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Simple permutations with order a power of two

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Abstract. Continuous maps from the real line to itself give, in a natural way, a partial ordering of permutations. This paper studies the structure of simple permutations which have order a power of two, where simple permutations are permutations corresponding to the simple orbits of Block.

0. *Introduction*

Šarkovskiĭ [**9**] proved the following theorem:

THEOREM. *Let* $\lhd$ *be the ordering of the positive integers*

$$3 \lhd 5 \lhd 7 \lhd \cdots \lhd 2\cdot 3 \lhd 2\cdot 5 \lhd 2\cdot 7 \lhd \cdots \lhd 2^2\cdot 3 \lhd 2^2\cdot 5 \lhd \cdots \lhd 2^2 \lhd 1.$$

Let f *be a continuous map from the real line to itself. If* f *has a periodic point of period* n *and if* m *satisfies* $n \lhd m$ *then* f *also has a periodic point of period* m.

Elegant proofs of this theorem using Markov graphs have been given in [**3**] and [**4**]. If the order in which points are permuted by a function is known then Markov graphs can give more information about the existence of other periodic points. For example, suppose f is a continuous map from the real line to itself such that there exist real numbers $x_1 < x_2 < x_3 < x_4$ and $f(x_i) = x_{\theta(i)}$, where θ is the permutation (1234). Šarkovskiĭ's theorem shows the existence of periodic points of periods one and two; but by looking at the Markov graph it is seen that f has periodic points of all periods. Further analysis of the graph shows that there exists points $y_1 < y_2 < y_3$ such that $f(y_i) = y_{\eta(i)}$, where $\eta = (123)$. This conclusion can also be drawn from the fact that there is no division for (x_1, x_2, x_3, x_4), see [**7**].

The important elements in the above example are the permutations. Continuous maps from the real line to itself induce a partial ordering on the set of permutations.

In this paper, the structure of simple permutations which have order a power of two is studied, where simple permutations denote permutations corresponding to the simple orbits of Block (see [**1**], [**2**], [**4**], [**5**]).

In the first section the basic concepts and notation is introduced. The second section shows that the partial ordering restricted to the above permutations gives rise to a tree. It shows what a permutation's immediate successors and predecessors are.

In the third section the number of critical points associated to a permutation is studied. This is of interest because there have been many papers considering unimodal maps (see for example [2], [6]) and because of theorem 1.5. It is shown that in the unimodal case there are only two simple permutations with order 2^n for each n; one corresponding to a map with a maximum and the other to a map with a minimum.

1. *Basics*

Throughout this paper $(S_n, \circ)$ will denote the group of permutations on n objects. All functions will be assumed to be continuous maps from the real line to itself.

Definition 1.1. Given a function, f, its *set of permutations* denoted Perm (f) is defined by the following. A permutation, θ, belongs to Perm (f) if there exist real $x_1 < x_2 < \cdots < x_n$ such that $f(x_i) = x_{\theta(i)}$.

Definition 1.2. Let θ and η be permutations. Say θ ***dominates*** η, denoted by $\theta \lhd \eta$, if $\{f | \theta \in \text{Perm}(f)\}$ is contained in $\{f | \eta \in \text{Perm}(f)\}$.

Definition 1.3. Suppose that θ belongs to Perm (f) and that $x_1, \ldots, x_n$ represent the reals such that $f(x_i) = x_{\theta(i)}$. Then a directed graph can be associated to θ and f in the following way. The graph has $n-1$ vertices $J_1, \ldots, J_{n-1}$, and an arrow is drawn from J_k to J_l if and only if $f([x_k, x_{k+1}]) \supseteq [x_l, x_{l+1}]$. This graph will be called the *Markov graph* associated to f and θ.

For basic facts about Markov graphs see [8], [4] (or [3], where they are called A-graphs).

Definition 1.4. The set which contains permutations consisting of exactly one cycle of order n will be denoted C_n.

