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ON A SUBCLASS OF BAZILEVIC FUNCTIONS

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Integral mean and coefficient bounds for some Bazilevic functions are determined.

1. INTRODUCTION

Let S denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are univalent in $|z| < 1$. Let $B(\alpha)$ denote the functions which can be written in the form

$$f(z) = \{\alpha \int_0^z g^\alpha(t)P(t)t^{-1}\mathrm{d}t\}^{1/\alpha}, \tag{2}$$

where $g(z)$ and $P(z)$ are subject to the conditions $g(0) = g'(0) - 1 = 0$, $\operatorname{Re} zg'(z)/g(z) \geqslant 0$ and $P(0) = 1$, $\operatorname{Re} P(z) \geqslant 0$ respectively. Then it is well-known that $B(\alpha) \subseteq S$ ([**13**, **6**, **7**, **8**]). The coefficients problem for $f(z)$ in $B(\alpha)$ and S has been settled by Leach [**7**] and de Branges [**1**], respectively.

In this paper we study the coefficients probem for $f(z) \in B(\alpha)$ when $P(t) = 1$. This type of function has been shown to be starlike in [**3**, Theorem 2]. We shall denote this type of function by $B_1(\alpha)$ and deduce some integral mean as well as coefficient bounds for the case $0 < \alpha \leqslant 1$. We shall be using the notation $f \prec F$ to mean that $f(z) = F(\phi(z))$ where $\phi(z)$ satisfies $\phi(0) = 0$ and $|\theta(z)| \leqslant 1$ in $|z| < 1$; $f(z)$ is said to be *subordinate* to $F(z)$ ([**2**, p.190], [**5**, p.178]). We shall also use the notation $\sum a_n z^n \ll \sum b_n z^n$ to mean that $|a_n| \ll b_n$ for $n = 1, 2, \ldots$ ([5, vol. 2, Theorem 5], [**6**]).

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2. Subordination

Theorem 1. *Let $f(z) \in B_1(\alpha)$. Then for $0 < \alpha \leqslant 1$ we have*

$$\log f'(z) \prec \log z^{-1} K(z),$$
$$f'(z) \prec z^{-1} K(z),$$

where $K(z) = z(1-z)^{-2}$ is the well-known Koebe function.

Proof: We see from (2) that

$$\log f'(z) = (1-\alpha) \log z^{-1} f(z) + \alpha \log z^{-1} g(z). \tag{3}$$

Now it is well-known (see question 13 of [**2**, p.213] and [**4**, p.118]) that

$$\log z^{-1} g(z) \prec \log z^{-1} K(z),$$
$$\log z^{-1} f(z) \prec \log z^{-1} K(z),$$

since $g(z)$ and $f(z)$ are both starlike.

These, together with (3), give, since the righthandside is a convex combination of Koebe functions, that

$$\log f'(z) \prec \log z^{-1} K(z)$$

as required in the first part of Theorem 1. The second part follows by exponentiation since subordination is preserved in this case (see [**9**, pp.23–24]). ∎

3. Integral mean bounds

Theorem 2. *Let $f(z) \in B_1(\alpha)$. Then for $z = re^{i\theta}$, with $0 < r < 1$, we have, for $\lambda > 0$, that*

$$\int_0^{2\pi} |f'(z)|^\lambda \, d\theta \leqslant \int_0^{2\pi} \left|z^{-1} K(z)\right|^\lambda d\theta,$$
$$\int_0^{2\pi} \log |f'(z)| \, d\theta \leqslant \int_0^{2\pi} \log \left|z^{-1} K(z)\right| d\theta,$$
$$\int_0^{2\pi} |\log f'(z)|^\lambda \, d\theta \leqslant \int_0^{2\pi} \left|\log z^{-1} K(z)\right|^\lambda d\theta.$$

Proof: This follows from Theorem 1 and [**2**, Theorem 6.1], [**5**, vol. 2, pp.178–181]. ∎

Remark 1. Using the Bernstein $\star$-function argument (see [**2**, Chapter 7], [**8**, **10**]), we can extend the first part of Theorem 2 to negative values of λ. Also the argument of [**4**, Theorem 1] may be applied for $0 < \lambda \leqslant 2$ in the third part of this theorem.

Remark 2. Using the coefficient formula and the first part of Theorem 2 we can see easily that $|a_n| < \frac{1}{2}e$ and this suggests that $|a_n| \leqslant 1$ for $f(z) \in B_1(\alpha)$, which we now prove.

4. Coefficient bounds

THEOREM 3. *Let $f(z) \in B_1(\alpha)$, let $0 < \alpha \leqslant 1$, and let (1) hold. Then for $n \geqslant 1$ we have*

$$|a_n| \leqslant 1.$$

PROOF: We see from (3) that

$$\log f'(z) \ll \log z^{-1} K(z) \tag{4}$$

by [**6**, Lemma 2], since both $f(z)$ and $g(z)$ are starlike and $0 < \alpha \leqslant 1$. In view of the fact that the coefficients of the righthandside of (4) are positive we deduce that

$$f'(z) \ll z^{-1} K(z),$$

since exponentiation preserves majorisation in this case. This gives Theorem 3 by the definition of $\ll$ above. ∎

The function $f(z) = z(1-z)^{-1}$ is in $B_1(\alpha)$ with respect to itself, since it is starlike and this shows that this bound is sharp.

5. Odd functions

THEOREM 4. *Let $f(z) \in B_1(\alpha)$, let $0 < \alpha \leqslant 1$, and let $F(z) = f(z^2)^{\frac{1}{2}} = z + a_3 z^3 + a_5 z^5 + \dots$. Then for $n \geqslant 1$ we have*

$$|a_{2n+1}| \leqslant \frac{1}{2n+1}. \tag{5}$$

PROOF: We see from (2) and [**6**, Lemma 2] that

$$\begin{aligned} F(z) = f^{\frac{1}{2}}(z^2) &= \int_0^z g^{\frac{1}{2}}(t^2) t^{-1} \mathrm{d}t \\ &\ll \int_0^z \frac{\mathrm{d}t}{1-t^2} \\ &= \frac{1}{2} \log \frac{1+z}{1-z} \end{aligned}$$

which gives (5) by the definition of $\ll$ above. ∎

The function $zF'(z) = z(1-z^2)^{-1}$ is in $B_1(\alpha)$ with respect to itself since it is starlike and this shows that (5) is sharp.

Remark 3. This theorem can also be proved by the method used in the proof of Theorem 2 of [**6**].

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