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HEDGEHOGS IN LEHMER'S PROBLEM

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To Gunther Cornelissen, with warm wishes, on the occasion of reaching the age
(for the first time!) that can be written as a sum of two positive squares in two different ways.
Niet elke egel is stekelig!

Abstract

Motivated by a famous question of Lehmer about the Mahler measure, we study and solve its analytic analogue.

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1. Introduction

Several deep arithmetic questions are known about polynomials with integer coefficients. One of them raised by Lehmer in the 1930s asks, for a monic irreducible polynomial $P(x) = \prod_{j=1}^{d}(x - \alpha_j) \in \mathbb{Z}[x]$, whether the quantity $\mathrm{M}(P(x)) = \prod_{j=1}^{d} \max\{1, |\alpha_j|\}$ can be made arbitrarily close to but greater than 1. The characteristic $\mathrm{M}(P(x))$ is known as the Mahler measure [1]; in spite of the name coined after Mahler's work in the 1960s, many results about it are rather classical. One of them, due to Kronecker, says that $\mathrm{M}(P(x)) = 1$ if and only if $P(x) = x$ or the polynomial is cyclotomic, that is, all its zeros are roots of unity.

A related question, usually considered as a satellite to Lehmer's problem, about the so-called house of a nonzero algebraic integer α defined through its minimal polynomial $P(x) \in \mathbb{Z}[x]$ as $\overline{|\alpha|} = \max_j |\alpha_j|$, was posed by Schinzel and Zassenhaus in the 1960s and answered only recently by Dimitrov [2]. He proved that $\overline{|\alpha|} \geq 2^{1/(4d)}$ for

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any nonzero algebraic integer α which is not a root of unity; the latter option clearly corresponds to $\lceil\overline{\alpha}\rceil = 1$.

Dimitrov's ingenious argument transforms the arithmetic problem into an analytic one. In this note we discuss the potential of Dimitrov's approach to Lehmer's problem.

2. Principal results

Consider a monic irreducible *noncyclotomic* polynomial $P(x) = \prod_{j=1}^{d}(x - \alpha_j)$ in $\mathbb{Z}[x]$ of degree $d > 1$ and assume that the polynomial $\prod_{j=1}^{d}(x - \alpha_j^2) \in \mathbb{Z}[x]$ is irreducible as well. (Otherwise the Mahler measure of $P(x)$ is bounded from below through the measures of irreducible factors of the latter polynomial.) As in [2], Dimitrov's cyclotomicity criterion together with Kronecker's rationality criterion and a theorem of Pólya imply that the *hedgehog*

$$K = K(\beta_1, \ldots, \beta_n) = \bigcup_{k=1}^{n}[0, \beta_j] = \bigcup_{j=1}^{d}[0, \alpha_j^2] \cup \bigcup_{j=1}^{d}[0, \alpha_j^4],$$

whose spines originate from the origin and end up at α_j^2, α_j^4 for $j = 1, \ldots, d$, has (logarithmic) capacity (or transfinite diameter) $t(K)$ at least 1. Then Dubinin's theorem [3] applies, which claims that $t(K) \le 4^{-1/n} \max_j |\beta_j|$ (with equality attained if and only if the hedgehog K is rotationally symmetric), and produces the estimate for $\lceil\overline{\alpha_1}\rceil = (\max_j |\beta_j|)^{1/4}$ since $n \le 2d$.

When dealing with Lehmer's problem instead, one becomes interested in estimating the 'Mahler measure of the hedgehog', namely the quantity $\prod_{j=1}^{n} \max\{1, |\beta_j|\}$, because any nontrivial (bounded away from 1) absolute estimate for it would imply a nontrivial estimate for the Mahler measure of $P(x)$. In this setting, Dubinin's theorem only implies the estimate $\prod_{j=1}^{n} \max\{1, |\beta_j|\} \ge 4^{1/n}$ for a hedgehog of capacity at least 1, which depends on n. The Mahler measure of the rotationally symmetric hedgehog on n spines, which is optimal in Dubinin's result, is equal to 4 (thus, independent of n), which certainly loses out to the Mahler measure $1.91445008\ldots$ of the 'Lehmer hedgehog' attached to the polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ but also to the measure $3.07959562\ldots$ of the hedgehog constructed on Smyth's polynomial $x^3 - x - 1$. The following question arises in a natural way.

