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ON A GENERALIZATION OF THE CATENOID

DAVID E. BLAIR

It is a classical result that the only surface of revolution in Euclidean space $\mathbf{E}^3$ which is minimal is the catenoid. Of course the surface is conformally flat, but if M^n, $n \geqq 4$, is a conformally flat hypersurface of Euclidean space $\mathbf{E}^{n+1}$, then M^n admits a distinguished direction [**2**] ("tangent to the meridians"). Thus we seek to characterize conformally flat hypersurfaces of $\mathbf{E}^{n+1}$ which are minimal. Specifically we prove the following

THEOREM. *Let M^n, $n \geqq 4$, be a conformally flat, minimal hypersurface immersed in $\mathbf{E}^{n+1}$. Then M^n is either a hypersurface of revolution $S^{n-1} \times M^1$ where S^{n-1} is a Euclidean sphere and M^1 is a plane curve whose curvature κ as a function of arc length s is given by $\kappa = -(n-1)\alpha$, $\alpha = -1/\nu^n$ and*

$$s = \int \frac{\nu^{n-1}\,d\nu}{\{A\nu^{2n-2} - 1\}^{\frac{1}{2}}}$$

where A is a constant, or M^n is totally geodesic.

In section 1 we give some preliminaries and in section 2 prove the theorem.

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1. Preliminaries. Let $\langle\ ,\ \rangle$ denote the usual Riemannian metric on $(n+1)$-dimensional Euclidean space $\mathbf{E}^{n+1}$ and D its Riemannian connexion. Let $\iota : M^n \to \mathbf{E}^{n+1}$ be an n-dimensional hypersurface immersed in $\mathbf{E}^{n+1}$ and let g denote the induced metric and ∇ its Riemannian connexion. Then the Gauss-Weingarten equations are

$$D_{\iota_* X}\iota_* Y = \iota_* \nabla_X Y + h(X, Y)N$$
$$D_{\iota_* X} N = -\iota_* HX$$

where N is a unit normal, h the second fundamental form and H the corresponding Weingarten map. The hypersurface M^n is said to be *quasi-umbilical* [**1**; **2**], if there exist on M^n two functions α, β and a unit vector field U with covariant form u such that

(1.1) $\quad h = \alpha g + \beta u \otimes u.$

In [**2**], B.-Y. Chen and K. Yano showed that if M^n, $n \geqq 4$ is a conformally flat hypersurface of a space form, then it is quasi-umbilical. If the space form

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has constant curvature k, the curvature tensor of M^n is

$$\begin{aligned} g(R_{XY}Z, W) = (k+\alpha^2)(g(X, W)g(Y, Z) - g(X, Z)g(Y, Z)) \\ + \beta\alpha(g(X, W)u(Y)u(Z) - g(Y, W)u(X)u(Z) \\ + g(Y, Z)u(X)u(W) - g(X, Z)u(Y)u(W)) \end{aligned}$$

by virtue of (1.1) and the Gauss equation, where R_{XY} denotes the curvature transformation $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The Ricci tensor S and the scalar curvature ρ are then given by

$$\begin{aligned} (1.2) \quad S &= ((n-1)(k+\alpha^2) + \alpha\beta)g + (n-2)\alpha\beta u \otimes u, \\ \rho &= n(n-1)(k+\alpha^2) + 2(n-1)\alpha\beta. \end{aligned}$$

Thus the tensor

$$L = -\frac{S}{n-2} + \frac{\rho g}{2(n-1)(n-2)}$$

becomes

$$L = -\tfrac{1}{2}(k+\alpha^2)g - \alpha\beta u \otimes u.$$

Now as M^n is conformally flat, we have

$$(\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z) = 0$$

and moreover we have the Codazzi equation

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = 0.$$

Thus, differentiating

$$L + \alpha h = \tfrac{1}{2}(\alpha^2 - k)g$$

we find that

$$(1.3) \quad \beta(d\alpha \wedge u) = 0.$$

We now discuss briefly a problem in the theory of submanifolds of codimension 2. Let D denote the Riemannian connexion of the ambient space, ∇' the Riemannian connexion of the induced metric g', ψ_* the differential of the immersion, and C_1, C_2 unit normals orthogonal to each other with h', k' the corresponding second fundamental forms and l the third fundamental form. Suppose $h' = ag'$, $k' = bg'$, $l = 0$ where a and b are constants on the submanifold; in particular the submanifold is umbilical. The Gauss-Weingarten equations are

