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THE ZEROES OF FUNCTIONS RELATED TO DIRICHLET L-FUNCTIONS

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Hecke, [3], has shown for χ a real Dirichlet character modulo q, the associated Dirichlet L-function $L(s, \chi)$ has infinitely many zeroes on the line $\mathrm{Re}(s)=\frac{1}{2}$.

Here, using a method of Polya, [5], we show that both the real and imaginary parts of a function associated to $L(s, \chi)$ through the functional equation, have infinitely many zeroes on any line $\mathrm{Re}(s)=\sigma_0$. We prove:

THEOREM 1. *Let χ be a primitive Dirichlet character modulo q, and let $L(s, \chi)$ denote the corresponding Dirichlet L-function. Define:*

$$\Phi(s)=\pi^{-(1/2)(s+a)}\Gamma(\tfrac{1}{2}(s+a))L(s, \chi)$$

$$\Phi^*(s)=\pi^{-(1/2)(s+a)}\Gamma(\tfrac{1}{2}(s+a))L(s, \bar{\chi})$$

where

$$a=\begin{cases}0 & \text{if} \quad \chi(-1)=1,\\ 1 & \text{if} \quad \chi(-1)=-1.\end{cases}$$

Then

$$\mathrm{Re}\,\Phi(\sigma_0+it)+\mathrm{Re}\,\Phi^*(\sigma_0+it)$$

has infinitely many zeroes, on any line $\mathrm{Re}(s)=\sigma_0$.

COROLLARY 1. *If χ is a real primitive Dirichlet character modulo q, then* $\mathrm{Re}\,\Phi(\sigma_0+it)$ *has infinitely many zeroes on any line* $\mathrm{Re}(s)=\sigma_0$.

THEOREM 2. $\mathrm{Im}\,\Phi(\sigma_0+it)+\mathrm{Im}\,\Phi^*(\sigma_0+it)$ *has infinitely many zeroes on any line* $\mathrm{Re}(s)=\sigma_0$.

COROLLARY 2. *If χ is a real primitive Dirichlet character modulo q, then* $\mathrm{Im}\,\Phi(\sigma_0+it)$ *has infinitely many zeros on any line* $\mathrm{Re}(s)=\sigma_0$.

Berlowitz, [1], has considered the case of the Riemann zeta-function.

LEMMA 1. *Let χ be a primitive Dirichlet character modulo q, such that* $\chi(-1)=1$.

Define

$$\psi(z, \chi)=\sum_{n=1}^{\infty}\chi(n)e^{-n^2\pi z/q},$$

where z is a complex variable.

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Then $\psi(z, \chi)$ is analytic for $\text{Re}(z)>0$, and satisfies the functional equation:

$$\tau(\bar{\chi})\psi(z,\chi)=\left(\frac{q}{z}\right)^{1/2}\psi\left(\frac{1}{z},\bar{\chi}\right),\qquad \text{Re}(z)>0.$$

Here $\tau(\chi)$ is the Gaussian sum:

$$\tau(\chi)=\sum_{n=1}^{q}\chi(n)e\left(\frac{n}{q}\right).$$

Proof. This is essentially proven on p. 70 of [2].

LEMMA 2. *Let χ be a primitive Dirichlet character modulo q, such that $\chi(-1)=1$. Then $\psi(z,\chi)$ and all its derivatives tend to 0 as $z\to\pm qi$ along any route in an angle $|\arg(z\mp qi)|<\pi/2$.*

Proof. The proof of this lemma is similar to that given on p. 215 of [6].

Proof of Theorem. 1. We shall assume $\chi(-1)=1$, the proof for $\chi(-1)=-1$ following similar lines.

We have:

$$\Gamma(\tfrac{1}{2}s)=\int_0^\infty e^{-y}y^{(s/2)-1}\,dy,\qquad \text{Re}(s)>0.$$

Setting $y=n^2\pi x$, and multiplying both sides of the above equation by $\chi(n)$ we obtain:

$$\pi^{-(s/2)}\Gamma(\tfrac{1}{2}s)\chi(n)n^{-s}=\int_0^\infty \chi(n)e^{-n^2\pi x}x^{(s/2)-1}\,dx.$$

Summing over n, we get:

$$\Phi(s)=\int_0^\infty \psi(qx,\chi)x^{(s/2)-1}\,dx=\int_0^1\psi(qx,\chi)x^{(s/2)-1}\,dx+\int_1^\infty\psi(qx,\chi)x^{(s/2)-1}\,dx$$
$$=\int_1^\infty\psi\left(\frac{q}{x},\chi\right)x^{-(s/2)-1}\,dx+\int_1^\infty\psi(qx,\chi)x^{(s/2)-1}\,dx.$$

This last expression represents, by Lemma 1, an entire function, and gives the analytic continuation of $\Phi(s)$ over the plane.

