δ_p with the subgroup

$$\Lambda = \{\chi \in \mathbf{T}:\chi = \exp[2\pi ik/p^n], k, n \in \mathbf{Z}\}$$

of $\mathbf{T}$ using the correspondence

$$\Lambda \times \Delta_p \to \mathbf{T}:(\lambda, x) \to \lambda^x,$$

where $x \to \chi^x$, $\chi \in \Lambda$ is the continuous extension from $\mathbf{Z}$ to Δ_p of the homomorphism $\mathbf{Z} \to \Lambda:n \to \chi^n$.

Let $\lambda \in \mathbf{T}$, $\chi \in \Lambda$. Then the irreducible σ_λ-representations of G which reduce to a multiple of χ on Δ_p are of the form $\chi'\pi'$, where χ' is the extension

$$\chi':G \to \mathbf{T}:(a, b, x) \to \chi^x$$

of the character χ of Δ_p and π is a $\omega_{\lambda\chi^{-1}}$-representation of $\mathbf{Z} \times \mathbf{Z}$. As χ ranges through Λ, we obtain all irreducible σ_λ-representations of G.

Note that $\lambda\chi^{-1}$ is rational if and only if λ is rational. Hence the irreducible σ_λ-representations of G, $\lambda \in \mathbf{T}$ are all infinite dimensional if λ is irrational and are all finite dimensional (but of arbitrarily high dimension) if λ is rational. It follows from Theorem 4.1 and 4.5 that $V(G, \sigma_\lambda)$ is type I finite if and only if λ is rational and the type I finite part of $V(G, \sigma_\lambda)$ is zero if λ is irrational.

REMARK 5.3. *As pointed out in* [**4**], *combining Theorem* 1.1 *of* [**4**] *with the proofs of Taylor* [**14**, *Theorem* 2] *and Moore* [**10**, *Theorem* 1] *yields the equivalence of the following statements*:

(i) *$V(G, \omega)$ is type $I_{\leqq k}$, that is non-zero maximal type I_n central projections occur in $V(G, \omega)$ only if $n \leqq k$.*

(ii) *G has an open abelian subgroup A of finite index in G such that $\omega|_{A \times A}$ is trivial.*

(iii) *The irreducible ω-representations of G are of dimension at most k.*

Thus in Example 5.2 we see that for λ rational, $V(G, \sigma_\lambda)$ admits a non-zero maximal type I_n part of arbitrary large n. This phenomenon does not occur in the case where G is discrete.

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