$$\begin{aligned} D_{\psi_*X}\psi_*Y &= \psi_*\nabla'_X Y + ag'(X, Y)C_1 + bg'(X, Y)C_2, \\ D_{\psi_*X}C_1 &= -a\psi_*X, \; D_{\psi_*X}C_2 = -b\psi_*X. \end{aligned}$$

Then setting

$$\bar{C}_1 = \frac{aC_1 + bC_2}{(a^2+b^2)^{\frac{1}{2}}} \quad \bar{C}_2 = \frac{bC_1 - aC_2}{(a^2+b^2)^{\frac{1}{2}}}$$

we have

$$D_{\psi_* X}\psi_* Y = \psi_*\nabla'_X Y + (a^2 + b^2)^{\frac{1}{2}} g'(X, Y)\bar{C}_1,$$

as well as, of course,

$$D_{\psi_* X}\bar{C}_1 = -(a^2 + b^2)^{\frac{1}{2}}\psi_* X, \qquad D_{\psi_* X}\bar{C}_2 = 0.$$

2. Proof of the Theorem. As pointed out in section 1, the conformal flatness of M^n, $n \geqq 4$, implies that M^n is quasi-umbilical and so by (1.1) the Weingarten map is given by

$$H = \alpha I + \beta u \otimes U,$$

where I denotes the identity. Since M^n is minimal, we have $trH = 0$ and hence $\beta = -n\alpha$, that is

$$H = \alpha I - n\alpha u \otimes U.$$

Now by (1.3) we have either $\beta = 0$ on M^n or $d\alpha = \gamma u$ for some function γ on M^n. $\beta = 0$ implies $\alpha = 0$ and hence $h = 0$, that is M^n is totally geodesic in E^{n+1}, the exceptional case of the theorem (M^n is a hyperplane and could of course, be thought of as $M^{n-1} \times M^1$ where M^1 is a line and M^{n-1} a space of constant curvature zero). Thus we consider the case $d\alpha = \gamma u$.

Differentiating H we have

$$\begin{aligned}(\nabla_X H)Y &= \nabla_X(\alpha Y - n\alpha u(Y)U) - \alpha\nabla_X Y + n\alpha u(\nabla_X Y)U \\ &= (X\alpha)Y - n(X\alpha)u(Y)U - n\alpha(\nabla_X u)(Y)U \\ &\qquad - n\alpha u(Y)\nabla_X U\end{aligned}$$

and so using $d\alpha = \gamma u$, the Codazzi equation becomes

$$\begin{aligned}(2.1)\quad 0 = (\nabla_X H)Y - (\nabla_Y H)X &= (X\alpha)Y - n(X\alpha)u(Y)U \\ &\quad - n\alpha(\nabla_X u)(Y)U - n\alpha u(Y)\nabla_X U - (Y\alpha)X + n(Y\alpha)u(X)U \\ &\quad + n\alpha(\nabla_Y u)(X)U + n\alpha u(X)\nabla_Y U \\ &= \gamma u(X)Y - 2n\alpha du(X, Y)U - n\alpha u(Y)\nabla_X U - \gamma u(Y)X \\ &\qquad + n\alpha u(X)\nabla_Y U.\end{aligned}$$

Taking the inner product of (2.1) with U we have $du = 0$; thus the distribution defined by $u = 0$ is integrable and so M^n is locally the product in the sense of separating coordinate systems of M^{n-1} and M^1 where U is tangent to M^1 and M^{n-1} is an integral submanifold of the distribution orthogonal to U.

Note that $d\alpha = \gamma u$ implies that $\gamma = U\alpha$ and $X\alpha = 0$ for X orthogonal to U. Moreover since $du = 0$ we have $d\gamma \wedge u = 0$ giving $d\gamma = (U\gamma)u$ and $X\gamma = 0$ for X orthogonal to U.