Now setting $x=e^{2t}$; $s=\sigma_0+iu$, we have:

$$\Phi(\sigma_0+iu)=2\int_0^\infty e^{-\sigma_0 t}\psi(qe^{-2t},\chi)e^{-iut}\,dt+2\int_0^\infty e^{\sigma_0 t}\psi(qe^{2t},\chi)e^{iut}\,dt.$$

Thus

$$\Phi(\sigma_0+iu)+\Phi^*(\sigma_0+iu)=2\int_0^\infty e^{-\sigma_0 t}[\psi(qe^{-2t},\chi)+\psi(qe^{-2t},\bar{\chi})]e^{-iut}\,dt$$
$$+2\int_0^\infty e^{\sigma_0 t}[\psi(qe^{2t},\chi)+\psi(qe^{2t},\bar{\chi})]e^{iut}\,dt.$$

Taking the real part of both sides, we obtain:

$$\operatorname{Re}\Phi(\sigma_0+iu)+\operatorname{Re}\Phi^*(\sigma_0+iu)=2\int_0^\infty \{e^{-\sigma_0 t}[\psi(qe^{-2t},\chi)+\psi(qe^{-2t},\bar{\chi})]$$

$$+e^{\sigma_0 t}[\psi(qe^{2t},\chi)+\psi(qe^{2t},\bar{\chi})]\}\cos ut\,dt$$

$$=2\int_0^\infty \Omega(t)\cos ut\,dt.$$

Now, we see $\Omega(t)$ and $\operatorname{Re}\Phi(\sigma_0+iu)+\operatorname{Re}\Phi^*(\sigma_0+iu)$ are even functions of t and u.

Thus

$$\operatorname{Re}\Phi(\sigma_0+iu)+\operatorname{Re}\Phi^*(\sigma_0+iu)=\int_{-\infty}^\infty \Omega(t)e^{iut}\,dt.$$

Now $\Omega(t)$ is seen to be in the Schwartz space, (see [4], pg. 245), and thus we may apply the Fourier inversion formula to get:

$$2\pi\Omega(-u)=\int_{-\infty}^\infty [\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it)]e^{iut}\,dt$$

$$=2\int_0^\infty [\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it)]\cos ut\,dt.$$

Now, by Stirling's formula, the left and right hand sides of the above equation, originally defined for u real, can be continued analytically throughout the region $\operatorname{Re}(e^{-2u})>0$, for u complex.

Thus, we may define:

$$\Omega(u)=\Omega(-u)=\sum_{n=0}^\infty a(n)u^n, \qquad |u|<\pi/4.$$

Here for k an integer, $k\geq 0$,

$$a(2k)=\frac{\Omega^{(2k)}(0)}{(2k)!}=\frac{(-1)^k}{\pi(2k)!}\int_0^\infty [\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it)]t^{2k}\,dt;$$

$$a(2k+1)=0.$$

Now let us assume $\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it)$ has only finitely many zeroes, on the line $\operatorname{Re}(s)=\sigma_0$. Assume that there exists a $T>0$, such that for $t>T$,

$$\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it)<0.$$

Then

$$(-1)^k\pi(2k)!\,a(2k)=\int_0^\infty[\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it)]t^{2k}\,dt$$

$$\leq\int_0^T|\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it)|\,t^{2k}\,dt$$

$$+\int_{T+1}^{T+2}(\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it))t^{2k}\,dt$$

$$\leq C_1(T)T^{2k}+C_2(T)(T+1)^{2k},$$

where

$$C_1(T)=\int_0^T|\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it)|\,dt\geq 0.$$

$$C_2(T)=\int_{T+1}^{T+2}(\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it))\,dt<0.$$

Thus for k sufficiently large, say $k\geq N=N(T)$, $(-1)^k\pi(2k)!\,a(2k)<0$, and thus $(-1)^ka(2k)<0$. This however, leads to a contradiction:

$$\Omega(-iz)=\sum_{n=0}^\infty(-1)^na(2n)z^{2n},\qquad |z|<\pi/4.$$

$$\Omega^{(2N)}(-iz)=\sum_{n=N}^\infty\frac{(2n)!}{(2n-2N)!}(-1)^na(2n)z^{2n-2N}.$$

This, implies, since $(-1)^na(2n)$ is negative for $n\geq N$, that $\Omega^{(2N)}(-iz)$ is negative and monotonically decreasing as z ranges through the values 0 to $\pi/4$. This, however, contradicts Lemma 2.

A similar argument applies if we assume $\operatorname{Re}\Phi(\sigma_0+it)+\operatorname{Re}\Phi^*(\sigma_0+it)>0$, for $t>T$.

Corollary 1 follows from the fact that if χ is real then

$$\Phi(\sigma_0+it)=\Phi^*(\sigma_0+it).$$

Theorem 2 follows the same argument as Theorem 1.

References

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