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ON THE MEROMORPHIC SOLUTIONS OF SOME FUNCTIONAL EQUATIONS

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Let $f(z)$ be a meromorphic function, let $P(z)$ and $Q(z)$ be two polynomials. We shall investigate the asymptotic behaviour of the ratio $T(r, f(P))/T(r, f(Q))$, and discuss the growth of the meromorphic solutions of some functional equations.

1. INTRODUCTION AND MAIN RESULTS

Let $f(z)$ be a meromorphic function in $\mathbf{C}$. We denote the order and the lower order of $f(z)$ by ρ_f and μ_f respectively.

There are many interesting works about the meromorphic solutions of functional equations(see [**4, 5, 7, 8**] et cetera). In this paper, we deal with the following functional equation:

$$R_1(z, f(g(z))) = R_2(z, f(P_m(z))), \tag{1}$$

where

$$R_j(z, w) = P_j(z, w)/Q_j(z, w),$$

and

$$P_j(z, w) = \sum_{i=0}^{p_j} a_{ij}(z)w^i, \quad Q_j(z, w) = \sum_{k=0}^{q_j} b_{kj}(z)w^k$$

are two polynomials of w which are mutually prime, $a_{ij}(z)$ and $b_{kj}(z)$ all are polynomials of z; $f(w)$ is a transcendental meromorphic function; $g(z)$ is an entire function; $P_m(z) = a_m z^m + \cdots + a_1 z + a_0$ $(a_m \neq 0)$ is a polynomial of degree m. Put $\partial R_j = max(p_j, q_j)$ $(j = 1, 2)$. We have the following:

THEOREM 1. *Let $R_j(z, w)$ $(j = 1, 2)$, $f(w)$, $g(z)$ and $P_m(z)$ satisfy the equation (1). Then $g(z)$ is a polynomial and its degree n lies between m and $(\partial R_2/\partial R_1)m$. Furthermore, put $g(z) = b_n z^n + \cdots + b_1 z + b_0$ $(b_n \neq 0)$, We have*

(1) *If $m \neq n$, then $\rho_f = 0$;*
(2) *If $m = n$ and $|a_m| \neq |b_n|$, then $\rho_f = \mu_f = \log(\partial R_1)/(\partial R_2)/\log|a_m/b_n|$;*
(3) *If $m = n$, $|a_m| = |b_n|$, and $\partial R_1 \neq \partial R_2$, then $\mu_f = \infty$.*

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REMARKS. (1) Let f be a meromorphic function, g be an entire function, P be a polynomial of degree k, and $f(g(z)) = f(P(z))$. Gross [**2**, p.542, Problem 31] has posed the following problem: Must g also be a polynomial of degree k? The above Theorem 1 can give an affirmative answer to this problem.

(2) In [**5**], Shimomura investigated the Schröder equation

$$f(cz) = Q(f(z)), \tag{2}$$

where c is a constant with $|c| > 1$, $Q(w)$ is a polynomial of degree n. He proved that any non-constant entire solution $f(z)$ of (2) has order $\rho_f = \log n/\log|c|$. The above Theorem 1 is a generalisation of this result.

(3) In [**8**], Yanagihara investigated the following functional equation:

$$f(z+1) = R(z, f(z)), \tag{3}$$

where $R(z,w)$ is a rational function of two variables. He proved that any transcendental meromorphic solution $f(z)$ of (3) has order $\rho_f = \infty$ if $\partial R > 1$. The above Theorem 1 is a generalisation of this result.

Qiao [**4**] has investigated the asymptotic behaviour of the ratio

$$T(r, f(\alpha z + \beta))/T(r, f(z)).$$

Let $P_m(z) = a_m z^m + \cdots + a_1 z + a_0$ and $Q_n(z) = b_n z^n + \cdots + b_1 z + b_0$ $(a_m b_n \neq 0)$ be two polynomials. Denote

$$\rho_f^* = \overline{\lim_{r\to\infty}} \log T(r,f)/\log\log r; \quad \mu_f^* = \underline{\lim}_{r\to\infty} \log T(r,f)/\log\log r.$$

In this paper, we deal with the ratio

$$\sigma(r, f, P_m, Q_n) = T(r, f(P_m))/T(r, f(Q_n)),$$

and prove the following:

THEOREM 2. *(1) If $m > n$, then*

$$\underline{\lim}_{r\to\infty} \sigma(r, f, P_m, Q_n) \leqslant \left(\frac{m}{n}\right)^{\mu_f^*} \leqslant \left(\frac{m}{n}\right)^{\rho_f^*} \leqslant \overline{\lim_{r\to\infty}} \sigma(r, f, P_m, Q_n); \tag{4}$$

(2) If $m = n$, and $|a_m| > |b_n|$, then

$$\underline{\lim}_{r\to\infty} \sigma(r, f, P_m, Q_n) \leqslant \left|\frac{a_m}{b_n}\right|^{\mu_f} \leqslant \left|\frac{a_m}{b_n}\right|^{\rho_f} \leqslant \overline{\lim_{r\to\infty}} \sigma(r, f, P_m, Q_n); \tag{5}$$

(3) If $m = n$, $|a_m| = |b_n|$, and $\mu_f < \infty$, then

$$\underline{\lim}_{r\to\infty} \sigma(r, f, P_m, Q_n) \leqslant 1 \leqslant \overline{\lim_{r\to\infty}} \sigma(r, f, P_m, Q_n). \tag{6}$$

REMARK. The proof of Theorem 1 mainly depends on Theorem 2.

2. The Proof of Theorem 2

Put

$$\Omega = \varliminf_{r\to\infty} \sigma(r, f, P_m, Q_n).$$

Firstly, we prove that $(m/n)^{\mu_f^*}, |a_m/b_n|^{\mu_f}$ and 1 are the upper bounds of Ω in the cases (1), (2) and (3) respectively. If $\Omega = 0$, this is obviously true. Below, we suppose $\Omega > 0$ (Ω may be infinity). Therefore, for any finite and positive number $\tau < \Omega$, there exists $r_1 > 0$, such that

$$T(r, f(P_m)) > \tau \cdot T(r, f(Q_n))$$

when $r \geqslant r_1$. Choose a complex number a which isnt a Valiron deficient value of $f(P_m), f(Q_n)$ and $f(z)$. Thus for any $\varepsilon > 0$, from the above inequality we deduce that there exists some $r_2 > r_1$, such that

$$N(r, f(P_m) = a) > \tau \cdot \frac{1-\varepsilon}{1+\varepsilon} N(r, f(Q_n) = a) \tag{7}$$

when $r \geqslant r_2$.

Now $|Q_n(z)| \sim |b_n|\,|z|^n$ as $z \to \infty$. For a positive number $\delta < min(|a_m|, |b_n|)$, put $A_1 = |b_n| - \delta$, and then there exists $R > 0$ such that $|Q_n(z)| \geqslant A_1 |z|^n$ when $|z| \geqslant R$. Therefore, all roots of $Q_n(z) = w$ must lie in $\{z : |z| < r\}$ when $r \geqslant R$ and $|w| < A_1 r^n$. This means that $n(r, Q_n = w) = n$ when $r \geqslant R$ and $|w| < A_1 r^n$. Denote the roots of $f(w) = a$ by $\{w_k\}$. Thus

$$\begin{aligned} n(r, f(Q_n)) &= \sum_{|w_k| \leqslant M(r, Q_n)} n(r, Q_n = w_k) \geqslant \sum_{|w_k| < A_1 r^n} n(r, Q_n = w_k) \\ &= n \cdot n(A_1 r^n, f = a) \end{aligned}$$

when $r \geqslant R$. It follows that

$$\begin{aligned} N(r, f(Q_n) = a) &= \int_0^r \frac{n(t, f(Q_n) = a) - n(0, f(Q_n) = a)}{t}\, dt \\ &\geqslant \int_R^r \frac{n(t, f(Q_n) = a)}{t}\, dt + O(1) \\ &\geqslant \int_R^r \frac{n \cdot n(A_1 t^n, f = a)}{t}\, dt + O(1) = \int_{A_1 R^n}^{A_1 r^n} \frac{n(t, f = a)}{t}\, dt + O(1) \\ &= N(A_1 r^n, f = a) - N(A_1 R^n, f = a) + O(1). \end{aligned}$$

We thus obtain

$$N(r, f(Q_n) = a) \geqslant N(A_1 r^n, f = a) + O(1), \ (r \to \infty). \tag{8}$$