Now since $du = 0$ and U is unit we have

$$g(\nabla_U U, X) = g(\nabla_X U, U) = 0,$$

that is the integral curves of U are geodesics on M^n. Setting $Y = U$ in (2.1) we have

$$(2.2)\quad n\alpha\nabla_X U = -\gamma X + \gamma u(X)U.$$

We have already noted the case $\alpha = 0$ everywhere on M^n. We now show that if α does not vanish identically, then α is nowhere zero. Suppose $\alpha = 0$ at a point $P \in M$ but not identically zero on any neighborhood of P. Taking X to be any vector orthogonal to U at P in (2.2) we have that $\gamma = 0$ at P. On the other hand differentiating (2.2) with respect to Y with X orthogonal to U we have

$$n(Y\alpha)\nabla_X U + n\alpha\nabla_Y\nabla_X U = -(Y\gamma)X - \gamma\nabla_Y X$$

and hence $Y\gamma = 0$ at P. Differentiating successively we find that all derivatives of α vanish at P. Now as M^n is minimal in $\mathbf{E}^{n+1}$, the coordinate functions are harmonic and hence α is real analytic on M^n. Thus if $\alpha = 0$ at P, $\alpha = 0$ everywhere, and for the non-totally geodesic case we have

$$(2.3)\quad \nabla_X U = -\frac{\gamma}{n\alpha}X + \frac{\gamma}{n\alpha}u(X)U.$$

We now focus our attention on the manifold M^{n-1}. Consider M^{n-1} as a submanifold of codimension 2 in $\mathbf{E}^{n+1}$ immersed first in M^n, that is

$$M^{n-1} \xrightarrow{\varphi} M^n \xrightarrow{\iota} \mathbf{E}^{n+1}.$$

Let ∇' denote the induced connexion and h' the second fundamental for φ corresponding to the unit normal U, that is

$$\nabla_{\varphi_*X}\varphi_*Y = \varphi_*\nabla'_X Y + h'(X, Y)U$$

for X, Y tangent to M^{n-1}. Now by (2.3)

$$\nabla_{\varphi_*X}U = -\frac{\gamma}{n\alpha}\varphi_*X,$$

and hence

$$D_{\iota_*\varphi_*X}\iota_*U = -\frac{\gamma}{n\alpha}\iota_*\varphi_*X + h(\varphi_*X, U)N = -\frac{\gamma}{n\alpha}\iota_*\varphi_*X.$$

Also

$$D_{\iota_*\varphi_*X}N = -\iota_*H\varphi_*X = -\alpha\iota_*\varphi_*X.$$

Thus the Weingarten maps for M^{n-1} in $\mathbf{E}^{n+1}$ are $H' = (\gamma/n\alpha)I$ and $K' = \alpha I$. But as $X\alpha = 0$ and $X\gamma = 0$ for X orthogonal to U, α and γ are constant on M^{n-1}, hence M^{n-1} is umbilical in $\mathbf{E}^{n+1}$ and therefore a Euclidean sphere S^{n-1}.

Next we show that M^1 is a plane curve. Since $\nabla_U U = 0$ we have

$$(2.4)\quad D_{\iota_*U}\iota_*U = h(U, U)N = -(n-1)\alpha N.$$

Similarly

$$D_{\iota_* U} N = -\iota_* HU = (n-1)\alpha\iota_* U.$$

Now let X be any vector field in $\mathbf{E}^{n+1}$ defined along an integral curve of U and orthogonal to both $\iota_* U$ and N. Then

$$\langle D_{\iota_* U} X, \iota_* U \rangle = \langle X, (n-1)\alpha N \rangle = 0,$$
$$\langle D_{\iota_* U} X, N \rangle = -\langle X, (n-1)\alpha\iota_* U \rangle = 0,$$

and therefore M^1 is a plane curve in the plane spanned by $\iota_* U$ and N which are normals to M^{n-1} in $\mathbf{E}^{n+1}$. Hence the centers of the M^{n-1}'s lie in this plane.

Now two such planes will intersect in either a point (the M^{n-1}'s would be concentric) or a line (giving M^n as a hypersurface of revolution). It remains to show that M^n is indeed a hypersurface of revolution and to identify the curve M^1. For both tasks we will need a differential equation for the function α.