QUESTION 1. What is the minimum of $\prod_{j=1}^{n} \max\{1, |\beta_j|\}$ taken over all hedgehogs $K = K(\beta_1, \ldots, \beta_n)$ of capacity at least 1?

Notice that answering this question for hedgehogs of capacity exactly 1 is sufficient, since the capacity satisfies $t(K_1) \le t(K_2)$ for any compact sets $K_1 \subset K_2$ in $\mathbb{C}$.

In order to approach Question 1 we use a different construction of hedgehogs outlined in Eremenko's post on the question in [5] with details set out in [6]. Any hedgehog $K = K(\beta_1, \ldots, \beta_n)$ of capacity *precisely* 1 is in a bijective correspondence (up to rotation!) with the set of points $z_1, \ldots, z_n$ on the unit circle with prescribed

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positive real weights $r_1, \dots, r_n$ satisfying $r_1 + \cdots + r_n = 1$. Namely, the mapping

$$F(z) = \prod_{k=1}^{n} ((z - z_k)(z^{-1} - \bar{z}_k))^{r_k}$$

is a Riemann mapping of the complement of the closed unit disk to the complement $\hat{\mathbb{C}} \setminus K$ of hedgehog. It is not easy to write down the corresponding β_j explicitly, but for their absolute values we get

$$|\beta_j| = \max_{z \in [z_{j-1}, z_j]} |F(z)| = \max_{z \in [z_{j-1}, z_j]} \prod_{k=1}^{n} |z - z_k|^{2r_k} \quad \text{for } j = 1, \dots, n,$$

where we conventionally take $z_0 = z_n$ and understand $[z_{j-1}, z_j]$ as *arcs* of the unit circle. This means that if $C \geq 1$ is the minimum of

$$\prod_{j=1}^{n} \max\Big\{1, \max_{z \in [z_{j-1}, z_j]} \prod_{k=1}^{n} |z - z_k|^{r_k}\Big\}$$

taken over all n and all possible weighted configurations $z_1, \dots, z_n$, then C^2 is the minimum in Question 1.

Furthermore, in the spirit of [4] observe that from continuity considerations it suffices to compute the required minimum C for *rational* positive weights $r_1, \dots, r_n$. Assuming the latter and writing $r_j = a_j/m$ for positive integers $a_1, \dots, a_n$ and $m = a_1 + \cdots + a_n$, we look for the mth root of the minimum of

$$\prod_{j=1}^{n} \max\Big\{1, \max_{z \in [z_{j-1}, z_j]} \prod_{k=1}^{n} |z - z_k|^{a_k}\Big\} = \prod_{j=1}^{m} \max\Big\{1, \max_{z \in [z'_{j-1}, z'_j]} \prod_{k=1}^{m} |z - z'_k|\Big\},$$

where $z'_1, z'_2, \dots, z'_m$ is the multi-set

$$\underbrace{z_1, \dots, z_1}_{a_1 \text{ times}}, \ \underbrace{z_2, \dots, z_2}_{a_2 \text{ times}}, \ \dots, \ \underbrace{z_n, \dots, z_n}_{a_n \text{ times}}$$

with prescribed weights all equal to 1. This means that it is enough to compute the minimum for the case of equal weights, $r_1 = \cdots = r_n = 1/n$, and we may give the following alternative formulation of Question 1.

QUESTION 2. What is the minimum C_n of

$$\prod_{j=1}^{n} \max\Big\{1, \max_{z \in [z_{j-1}, z_j]} \prod_{k=1}^{n} |z - z_k|\Big\}^{1/n}$$

taken over all configurations of points $z_1, \dots, z_n$ on the unit circle $|z| = 1$? The points are not required to be distinct and $[z_{j-1}, z_j]$ is understood as the corresponding arc of the circle, z_0 is identified with z_n.

Though there is no explicit requirement on the order of precedence, the minimum corresponds to the successive locations of $z_1, \dots, z_n$ on the circle.

A comparison with Dubinin's result suggests that good candidates for the minima in Question 2 may originate from configurations in which all factors in the defining product but one are equal to 1. In our answer to the question we show that this is essentially the case by computing the related minima C_n^* explicitly.