On the other hand, for sufficiently large r, we have

$$n(r, f(P_m) = a) = \sum_{|w_k| \leqslant M(r,P_m)} n(r, P_m = w_k) \leqslant m \cdot n(A_2 r^m, f = a),$$

where $A_2 = |a_m| + \delta$. It follows that

$$N(r, f(P_m) = a) \leqslant N(A_2 r^m, f = a) + O(1), \ (r \to \infty). \tag{9}$$

Since a isnt a Valiron deficient value of $f(z)$, it follows from (7), (8) and (9) that there exists $r_3 > r_2$, such that

$$T(A_2 r^m, f) \geqslant \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^3 \tau T(A_1 r^n, f) \tag{10}$$

when $r \geqslant r_3$. Put $c = ((1-\varepsilon)/(1+\varepsilon))^3 \tau$, $t = A_2 / A_1^{m/n}$, and $R_1 = A_1 r_3^n$. Hence it follows from (10) that

$$T\left(t r^{m/n}, f\right) \geqslant c T(r, f), \ (r \geqslant R_1). \tag{11}$$

We discuss the following three cases:

(1) If $m > n$. For any $\varepsilon > 0$, put $\alpha = m/n + \varepsilon$ and assume $t/r^{\varepsilon} < 1$ when $r \geqslant R_1$ (otherwise, we choose a larger R_1). By (11), we obtain that

$$T(r^{\alpha}, f) \geqslant c_1 T(r, f), \ (r \geqslant R_1).$$

It follows that

$$T\left(R_1^{\alpha^k}, f\right) \geqslant c_1^k T(R_1, f), \ (k = 1, 2, 3, \cdots). \tag{12}$$

For arbitrary real number $r \geqslant R_1$, since $\alpha > 1$, we assume $r \in \left[R_1^{\alpha^p}, R_1^{\alpha^{p+1}}\right)$ for some natural number p. By (12) we deduce

$$T(r, f) \geqslant T\left(R_1^{\alpha^p}, f\right) \geqslant c^p T(R_1, f) > \lambda_1 c^{\log\log r / \log \alpha},$$

where λ_1 is a positive number. It follows immediately that $\mu_f^* \geqslant \log c / \log \alpha$. Let $\varepsilon \to 0$, then $c \to \tau, \alpha \to m/n$. Thus $\tau \leqslant (m/n)^{\mu_f^*}$. Let $\tau \to \Omega$, then $\Omega \leqslant (m/n)^{\mu_f^*}$.

(2) If $m = n$ and $|a_m| > |b_n|$. Let δ be sufficiently small such that $A_2 > A_1$. Since $t > 1$, it follows from (11) that

$$T(t^k R_1, f) \geqslant c^k T(R_1, f), \ (k = 1, 2, 3, \cdots). \tag{13}$$

For arbitrary real number $r \geqslant R_2$, we assume $r \in [R_1 t^p, R_1 t^{p+1})$ for some natural number p. By (13) we deduce

$$T(r,f) \geqslant T(R_1 t^p, f) \geqslant c^p T(R_1, f) > \mu_1 c^{\log r/\log t},$$

where μ_1 is a positive number. It follows that $\mu_f \geqslant \log c/\log t$. Let $\varepsilon \to 0$, then $c \to \tau$, and $t \to |a_m/b_n|$. Thus $\tau \leqslant |a_m/b_n|^{\mu_f}$. Let $\tau \to \Omega$, then $\Omega \leqslant |a_m/b_n|^{\mu_f}$.

(3) If $m = n$, $|a_m| = |b_n|$ and $\mu_f < \infty$. Since $A_2 > A_1$,we have $t > 1$. We can deduce $\mu_f \geqslant \log c/\log t$ by the same method as in case 2). Let $\varepsilon \to 0$, then $c \to \tau$ and $t \to 1$. Thus $\tau \leqslant 1$. Let $\tau \to \Omega$, then $\Omega \leqslant 1$.

