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MULTIPLIERS BETWEEN SOBOLEV SPACES

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ABSTRACT. A sufficient condition for the boundedness of a multiplier from a Sobolev space of index $t > 1/4$ to one of opposite index $-t$ is obtained. The condition relates the indices of the Sobolev spaces to which the multiplier belongs to the pairs of Sobolev spaces between which the multiplier is bounded. The result is applied to homogeneous multipliers and a description of these multipliers in this setting is presesented. Extensions to higher dimensions are indicated.

Multipliers form an important class of densely defined operators on weighted L^p spaces. Obtaining sufficient and in addition possibly necessary conditions for the boundedness of these operators has an interesting history with powerful results depending on sensitive integral estimates and special Hörmander type conditions. For instance, papers [3] of Muckenhoupt, Wheeden and Young, and [4] of E. Sawyer consider multipliers of Hörmander type on power weighted L^p spaces. The results are very sharp and in some cases necessary and sufficient. The Hörmander condition of type α restricts the multiplier to be essentially pointwise bounded and have its α^{th} derivative have average square growth grow not too quickly at ∞. Other papers have dealt with studying the classes of multipliers giving bounded operators on weighted L^p spaces. Moreover, the subject has focused primarily on conditions for boundedness on a fixed weighted L^p space and not on the more general question of determining when the operator is bounded between different spaces having different weights.

In this paper we determine sufficient boundedness conditions for multipliers between Sobolev spaces of different indices. Though necessary and sufficient conditions have been described in this context, these conditions relate to the boundedness of associated multipliers between the space and L^2 and local properties of the multiplier. Indeed, Maz'ya and Shaposhnikova [2, 2.2.7, Theorem 1] present precisely these types of necessary and sufficient conditions for a multiplier to be bounded. The sufficiency conditions we present depend only on the Sobolev space to which the multiplier belongs. Indeed, our results, though not close to being necessary, are quite general and applicable to fairly unrestricted families of multipliers. Essentially knowing the highest Sobolev index to which the multiplier belongs determines pairs of Sobolev spaces for which the operator is bounded. The estimates for the bounds are made in terms of the corresponding convolution operators. Moreover, the arguments are clean and elementary.

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As an application we consider the multipliers which arise from convolution operators homogeneous of degree d. These operators play fundamental roles in the representation theory of simple Lie groups and related questions in automorphic forms and have important applications in abelian Fourier analysis. The distributions which define these operators are well known and much is known about their Fourier transforms. For completeness we present a description of their Fourier transforms for $d > 0$ and then use our boundedness results to obtain Sobolev spaces between which these operators are bounded. Finally we indicate the generalizations of these results to the n dimensional case. Similar generalization should be possible for the L^p Sobolev spaces.

Multipliers on Sobolev spaces. Let $\mathcal{S}$ be the Schwartz space of C^∞ rapidly decreasing functions on $\mathbf{R}$ and let $\mathcal{S}'$ be the space of all tempered distributions. The Fourier transform is defined on $\mathcal{S}$ by $\hat{f}(x) = \mathcal{F}f(x) = \frac{1}{\sqrt{2\pi}} \int f(y)e^{-ixy}\,dy$. Its inverse is $\mathcal{F}^{-1}(f) = \mathcal{F}(\check{f})$ where $\check{f}(x) = f(-x)$. We shall denote the value of a distribution U at f by $U(f)$, (f, U) or by $\int dU(x)f(x)$. The mapping $U \mapsto \mathcal{F}U$ defined by $(f, \mathcal{F}U) = (\mathcal{F}f, U)$ is an extension of $\mathcal{F}$ to an isomorphism of $\mathcal{S}'$. A multiplication is defined between functions f in $\mathcal{S}$ and distributions U in $\mathcal{S}'$ by $(h, fU) = (hf, U)$. Recall that differentiation is given on $\mathcal{S}'$ by $(f, DU) = (-f', U)$.