For X a unit vector orthogonal to U we compute the sectional curvature of a plane section containing U. Using (2.3) we have

$$\begin{aligned} &g(\nabla_U \nabla_X U - \nabla_X \nabla_U U - \nabla_{[U,X]} U, X) \\ &= g\left(\nabla_U\left(-\frac{\gamma}{n\alpha} X\right) + \frac{\gamma}{n\alpha}[U, X], X\right) = U\left(-\frac{\gamma}{n\alpha}\right) + \frac{\gamma^2}{n^2\alpha^2} \\ &= -\frac{1}{n\alpha} U\gamma + \frac{(n+1)\gamma^2}{n^2\alpha^2}. \end{aligned}$$

Now by (1.2) the Ricci curvature in the direction of U is $-\alpha^2(n-1)^2$ and therefore

$$(n-1)\left(\frac{1}{n\alpha} U\gamma - \frac{(n+1)\gamma^2}{n^2\alpha^2}\right) = -\alpha^2(n-1)^2.$$

Thus letting s denote arc length along an integral curve of U we have

$$\frac{1}{n\alpha}\frac{d^2\alpha}{ds^2} - \frac{(n+1)}{n^2\alpha^2}\left(\frac{d\alpha}{ds}\right)^2 + (n-1)\alpha^2 = 0 \tag{2.5}$$

along the integral curve.

We now show that M^n is a hypersurface of revolution. Let

$$\bar{C}_1 = \frac{(\gamma/n\alpha)\iota_* U + \alpha N}{\{\alpha^2 + \gamma^2/n^2\alpha^2\}^{\frac{1}{2}}}, \quad \bar{C}_2 = \frac{\alpha\iota_* U - (\gamma/n\alpha)N}{\{\alpha^2 + \gamma^2/n^2\alpha^2\}^{\frac{1}{2}}}.$$

Then $\bar{C}_1$ is a unit normal to an M^{n-1} in a hyperplane $\mathbf{E}^n$ and $\bar{C}_2$ is a unit normal to E^n in E^{n+1}. Now a straightforward computation using (2.5) shows that

$$D_{\iota_* U} \bar{C}_2 = 0,$$

that is the $\mathbf{E}^n$'s form a parallel family of hyperplanes and hence the planes of the integral curves of U intersect in a line.

Finally let V be a unit parallel vector field in $\mathbf{E}^{n+1}$ in the direction of $\bar{C}_2$ (i.e. parallel to the line of centers) and let $\langle V, \iota_* U\rangle = \cos\theta$ choosing the orientation such that $\langle V, N\rangle = -\sin\theta$. Then

$$\iota_* U\langle V, \iota_* U\rangle = \langle V, N\rangle \frac{d\theta}{ds}$$

and using (2.4),

$$\iota_* U\langle V, \iota_* U\rangle = -(n-1)\alpha\langle V, N\rangle.$$

Thus the curvature $\kappa = d\theta/ds$ of M^1 is given by

$$\kappa = -(n-1)\alpha,$$

but α as a function of s is given by (2.5). Setting $\alpha = 1/\nu^n$, (2.5) becomes

$$\frac{d^2\nu}{ds^2} - (n-1)\frac{1}{\nu^{2n-1}} = 0.$$

Integrating once we have

$$\frac{d\nu}{ds} = \{-1/\nu^{2n-2} + A\}^{\frac{1}{2}}.$$

where A is a constant. Hence the plane curve M^1 is given by its curvature $\kappa(s) = -(n-1)\alpha(s)$ where $\alpha = -1/\nu^n$ and

$$s = \int \frac{\nu^{n-1} d\nu}{\{A\nu^{2n-2} - 1\}^{\frac{1}{2}}}$$

completing the proof.

Remark. For $n = 3$, the conformal flatness of M^n does not yield quasi-umbilicity. If however we assume that M^3 is a minimal, quasi-umbilical hypersurface of $\mathbf{E}^4$, then the conclusion of the theorem also holds with the same proof.

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Michigan State University,
East Lansing, Michigan