THEOREM 3. *For the quantity C_n we have the inequality $C_n \le C_n^*$, where $C_n^* = (T_n(2^{1/n}))^{1/n}$ and*

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2 - 1)^k x^{n-2k}$$

denotes the nth Chebyshev polynomial of the first kind.

THEOREM 4. *For the quantity C_n^* in Theorem 3 we have the asymptotic expansion*

$$C_n^* = 1 + \nu - \frac{1}{4}\nu^3 + \frac{5}{96}\nu^5 - \frac{1}{128}\nu^7 + O(\nu^9)$$

in terms of $\nu = \sqrt{(\log 4)/n}$, as $n \to \infty$. In particular, $(C_n^*)^{\sqrt{n}} \to e^{\sqrt{\log 4}}$ and $C_n^* \to 1$ as $n \to \infty$.

Thus, our results imply that the minimum in Question 1 is equal to 1, meaning that an analogue of Lehmer's problem in an analytic setting is trivial. This has no consequences for Lehmer's problem itself, as we are not aware of a recipe to cook up polynomials in $\mathbb{Z}[x]$ from optimal (or near optimal) configurations of $z_1, \dots, z_n$ on the unit circle.

3. Proofs

PROOF OF THEOREM 3. We look for a configuration of the points $z_1, \dots, z_n$ on the unit circle such that the maximum of $|Q(z)|$, where $Q(z) = (z - z_1) \cdots (z - z_n)$, on all the arcs $[z_{j-1}, z_j]$ but one is equal to 1:

$$\max_{z \in [z_{j-1}, z_j]} |Q(z)| = |Q(z_j^*)| = 1 \quad \text{for } z_j^* \in (z_{j-1}, z_j), \quad \text{where } j = 2, \dots, n.$$

At the same time, the kth Chebyshev polynomial $T_k(x) = 2^{k-1}x^k + \cdots$ is known to satisfy $|T_k(x)| \le 1$ on the interval $-1 \le x \le 1$, with *all* the extrema on the interval being either -1 or 1. Note that $T_k(x)$ has k distinct real zeros on the open interval $-1 < x < 1$ and satisfies $T_k(1) = (-1)^k T_k(-1) = 1$. Therefore, for $n = 2k$ even,

$$Q(z) = z^k \, T_k\Big(2^{1/k}\Big(\frac{z + z^{-1}}{2} - 1\Big) + 1\Big),$$

we get a *monic* polynomial of degree n with the desired properties; its zeros $z_1, \dots, z_n$ ordered in pairs, so that $z_{n-j} = \bar{z}_j = z_j^{-1}$ for $j = 1, \dots, k$, correspond to the real zeros $2^{1/k}((z_j + z_j^{-1})/2 - 1) + 1$ of the polynomial $T_k(x)$ on the interval $-1 < x < 1$. Then

$$\max_{z \in [z_n, z_1]} |Q(z)| = \max_{|z|=1} |Q(z)| = |Q(-1)| = |T_k(1 - 2^{1+1/k})| = T_k(2^{1+1/k} - 1) = T_{2k}(2^{1/(2k)}),$$

where the duplication formula $T_k(2x^2 - 1) = T_{2k}(x)$ was applied.

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The duplication formula in fact allows one to write the very same polynomial $Q(z)$ in the form

$$Q(z) = \pm(-z)^{n/2}\, T_n(2^{1/n-1}\sqrt{2-(z+z^{-1})}\,),$$

and this formula gives the desired polynomial, monic and of degree n, for n of any parity. If we set $k = \lfloor (n+1)/2 \rfloor$, the zeros $z_1, \dots, z_n$ of $Q(z)$ pair as before, that is, $z_{n-j} = \bar{z}_j = z_j^{-1}$ for $j = 1, \dots, k$, with the two zeros merging into one, $z_{(n+1)/2} = 1$ for $j = k$ when n is odd, so that $2^{1/n-1}\sqrt{2-(z_j+z_j^{-1})}$ for $j = 1, \dots, k$ are precisely the k real zeros of the polynomial $T_n(x)$ on the interval $0 \le x < 1$. This leads to the estimate

$$\max_{z\in[z_n,z_1]} |Q(z)| = \max_{|z|=1} |Q(z)| = |Q(-1)| = T_n(2^{1/n})$$

for both even and odd values of n.