The s^{th} Sobolev space $\hat{\mathbf{H}}_s$ is the space of distributions obtained by Fourier transforming the space $\mathbf{H_s} = \mathbf{L^2}\big((\mathbf{1} + \mathbf{x^2})^{\mathbf{s}}\,\mathbf{dx}\big)$. This space is a Hilbert space with norm defined by $\|\mathcal{F}f\|_s = \{\int |f(x)|^2(1+x^2)^s\,dx\}^{\frac{1}{2}}$. One has $\hat{\mathbf{H}}_s \subset \hat{\mathbf{H}}_t$ and $\|U\|_s \geq \|U\|_t$ for $s > t$. Moreover, $\|U\|_s^2 = \|U\|_{s-1}^2 + \|DU\|_{s-1}^2$ for $U \in \hat{\mathbf{H}}_s$. Furthermore, if $0 \leq m \leq s - 1/2$, then the distribution $D^m U$ is a continuous function and the mapping $U \mapsto D^m U$ is a continuous transformation from $\hat{\mathbf{H}}_s$ into the space of continuous functions on $\mathbf{R}$ equipped with the topology of uniform convergence.

The natural pairing $\langle\ ,\ \rangle$ between the L^2 spaces $L^2\big((1+x^2)^s\,dx\big)$ and $L^2\big((1+x^2)^{-s}\,dx\big)$ defined by $\langle f, g\rangle = \int f(x)g(x)\,dx$ defines a pairing between the Sobolev spaces $\hat{\mathbf{H}}_s$ and $\hat{\mathbf{H}}_{-s}$. This pairing satisfies

(1) $|\langle U, V\rangle| \leq \|U\|_s \|V\|_{-s}$

(2) If L is a continuous linear functional on $\hat{\mathbf{H}}_s$, then there exists a unique $V \in \hat{\mathbf{H}}_{-s}$ such that $\|L\| = \|V\|_{-s}$ and $L(U) = \langle U, V\rangle \quad \forall U \in \hat{\mathbf{H}}_s$.

The multiplication operators between Sobolev spaces $\hat{\mathbf{H}}_s$ and $\hat{\mathbf{H}}_t$ correspond to convolution operators between the L^2 spaces $\mathbf{H}_s$ and $\mathbf{H}_t$. We begin by analyzing these operators.

We first note the following:

(3) The maximum and minimum of the function $y \mapsto \frac{1+y^2}{1+(y-x)^2}$ are $M_{|x|}$ and $M_{-|x|}$ where $M_r = 1 + 1/2r^2 + 1/2r(r^2+4)^{\frac{1}{2}}$.

LEMMA 1. *Let $\tau_x f(y) = f(y-x)$. Then τ_x is a bounded operator on $\mathbf{H}_s$. In fact, $\|\tau_x\| = (M_{|x|})^{s/2}$ if $s \geq 0$ and $\|\tau_x\| = (M_{-|x|})^{s/2} = (M_{|x|})^{-s/2}$ if $s < 0$.*

PROOF. The operator J_s defined by $J_s f(x) = g_s(x)f(x)$ where $g_s(x) = (1+x^2)^{s/2}$ is an isomorphism of $\mathbf{H}_t$ onto $\mathbf{H}_{t-s}$. Thus the operator B on $L^2(\mathbf{R})$ defined by $B = J_s\tau_x J_s^{-1}$ has the same norm as τ_x. But $B = A\tau_x$ where A is the multiplication operator defined by

$Af(y) = \frac{g_s(y)}{g_s(y-x)} f(y)$. Since τ_x is unitary on $L^2(\mathbf{R})$, the norm of τ_x on $\mathbf{H}_s$ is the maximum of the function $\{\frac{1+y^2}{1+(y-x)^2}\}^{s/2}$. The result follows by (3). ∎