Definition 1.5. Given a permutation θ belonging to S_n the *primitive function*, $\bar{f}$, associated to θ is defined by the following:

(1) $\bar{f}(k) = \theta(k)$;
(2) $\bar{f}(tk + (1-t)(k+1)) = t\theta(k) + (1-t)\theta(k+1)$;
(3) $\bar{f}(x) = \theta(1) \quad \text{if } x < 1$;
(4) $\bar{f}(x) = \theta(n) \quad \text{if } x > n$;

where $k = 1, \ldots, n$ and $0 \le t \le 1$.

Definition 1.6. The Markov graph associated to θ and its primitive function will be called the Markov graph of θ.

The following lemma follows from the definition of primitive function.

LEMMA 1.7. *Let θ belong to C_n and let $\bar{f}$ be its primitive function. If η belongs to $C_m \cap \text{Perm}(\bar{f})$ and if $\eta \neq \theta$ then the Markov graph of θ has a non-repetitive loop of length m corresponding to η.*

If θ belongs to Perm (f) then the Markov graph associated to θ and f contains, in a natural way, the Markov graph of θ, (see [4]). Thus an easy consequence of the above lemma is the following.

LEMMA 1.8. *Let θ belong to C_n and η to C_m and $\theta \neq \eta$. Then θ dominates η if and only if the Markov graph of η has a non-repetitive loop of length m corresponding to η.*

The following is an extension of Šarkovskiĭ's theorem.

ŠARKOVSKIĬ'S EXTENDED THEOREM. *If θ belongs to C_n then for any integer m, satisfying $m \rhd n$ there exists an $\eta \in C_m$ such that $\eta \rhd \theta$.*

Proof. From θ construct its primitive function $\bar{f}$. Since $\bar{f}$ has a periodic point of period n, Šarkovskiĭ's theorem shows that $\bar{f}$ has a periodic point of period m, and so there exists an η belonging to Perm $(\bar{f}) \cap C_m$. Lemma 1.7 shows that the Markov graph of θ has a loop corresponding to η and Lemma 1.8 completes the proof. □

Block has strengthened Šarkovskiĭ's theorem by considering simple orbits, see [**1**], [**2**]. Ho has also studied simple orbits see [**4**] and [**5**].

Definition 1.9. Let m and n be positive integers. Let S denote the set $\{x \in \mathbb{Z} | 1 \leq x \leq mn\}$. Then there is a natural way of partitioning S into subsets each of size n by choosing the first n elements, then the second n elements and so on. Define

$$P(mn, m, k) := \{x \in \mathbb{Z} | (k-1)n < x \leq kn\},$$

where k is an integer satisfying $1 \leq k \leq m$.

Definition 1.10. A permutation belonging to C_{2k-1} is *simple* if when expressed in cycle notation it is equal to either

$$[k(k-1)(k+1)(k-2)(k+2)\cdots(k-j)(k+j)\cdots 1(2k-1)]$$

or

$$[k(k+1)(k-1)(k+2)\cdots(k+j)(k-j)\cdots(2k-1)1].$$

Definition 1.11. An element θ of C_2n is *simple* if for every k satisfying $0 \leq k \leq n-1$ it satisfies the following two conditions:

(i) $\theta^{2^k}[P(2^n, 2^k, j)] = P(2^n, 2^k, j))$;

(ii) $\theta^{2^k}[P(2^n, 2^{k+1}, j)]$ has empty intersection with $P(2^n, 2^{k+1}, j)$.

Definition 1.12. An element θ of C_{r2^m} is *simple* if it satisfies the following conditions:

(i) $\theta[P(r2^m, 2^m, j)] = P(r2^m, 2^m, \sigma(j))$, where σ is a simple element of C_{2^r};

(ii) θ^{2^m} restricted to $P(r2^n, 2^m, j)$ is simple for every j.

Definition 1.13. The set of simple elements of C_k will be denoted Sim (k).

Example 1.14. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 8 & 10 & 6 & 9 & 2 & 3 & 5 & 1 & 4 \end{pmatrix}.$$

Then it is easily checked that α belongs to Sim (6) and β to Sim (10).

Block and Hart [**2**] have shown that if a function has a periodic point of period n then it has a simple periodic point of period n. By an argument analogous to the proof of Šarkovskiĭ's Extended Theorem the following can be proved.