Finally, we remark that the *uniqueness* of $Q(z)$, up to rotation, follows from the extremal properties of the Chebyshev polynomials. □

PROOF OF THEOREM 4. For this part we cast the Chebyshev polynomial $T_n(x)$ in the form

$$T_n(x) = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2} = \frac{x^n}{2}\cdot((1+\sqrt{1-x^{-2}})^n + (1-\sqrt{1-x^{-2}})^n)$$

leading to

$$T_n(2^{1/n}) = (1+\sqrt{1-e^{-\nu^2}})^n + (1-\sqrt{1-e^{-\nu^2}})^n$$

in the notation $\nu = \sqrt{(\log 4)/n}$. Since

$$\begin{aligned}\sqrt{1-e^{-\nu^2}} &= \bigg(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\nu^{2k}}{k!}\bigg)^{1/2} = \nu\bigg(1+\sum_{k=2}^{\infty} \frac{(-1)^{k-1}\nu^{2k-2}}{k!}\bigg)^{1/2} \\ &= \nu\cdot\bigg(1-\frac{1}{4}\nu^2+\frac{5}{96}\nu^4-\frac{1}{128}\nu^6+\frac{79}{92160}\nu^8-\frac{3}{40960}\nu^{10}+O(\nu^{12})\bigg),\end{aligned}$$

we conclude that the term $(1-\sqrt{1-e^{-\nu^2}})^n = O(\varepsilon^n)$ for any choice of positive $\varepsilon < 1$, hence

$$(T_n(2^{1/n}))^{1/n} = (1+\sqrt{1-e^{-\nu^2}})\cdot(1+O(\varepsilon^n)),$$

and the required asymptotics follows. □

4. Speculations

Dimitrov's estimate $t(K) \ge 1$ for the capacity of the hedgehog $K = K(\beta_1, \dots, \beta_n)$ assigned to a polynomial in $\mathbb{Z}[x]$ is not necessarily sharp, and one would rather expect to have $t(K) \ge t$ for some $t > 1$. By replacing the polynomial in the proof of Theorem 3 with

$$Q(z) = \pm(-z)^{n/2}\, T_n(2^{1/n-1}t\sqrt{2-(z+z^{-1})}\,)$$

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and assuming (or, better, believing!) that the corresponding minimum in Question 2 is indeed attained in the case when all but one of the factors are equal to 1, we conclude that the minimum is equal to $(T_n(2^{1/n}t))^{1/n}$. The asymptotics of the Chebyshev polynomials then converts this result into the answer

$$\inf_{\substack{n=1,2,\ldots\\ K=K(\beta_1,\ldots,\beta_n)\\ t(K)\ge t}} \prod_{j=1}^{n} \max\{1,|\beta_j|\} \ge t+\sqrt{t^2-1}$$

to the related version of Question 1. This is slightly better, when $t>1$, than the trivial estimate of the infimum by t from below.

In another direction, one may try to associate hedgehogs K to polynomials in a different (more involved!) way, to achieve some divisibility properties for the Hankel determinants A_k that appear in the estimation $t(K)\ge \limsup_{k\to\infty}|A_k|^{1/k^2}$ of the capacity on the basis of Pólya's theorem. Such an approach has the potential to lead to some partial ('Dobrowolski-type') resolutions of Lehmer's problem. Notice, however, that the bound for $t(K)$ in Pólya's theorem is not sharp: numerically, the Hankel determinants $A_k=\det_{0\le i,j<k}(a_{i+j})$ constructed on (Dimitrov's) irrational series

$$\begin{aligned}\sum_{k=0}^{\infty} a_k x^k &= \sqrt{(x-\alpha_1^2)(x-\alpha_2^2)(x-\alpha_3^2)\,(x-\alpha_1^4)(x-\alpha_2^4)(x-\alpha_3^4)}\\ &= \sqrt{(1-x+2x^2-x^3)(1+3x+2x^2-x^3)}\in\mathbb{Z}[[x]]\end{aligned}$$

for Smyth's polynomial $x^3-x-1=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$ satisfy $|A_k|\le C^k$ for some $C<2.5$ and all $k\le 150$, so that it is likely that $\limsup_{k\to\infty}|A_k|^{1/k^2}=1$ in this case.

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