LEMMA 2. *Suppose $s - t < 1/2$ where $t > 1/2$. Then*

$$\iint g_{s-1}^2(y+x)(1+x^2)^{-t}(1+y^2)^{-t}\,dx\,dy < \infty.$$

PROOF. If $s \geq 1$, then by (3)

$$\begin{aligned}\iint \left(1+(x+y)^2\right)^{s-1}(1+x^2)^{-t}(1+y^2)^{-t}\,dx\,dy \\ = \iint \left(\frac{1+(y+x)^2}{1+x^2}\right)^{s-1}(1+x^2)^{s-t-1}(1+y^2)^{-t}\,dx\,dy \\ \leq \iint (M_{|y|})^{s-1}(1+x^2)^{s-t-1}(1+y^2)^{-t}\,dx\,dy\end{aligned}$$

But $M_{|y|} \leq 1 + 1/2y^2 + 1/2|y|(y^2+4|y|+4)^{1/2} = 1 + |y| + y^2 \leq 2(1+y^2)$. Thus

$$\begin{aligned}\iint g_{s-1}^2(x+y)(1+x^2)^{-t}(1+y^2)^{-t}\,dx\,dy \leq \\ 2^{s-1}\iint (1+x^2)^{s-t-1}(1+y^2)^{s-t-1}\,dx\,dy < \infty\end{aligned}$$

provided $s - t < 1/2$.

If $s < 1$, then $\iint g_{s-1}(x+y)^2(1+x^2)^{-t}(1+y^2)^{-t}\,dx\,dy < \iint(1+x^2)^{-t}(1+y^2)^{-t}\,dx\,dy < \infty$ when $t > 1/2$. ∎

PROPOSITION 1. *Suppose $s-t < 1/2$ where $t > 1/2$. Then the convolution operator $f \longmapsto f * g_{s-1}$ is a bounded operator from $\mathbf{H}_t$ to $\hat{\mathbf{H}}_{-t}$.*

PROOF. Note $f * g_{s-1}(x) = \langle f, \tau_x g_{s-1}\rangle$. Thus $|f * g_{s-1}(x)| \leq \|f\|_t \|\tau_x g_{s-1}\|_{-t}$. It therefore suffices to show that the function $H(x) = \|\tau_x g_{s-1}\|_{-t}$ is in $\hat{\mathbf{H}}_{-t}$. But

$$\begin{aligned}\|H\|_{-t}^2 &= \int H(x)^2(1+x^2)^{-t}\,dx \\ &= \iint \left(\tau_x g_{s-1}(y)\right)^2(1+y^2)^{-t}(1+x^2)^{-t}\,dy\,dx.\end{aligned}$$

Thus by Lemma 2, H is in $\mathbf{H}_{-t}$. ∎

Define the *left Sobolev space* $\hat{\mathbf{H}}_{-s}$ by $\hat{\mathbf{H}}_{-s} = \bigcap_{t<s}\hat{\mathbf{H}}_t$. This is the space obtained by Fourier transforming $\mathbf{H}_{s-} = \bigcap_{t<s}\mathbf{H}_t$. We use the following lemma to establish our central theorem.

LEMMA 3. *Suppose $u \in \mathbf{H}_{1/2-}$ and $f \in \mathbf{H}_0 = L^2(\mathbf{R})$. Then the function $H(x) = \int \frac{g_s(y-x)}{g_t(y)} u(y-x) f(y)\,dy$ is in $\hat{\mathbf{H}}_{-t}$ whenever $t > 1/4$ and $t > s - 1/4$. In this case there is a constant K with $\|H_{-t}\| \leq K\|f\|_0$.*

PROOF. We may assume $u \geq 0$ and $f \geq 0$. Then

$$H(x) = \int g_{r-\frac{1}{2}}(y-x)g_{-t}(y)f(y)^{1/2}g_{\frac{1}{2}-r+s}(y-x)u(y-x)f(y)^{1/2}\,dy$$
$$\leq \Big(\int g_{2r-1}(y-x)g_{-2t}(y)f(y)\,dy\Big)^{1/2}$$
$$\Big(\int (g_{1-2r+2s}(y-x)u(y-x)^2 f(y)\,dy\Big)^{1/2}.$$

Therefore $H(x)^2 \leq A(x)B(x)$ where $A(x) = g_{2r-1} * (g_{-2t}f)(x)$, and $B(x) = (g_{\frac{1}{2}-r+s}\check{u})^2 * f(x)$.