BLOCK AND HART'S EXTENDED THEOREM. *If θ belongs to C_n then for any integer m satisfying $m \rhd n$ or $m = n$ there exists η an element of Sim (m) such that $\eta \rhd \theta$.*

Definition 1.15. Let θ belong to C_n. Say θ has a *relative maximum at k* if both $\theta(k-1)$ and $\theta(k+1)$ are defined and $\theta(k-1)<\theta(k)$ and $\theta(k+1)<\theta(k)$.

Similarly, θ has a *relative minimum* at k if both $\theta(k-1)$ and $\theta(k+1)$ are defined and both $\theta(k+1)$ and $\theta(k-1)$ are greater than $\theta(k)$.

Definition 1.16. The *number of relative maxima* of a permutation θ or a function f will be denoted $\gamma_{\max}(\theta)$, $\gamma_{\max}(f)$ respectively. The *number of relative minima* of a permutation θ, or a function f, will be denoted $\gamma_{\min}(\theta)$, $\gamma_{\min}(f)$ respectively. Let $\gamma(\theta)=\gamma_{\max}(\theta)+\gamma_{\min}(\theta)$.

The following lemma follows easily from the definitions

LEMMA 1.17. *Let f be a function and θ an element of* Perm (f). *Then*
(i) $\gamma_{\max}(f)\geq\gamma_{\max}(\theta)$;
(ii) $\gamma_{\min}(f)\geq\gamma_{\min}(\theta)$;
(iii) $\gamma(f)\geq\gamma(\theta)$.

LEMMA 1.18. *If θ is a permutation there exists a function f such that* $\gamma(f)=\gamma(\theta)$.

Proof. Clearly the primitive function of θ is such a map. □

The following lemma follows trivially from the above lemmas and definitions.

LEMMA 1.19. *If θ and η are two permutations with $\theta\lhd\eta$ then*
(i) $\gamma_{\max}(\theta)\geq\gamma_{\max}(\eta)$;
(ii) $\gamma_{\min}(\theta)\geq\gamma_{\min}(\eta)$;
(iii) $\gamma(\theta)\geq\gamma(\eta)$.

Remark. It is easily checked that if θ belongs to Sim$(2k+1)$ then $\gamma(\theta)=1$. This observation combined with Block and Hart's theorem shows the following.

THEOREM 1.20. *If θ belongs to C_{2k+1} then for any m with $m\rhd(2k+1)$ there exists η an element of* Sim (m) *such that*
(i) $\eta\rhd\theta$; *and*
(ii) $\gamma(\eta)=1$.

Remarks. Notice that $\gamma(\alpha)=4$ and $\gamma(\beta)=6$ where α and β are as in example 1.14.

Clearly if θ belongs to C_k then $\gamma(\theta)\leq k-2$, because 1 and k cannot be critical points. The simple permutation α is an example where $\gamma(\alpha)$ equals $6-2$. However, it will be shown in § 3 that if θ belongs to Sim (2^n), for $n\geq2$, then $\gamma(\theta)\leq2^n-3$.

It is interesting to note that both α and β are maximal, in the sense that no simple permutation dominates α other than itself and no simple permutation dominates β other than itself. Thus if the intersection of Perm (f) and C_{10} contains only β the function f has periodic points only for periods m where $m>10$.

2. *Partial ordering*

In this section it is shown that the partial ordering restricted to $\bigcup_n$ Sim (2^n) gives rise to a tree. Theorem 2.10 shows what are the immediate predecessors and successors of a given permutation.

The following lemma was proved by C. Ho in [**4**].

LEMMA 2.1. *There exist* $2^{2^n-(n+1)}$ *simple permutations of period* 2^n.

LEMMA 2.2. *If* θ *belongs to* Sim (2^n), *then for any integer m*, θ^{2^m+1} *belongs to* Sim (2^n).