Now

$$\overline{\lim_{r\to\infty}}\, \sigma(r,f,P_m,Q_n) = 1/\Big(\underline{\lim_{r\to\infty}}\, \sigma(r,f,Q_n,P_m)\Big).$$

Therefore, by the above discussion, we know that $\overline{\lim_{r\to\infty}}\, \sigma(r,f,P_m,Q_n)$ has the lower bounds as stated in Theorem 2. The proof of Theorem 2 is thus complete. ∎

3. The proof of Theorem 1

In order to prove Theorem 1, we need the following results:

Lemma 1. [3] *Let $g(z)$ be a transcendental entire function, q be a natural number. Then for any $M > 0$, there exists $R_0 > 0$ and $R_n \to \infty$ (here $R_0 < R_1 < R_2 < \cdots < R_n < \cdots$), such that*

$$N(r, g(z) = w) > M \tag{14}$$

when $r \in [R_n, R_n^2]$ and $w \in \{w : R_0 \leqslant |w| \leqslant r^q\}$.

Lemma 2. *Let $g(z)$ be a transcendental entire function, $P(z)$ be a polynomial and $f(w)$ be a meromorphic function. Then*

$$\overline{\lim_{r\to\infty}}\, T(r,f(g))/T(r,f(P)) = \infty.$$

Proof: Denote the degree of $P(z)$ by m. We choose a natural number $q > m$ and real number $M > 0$. By Lemma 1, there exist R_0 and $R_n \to \infty$ (here $R_0 < R_1 < R_2 < \cdots < R_n < \cdots$), such that the inequality (14) holds when $r \in [R_n, R_n^2]$ and $w \in \{w : R_0 \leqslant |w| \leqslant r^q\}$. Now we can deduce the following inequality by a method similar to that used in [1] to prove that the superior limit of $T(r,f(g))/T(r,f)$ is infinite:

$$T(R_n^2, f(g)) > \frac{M}{2q}(1+o(1))T(R_n^{2q}, f(z)),\ (n \to \infty). \tag{15}$$

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Choose a complex number a which is not a Valiron deficient value of $f(g)$. We deduce from (10) and (15) that

$$T(R_n^2, f(g)) > \frac{M}{2q}(1+o(1))T(R_n^2, f(P)),\ (n \to \infty).$$

Let $M \to \infty$, then the proof of Lemma 2 is complete. ∎

LEMMA 3. [6] *Let $R(z,w)$ be a rational function of two variables, and let f be a meromorphic function. Then*

$$T(r, R(z, f(z))) = \partial R \cdot T(r, f) + O(\log r),\ (r \to \infty).$$

THE PROOF OF THEOREM 1: Firstly, we can deduce from the equality (1) and Lemma 3 that

$$\lim_{r\to\infty} T(r, f(g))/T(r, f(P_m)) = \frac{\partial R_2}{\partial R_1} \neq \infty. \tag{16}$$

It follows from Lemma 2 that $g(z)$ is a polynomial. If the degree n of $g(z)$ is not equal to m, by Theorem 2 and (16) we obtain

$$(n/m)^{\rho_f^*} = \frac{\partial R_2}{\partial R_1}. \tag{17}$$

Since f is not a constant, we have $\rho_f^* \geqslant 1$. By (17) we know : If $n < m$, then $n/m \geqslant (n/m)^{\rho_f^*} = (\partial R_2)/(\partial R_1)$, thus $n \geqslant ((\partial R_2)/(\partial R_1)) \cdot m$; If $n > m$, then $n/m \leqslant (n/m)^{\rho_f^*} = (\partial R_2)/(\partial R_1)$, thus $n \leqslant ((\partial R_2)/(\partial R_1)) \cdot m$. Hence n lies between m and $\left((\partial R_2)\middle/(\partial R_1)\right) \cdot m$. Below, we discuss three cases:

(1) If $m \neq n$, (17) implies $\rho_f^* < \infty$, thus $\rho_f = 0$.

(2) If $m = n$ and $|a_m| \neq |b_n|$, without loss of generality, we suppose $|a_m| > |b_n|$. By Theorem 2 and (16),

$$\left|\frac{a_m}{b_n}\right|^{\mu_f} = \left|\frac{a_m}{b_n}\right|^{\rho_f} = \frac{\partial R_1}{\partial R_2}.$$

Hence $\partial R_1 \geqslant \partial R_2$ and

$$\rho_f = \mu_f = \log \frac{\partial R_1}{\partial R_2} / \log \left|\frac{a_m}{b_n}\right|.$$

(3) If $m = n, |a_m| = |b_n|$ and $\partial R_1 \neq \partial R_2$, by Theorem 2 and (16) we have $\mu_f = \infty$. The proof of Theorem 1 is complete. ∎

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