Choose r with $t > r - 1/4 > s - 1/4$. Since $f \in \mathbf{H}_0$, $g_{-2t}f \in \mathbf{H}_{2t}$. It follows by Proposition 1 that $A \in \mathbf{H}_{-2t}$ and $\|A\|_{-2t} \leq K_1\|g_{-2t}f\|_{2t} = K_1\|f\|_0$ for some constant K_1.

Next note that $g_{\frac{1}{2}-r+s}\check{u} \in \mathbf{H}_{r-s-} \subset L^2(\mathbf{R})$ since $u \in \mathbf{H}_{1/2-}$. Hence $B = h * f$ with $h = (g_{\frac{1}{2}-r+s}\check{u})^2 \in L^1(\mathbf{R})$. Therefore $\|B\|_0 \leq \|h\|_{L_1}\|f\|_0$ is finite.

Finally

$$\begin{aligned}\|H\|_{-t}^2 &\leq \int A(x)B(x)(1+x^2)^{-t}\,dx\\ &= \langle g_{-2t}B, A\rangle\\ &\leq \|g_{-2t}B\|_{2t}\|A\|_{-2t}\\ &= \|B\|_0\|A\|_{-2t}\\ &\leq K_1\|f\|_0\|h\|_{L_1}\|f\|_0.\end{aligned}$$

Hence $H \in \mathbf{H}_{-t}$ and $\|H\|_{-t} \leq K\|f\|_0$ where $K = \sqrt{K_1\|h\|_{L_1}}$. ■

THEOREM 1. *Suppose $u \in \mathbf{H}_{1/2-s-}$. Then $f \mapsto f * u$ is a bounded operator from $\mathbf{H}_t$ to $\mathbf{H}_{-t}$ if $t > 1/4$ and $t > s - 1/4$.*

PROOF. Let $\tilde{u} = g_{-s}\check{u}$ and $\tilde{f} = g_t f$. Then $\tilde{u} \in g_{-s}\mathbf{H}_{\frac{1}{2}-s-} = \mathbf{H}_{\frac{1}{2}-}$ and $\|\tilde{f}\|_0 = \|f\|_t$. Thus $f * u(x) = \int \frac{g_s(y-x)}{g_t(y)}\tilde{u}(y-x)\tilde{f}(y)\,dy$. By Lemma 3, $f * u \in \mathbf{H}_{-t}$ and $\|f * u\|_{-t} \leq K\|\tilde{f}\|_0 = K\|f\|_t$. ■

A *multiplication operator* from $\hat{\mathbf{H}}_s$ to $\hat{\mathbf{H}}_t$ is a bounded linear operator L between these two spaces for which there is a tempered distribution U with $Lf = fU$ for all f in $\mathcal{S}$. Such a distribution U will be called a *bounded multiplier.*

Hence a distribution U is a bounded multiplier between $\hat{\mathbf{H}}_s$ and $\hat{\mathbf{H}}_t$ provided the mapping $f \mapsto fU$ for f in $\mathcal{S}$ has a bounded extension to $\hat{\mathbf{H}}_s$. In this case the value of the extension at a distribution V in $\hat{\mathbf{H}}_s$ will be denoted by VU. Theorem 1 has the following corollary.