Proof. This follows directly from definition 1.11. □

Definition 2.3. If θ belongs to Sim (2^n) then θ^*, an element of $S_{2^{n+1}}$, is defined by

$$\theta^*(2k)=2\theta(k), \qquad \theta^*(2k-1)=2\theta(k)-1.$$

Remarks. The permutation θ^* consists of two 2^n-cycles. It is clear that θ^* dominates θ.

Definition 2.4. Let ρ_s denote the transposition

$$\begin{pmatrix} 2s-1 & 2s \\ 2s & 2s-1 \end{pmatrix}.$$

LEMMA 2.5. *If* θ *belongs to* Sim (2^n) *then*

$$\theta^* \circ \rho_{i_1} \circ \rho_{i_2} \circ \cdots \circ \rho_{i_{2m-1}}$$

belongs to Sim (2^{n+1}) *for any positive integers m*, i_j *where* $1 \le i_j \le 2^n$ *for* $1 \le j \le 2m-1$.

Proof. Let η denote $\theta^* \circ \rho_{i_1} \circ \rho_{i_2} \circ \cdots \circ \rho_{i_{2m-1}}$. First, it will be shown that η belongs to $C_{2^{n+1}}$. Since θ belongs to C_{2^n} the set $\eta^k(\{1,2\})$ has empty intersections with $\{1,2\}$ for $1 \le k < 2^n$ and $\eta^{2^n}(\{1,2\})=\{1,2\}$. So, either $\eta^{2^n}(1)=1$ or $\eta^{2^{n+1}}(1)=1$. Now $\eta^{2^n}|_{\{1,2\}}=\rho_1^{2m-1}|_{\{1,2\}}$, thus $\eta^{2^n}(1)=2$ and consequently η belongs to $C_{2^{n+1}}$. Next it will be shown that η satisfies the conditions given in definition 1.11. Since θ is simple it follows from the construction of η that for every k satisfying $0 \le k \le n-1$,

(i) $\eta^{2^k}[P(2^{n+1},2^k,j)]=P(2^{n+1},2^k,j)$;

(ii) $\eta^{2^k}P(2^{n+1},2^{k+1},j)]$ has empty intersection with $P(2^{n+1},2^{k+1},j)$.

Putting $k=n-1$ in both of the above conditions shows that

$$\eta^{2^n}[P(2^{n+1},2^n,j)]=P(2^{n+1},2^n,j),$$

and because η belongs to $C_{2^{n+1}}$ it is clear that $\eta^{2^n}[P(2^{n+1},2^{n+1},j)]$ has empty intersection with $P(2^{n+1},2^{n+1},j)$. □

LEMMA 2.6. *If* η *belongs to* Sim (2^{n+1}) *then there exists* θ *belonging to* Sim (2^n) *and transpositions* $\rho_{i_1},\ldots,\rho_{i_{2k-1}}$ *such that* $\eta=\theta^* \circ \rho_{i_1} \circ \cdots \circ \rho_{i_{2k-1}}$.

Proof. First notice that if $\theta_1^* \circ \rho_{i_1} \circ \cdots \circ \rho_{i_{2k-1}} = \theta_2^* \circ \rho_{j_1} \circ \cdots \circ \rho_{j_{2m-1}}$ then $\theta_1^*=\theta_2^*$, and if the strings of transpositions contain no repetitions $\{\rho_{i_1},\ldots,\rho_{i_{2k-1}}\}=\{\rho_{j_1},\ldots,\rho_{j_{2m-1}}\}$.

Lemma 2.1 shows that there are $2^{2^n-(n+1)}$ elements in Sim (2^n). The number of ways of choosing an odd length string of transpositions is 2^{2^n-1}, if all the transpositions in the string are distinct. Thus there are $(2^{2^n-(n+1)})(2^{2^n-1})$ ways of choosing η, by lemma 2.5 each of the choices corresponds to an element of Sim (2^{n+1}) and lemma 2.1 completes the proof. □

Definition 2.7. If θ belongs to Sim (2^n) define θ_* an element of $S_{2^{n-1}}$ by

$$\theta_*(k)=\text{Int}\,[\tfrac{1}{2}\theta(2k)]$$

where Int [ ] means round up to the nearest integer.