COROLLARY 1. *Let U be a distribution in the Sobolev space $\hat{\mathbf{H}}_{\frac{1}{2}-s-}$. Then U is a bounded multiplier between $\hat{\mathbf{H}}_t$ and $\hat{\mathbf{H}}_{-t}$ whenever t is larger than both $1/4$ and $s - 1/4$.*

PROOF. By Theorem 1, the operator $V \mapsto \mathcal{F}^{-1}V \mapsto \mathcal{F}^{-1}V * \mathcal{F}^{-1}U \mapsto \mathcal{F}(\mathcal{F}^{-1}V * \mathcal{F}^{-1}U)$ is a bounded operator from $\hat{\mathbf{H}}_t$ to $\hat{\mathbf{H}}_{-t}$. Moreover, if $V = f \in \mathcal{S}$ and $h \in \mathcal{S}$, then:

$$\begin{aligned}\big(h, \mathcal{F}(\mathcal{F}^{-1}f * \mathcal{F}^{-1}U)\big) &= (\mathcal{F}h, \mathcal{F}^{-1}f * \mathcal{F}^{-1}U) \\ &= \iint \mathcal{F}(h)(y)\mathcal{F}^{-1}f(y-x)\,dy\,\mathcal{F}^{-1}U(x)\,dx \\ &= \iint \mathcal{F}(h)(y)(\mathcal{F}f)(x-y)\,dy\,\mathcal{F}^{-1}U(x)\,dx \\ &= \int (\mathcal{F}h * \mathcal{F}f)(x)\mathcal{F}^{-1}U(x)\,dx \\ &= (\mathcal{F}(hf), \mathcal{F}^{-1}U) \\ &= (hf, U) \\ &= (h, fU).\end{aligned}$$

∎

Homogeneous Multiplication Operators. In the remaining part of this paper we will study all those operators A commuting with translations and homogeneous with respect to dilations on the group $\mathbf{R}$. Specifically, we assume from the fact that A commutes with translations that $Af = f * U$ for some tempered distribution U. Moreover, A is homogeneous of degree d if $A({}_af) = a^d\big({}_aA(f)\big)$ where ${}_af(x) = f(a^{-1}x)$, $a > 0$. From this it follows that the distribution V defined by $V(f) = \int dU(y)|y|^{-d}f(y)$ is invariant under the action $y \longmapsto ay$ for $a > 0$. Since this action has two orbits, the positive and negative reals, it follows that two scalars c_+ and c_- along with a Haar measure on $\mathbf{R}^+$ uniquely determine V. Namely $V(f) = c_+\int_0^\infty f(y)\,\frac{dy}{|y|} + c_-\int_{-\infty}^0 f(y)\frac{dy}{|y|}$. From this we see that U has form

$$U(f) = c_+\int_{\mathbf{R}^+} f(y)|y|^{d-1}\,dy + c_-\int_{\mathbf{R}^-} f(y)|y|^{d-1}\,dy.$$

Next define functions ϵ_+ and ϵ_- by $\epsilon_+(x) = 1$ and $\epsilon_-(x) = \operatorname{sgn}(x)$. It follows that any distribution U of the above form is a linear combination of the distributions defined by the functions $U(d,\pm)(x) = \epsilon_\pm(x)|x|^{d-1}$. Thus any operator commuting with translations and homogeneous of degree d relative to dilations on the space of Schwartz functions is a combination of the operators $A(d,\pm)$ defined by the following:

$$A(d,\pm)f(x) = \int |y|^{d-1}\epsilon_\pm(y)f(x-y)\,dy. \tag{4}$$

In general these operators are not bounded on $L^2(\mathbf{R})$. We thus consider them in the context of Theorem 1. That is they are bounded convolution operators between spaces $\mathbf{H}_t = L^2\big((1+x^2)^t\,dx\big)$ and $\mathbf{H}_{-t} = L^2\big((1+x^2)^{-t}\,dx\big)$ for appropriate values of t. It then follows that the Fourier transforms $\hat{A}(d,\pm) = \mathcal{F}A(d,\pm)\mathcal{F}^{-1}$ are multiplication operators between the Sobolev spaces $\hat{\mathbf{H}}_t$ and $\hat{\mathbf{H}}_{-t}$ defined by the distribution $\hat{U}(d,\pm)$. That is:

$$\hat{A}(d,\pm)f = f\hat{U}(d,\pm). \tag{5}$$

THEOREM 2. *For $d > 1/2$, the functions $U(d,\pm)$ are in $\mathbf{H}_{\frac{1}{2}-d-}$. In particular the operators $\hat{A}(d,\pm)$ are bounded multiplication operators between the Sobolev spaces $\hat{\mathbf{H}}_s$ and $\hat{\mathbf{H}}_{-s}$ for $s > 1/4$ and $s > d - 1/4$.*

PROOF. Note $\int |U(d,\pm)(x)|^2(1+x^2)^s\,dx = \int |x|^{2d-2}(1+x^2)^s\,dx < \infty$ iff $d > 1/2$ and $s < 1/2 - d$. Thus $U(d,\pm) \in \mathbf{H}_{\frac{1}{2}-d-}$ for $d > 1/2$. The final statement follows from Corollary 1. ∎

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The remaining part of the paper will determine the homogeneous distributions $\hat{U}(d,\pm)$. Results concerning homogeneous distributions are known in various parts of the literature. Our discussion of these distributions will complete the description of the operators $\hat{A}(d,\pm)$.

Note that the distributions $\hat{U}(d,\pm)$ are tempered when $d > 0$ and are weakly analytic in d.

PROPOSITION 2. *Suppose f is a Schwartz function vanishing in a neighborhood of 0. Then*

$$\big(f, \hat{U}(d,\pm)\big) = c(d,\pm)\int \epsilon_\pm(x)|x|^{-d} f(x)\,dx \text{ where}$$

$$c(d,+) = 2^{d-\frac{1}{2}}\frac{\Gamma(d/2)}{\Gamma\big((1-d)/2\big)} \text{ and}$$

$$c(d,-) = -2^{d-\frac{1}{2}}\frac{\Gamma\big((d+1)/2\big)}{\Gamma(1-d/2)}i.$$

PROOF. Let $\mathbf{R}^*$ be the multiplicative group of nonzero real numbers. Then, since $\mathcal{F}({}_af)(y) = |a|\mathcal{F}(f)(ay)$, the distributions $\hat{U}(d,\pm)$ restricted to $C_c^\infty(\mathbf{R}^*)$ satisfy $\big({}_af, \hat{U}(d,\pm)\big) = \epsilon_\pm(a)|a|^{1-d}\big(f, \hat{U}(d,\pm)\big)$. Define functions $w_\pm$ by $w_\pm(x) = \epsilon_\pm(x)|x|^{d-1}$. Then the distributions $w_\pm\hat{U}(d,\pm)$ are invariant under multiplication on $\mathbf{R}^*$. Indeed,

$$\begin{aligned}\big({}_af, w_\pm\hat{U}(d,\pm)\big) &= \big(w_\pm({}_af), \hat{U}(d,\pm)\big)\\ &= \epsilon_\pm(a)|a|^{d-1}\big({}_a(w_\pm f), \hat{U}(d,\pm)\big)\\ &= \big(w_\pm f, \hat{U}(d,\pm)\big)\\ &= \big(f, w_\pm\hat{U}(d,\pm)\big).\end{aligned}$$

Thus the distributions $w_\pm\hat{U}(d,\pm)$ are multiples of Haar measure $\frac{dx}{|x|}$ on $\mathbf{R}^*$. Thus there are constants $c(d,\pm)$ with

$$\big(f, w_\pm\hat{U}(d,\pm)\big) = c(d,\pm)\int\frac{f(x)}{|x|}\,dx.$$

From this it follows that $\big(f, \hat{U}(d,\pm)\big) = c(d,\pm)\int \epsilon_\pm(x)\frac{f(x)}{|x|^d}\,dx$ for all Schwartz functions f vanishing near 0.

Finally to determine the constants $c(d,\pm)$, we note these are analytic functions in d. Moreover, for $0 < d < 1/2$, $U(d,\pm)$ are the sums of an L^1 function and an L^2 function. In fact the functions $\chi_{[-1,1]}U(d,\pm)$ are integrable and the functions $(1-\chi_{[-1,1]})U(d,\pm)$ are square integrable. Therefore, the distributions $\hat{U}(d,\pm)$ are functions and thus must be given everywhere on $\mathcal{S}$ by

$$\big(f, \hat{U}(d,\pm)\big) = c(d,\pm)\int \epsilon_\pm(x) f(x)|x|^{-d}\,dx.$$

Using the fact that $\mathcal{F}(e^{-x^2/2}) = e^{-x^2/2}$, one sees that

$$\int_0^\infty |x|^{d-1}e^{-x^2/2}\,dx = c(d,+)\int_0^\infty |x|^{-d}e^{-x^2/2}\,dx.$$

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Replacing x by $\sqrt{2x}$, we finally obtain $c(d,+) = 2^{d-1/2}\frac{\Gamma(d/2)}{\Gamma((1-d)/2)}$ for $0 < d < 1/2$. This formula holds for all d by analyticity.

To find $c(d,-)$ a similar argument is used using the fact that $\mathcal{F}(xe^{-x^2/2}) = -ixe^{-x^2/2}$. ∎

COROLLARY 2. *If $0 < d < 1$, then*

$$\big(f, \hat{U}(d,\pm)\big) = c(d,\pm)\int f(x)\epsilon_\pm(x)|x|^{-d}\,dx$$

for all Schwartz functions f.

PROOF. In the proof of Theorem 3 we showed the equalities hold for $0 < d < 1/2$. Since both sides are analytic in d on the interval $(0,1)$, the equalities also hold on this interval. ∎

COROLLARY 3. *Suppose f is a Schwartz function and $f^{(k)}(0) = 0$ for $k \le [d-1]$. Then $\big(f, \hat{U}(d,\pm)\big) = c(d,\pm)\int f(x)\epsilon_\pm(x)|x|^{-d}\,dx$.*

PROOF. The equalities hold for $0 < d < 1$. Both sides are analytic as long as $f(x)|x|^{-d}$ is integrable on the interval $[-1,1]$. But if $f^{(k)}(0) = 0$ for $k \le [d-1]$, then there is a K such that $|f(x)| \le K|x|^{[d]}$ for x in this interval. It follows that the equalities hold at d. ∎

PROPOSITION 3. *Suppose n is a natural number. Then*

(*a*) $\hat{U}(d,\pm) = (-1)^{\frac{n}{2}}D^n\hat{U}(d-n,\pm)$ *if n is even*

(*b*) $\hat{U}(d,\pm) = (-1)^{\frac{n-1}{2}}iD^n\hat{U}(d-n,\mp)$ *if n is odd.*

PROOF. Note that $U(d,\pm) = x^nU(d-n,\pm)$ if n is even and $U(d,\pm) = x^nU(d-n,\mp)$ if n is odd. The result now follows from the fact that $\mathcal{F}(xU) = iD(\mathcal{F}U)$ for any tempered distribution U. ∎

The formulas given in Corollary 2 and Proposition 3 determine all the distributions $\hat{U}(d,\pm)$ for any positive noninteger d. To determine $\hat{U}(d,\pm)$ for integer values of d, it suffices to obtain the distributions $\hat{U}(1,\pm)$.

PROPOSITION 4. *Let f be a Schwartz function. Then:*

$$\big(f, \hat{U}(1,+)\big) = \sqrt{2\pi}f(0)$$

$$\big(f, \hat{U}(1,-)\big) = -i\sqrt{\frac{2}{\pi}}\left(\int_{|x|\ge 1}\frac{\epsilon_-(x)f(x)}{|x|}\,dx + \int_{|x|\le 1}\frac{\epsilon_-(x)\big(f(x)-f(0)\big)}{|x|}\,dx\right).$$

PROOF. (a) follows from the identity $\frac{1}{2\pi}\int\hat{f}(x)\,dx = f(0)$.

For (b) note by Corollary 2 that

$$\big(f, \hat{U}(1,-)\big) = -i\Big(\frac{2}{\pi}\Big)^{1/2}\lim_{s\to 1-}\int\frac{\epsilon_-(x)f(x)}{|x|^s}\,dx.$$

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The result follows since

$$\lim_{s\to 1-}\int_{-1}^{1}\frac{\epsilon_-(x)f(x)}{|x|^s}\,dx=\lim_{s\to 1-}\int_{-1}^{1}\frac{\epsilon_-(x)\big(f(x)-f(0)\big)}{|x|^s}\,dx$$
$$=\int_{-1}^{1}\frac{\epsilon_-(x)\big(f(x)-f(0)\big)}{|x|}\,dx$$

for $|f(x)-f(0)|\leq K|x|$ for some constant K. ∎

From Propositions 3 and 4 we see the distributions $\hat{U}(n,+)$ for n odd and $\hat{U}(n,-)$ for n even are distributions supported at 0 of particularly simple form. Indeed, if δ is the Dirac unit mass at 0, then:

$$\text{(6)}\qquad \begin{aligned}\hat{U}(n,+)&=(-1)^{\frac{n-1}{2}}\sqrt{2\pi}D^{n-1}\delta \text{ if } n \text{ is odd,}\\ \hat{U}(n,-)&=-i(-1)^{\frac{n}{2}}\sqrt{2\pi}D^{n-1}\delta \text{ if } n \text{ is even.}\end{aligned}$$

In particular, since $\hat{A}(d,\pm)V=V\hat{U}(d,\pm)$, we see for f in $\hat{\mathbf{H}}_n$ that:

$$\text{(7)}\qquad \begin{aligned}\hat{A}(n,+)f&=(-1)^{\frac{n-1}{2}}\sqrt{2\pi}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}f^k(0)D^{n-1-k}\delta &&\text{if } n \text{ is odd, and}\\ \hat{A}(n,-)f&=i(-1)^{\frac{n}{2}}\sqrt{2\pi}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}f^k(0)D^{n-1-k}\delta &&\text{if } n \text{ is even.}\end{aligned}$$

The boundedness of the multiplication operators between the Sobolev spaces $\hat{\mathbf{H}}_t$ and $\hat{\mathbf{H}}_{-t}$ in the one dimensional Euclidean case have natural extensions to the case $\mathbf{R}^n$. Specifically, if $\hat{\mathbf{H}}_s$ is the Sobolev space obtained by Fourier transforming the space $L^2(\mathbf{R}^n,(1+\|x\|^2)^s\,dx)$, where $\|x\|^2=\sum x_i^2$, then the distribution U is a bounded multiplication operator from $\hat{\mathbf{H}}_t$ into $\hat{\mathbf{H}}_{-t}$ provided $U\in\hat{\mathbf{H}}_{\frac{1}{2}-s-}$ and t is greater than both $n/4$ and $s+n/4-1/2$. We leave the details to the reader.

Moreover, the analysis on homogeneous distributions extends to $\mathbf{R}^n$. These distributions which were essentially determined by the analytic functions $c(d,\pm)$ are now determined by functions $c(d,\cdot)$ or more generally by distributions on the sphere $\mathbf{S}^{n-1}$. The behavior near the origin may be very complex, but away from the origin, their transforms behave as in Proposition 2.

Specifically

$$\hat{U}(f)=\int_{\mathbf{S}^{n-1}}dV(u)\int_0^\infty f(ru)r^{n-1-d}\,dr$$

for f which vanish near the origin.

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