Remark. It is clear that if θ belongs to Sim (2^n) then $(\theta^* \circ \rho_{i_1} \circ \cdots \circ \rho_{i_{2k-1}})_* = \theta$, and so the following is obtained trivially.

LEMMA 2.8. *If* θ *belongs to* Sim (2^n) *then*

(i) $\theta_* \in \text{Sim}(2^{n-1})$;

(ii) $\theta \lhd \theta_*$.

LEMMA 2.9. *If* θ *belongs to* Sim (2^n) *and* θ *dominates both* η_1 *and* η_2, *where* η_1 *and* η_2 *are elements of* Sim (2^{n-1}), *then* $\eta_1 = \eta_2$.

Proof. Consider the Markov graph associated to θ. It has $2^n - 1$ vertices. It will be shown that there exists only one loop of length 2^{n-1}.

In the graph there exists at least one loop of 2^{n-k} vertices corresponding to a periodic point of period 2^{n-k} for each k, $1 \le k \le n$. These loops must be distinct or else η would dominate an infinite number of permutations. However, $\sum_{k=1}^{n} 2^{n-k} = 2^n - 1$ and so there exists exactly one loop of length 2^{n-k} for each k. □

The following theorem has now been proved.

THEOREM 2.10. *Suppose* θ *belongs to* Sim (2^n).

(i) *If* $\eta \lhd \theta$ *and* η *belongs to* Sim $(2^n + 1)$ *then there exist transpositions* $\rho_{i_1}, \ldots, \rho_{i_{2k-1}}$ *such that* $\eta = \theta^* \circ \rho_{i_1} \circ \cdots \circ \rho_{i_{2k-1}}$.

(ii) *If* $\phi \rhd \theta$ *and* ϕ *belongs to* $C_{2^{n-1}}$ *then* $\phi = \theta_*$.

3. *Critical points*

In this section the following theorem will be proved.

THEOREM 3. *For* $n \ge 2$ *the following hold.*

(1) *If* θ *belongs to* Sim (2^n) *and* m *is an integer satisfying* $\gamma(\theta) \le m \le 2^{n+1} - 2 - \gamma(\theta)$, *then there exists* η *belonging to* Sim (2^{n+1}) *such that* η *dominates* θ *and* $\gamma(\eta) = m$.

(2) *If* θ *belongs to* Sim (2^n) *then* $\theta^{2^{n-1}+1}$ *belongs to* Sim (2^n) *and*

$$\gamma(\theta) + \gamma(\theta^{2^{n-1}+1}) = 2^n - 2.$$

(3) *There are exactly two elements of* Sim (2^n) *that have only one critical point.*

LEMMA 3.1. *Let* θ *belong to* Sim (2^n).

(i) *If* k *is a critical point of* θ_* *then one of* $2k$ *or* $2k - 1$ *is a critical point of* θ, *but not both.*

(ii) *If* k *is not a critical point of* θ_*, *where* $1 < k < 2^n$, *then either*

(a) *both* $2k$ *and* $2k - 1$ *are critical points of* θ; *or*

(b) *neither* $2k$ *nor* $2k - 1$ *are critical points of* θ.

Proof. The case when θ_* has a maximum at k will be proved, the other cases can be proved similarly.

If $\theta_*(k) > \theta_*(k-1)$ and $\theta_*(k) > \theta_*(k+1)$ then $\theta(2k-1) > \theta(2k-2)$ and $\theta(2k) > \theta(2k+1)$. If $\theta(2k-1) > \theta(2k)$ then θ has a maximum at $2k-1$ and $2k$ is not a critical point. Similarly, if $\theta(2k-1) < \theta(2k)$ then θ has a maximum at $2k$ and $2k-1$ is not a critical point. □

LEMMA 3.2. *Suppose* θ *belongs to* Sim (2^n), $n \ge 2$. *Let* m *be an integer satisfying* $\gamma(\theta) \le m \le 2^{n+1} - 2 - \gamma(\theta)$. *Then there exists* η *belonging to* Sim (2^{n+1}) *such that* $\eta \lhd \theta$ *and* $\gamma(\eta) = m$.

Proof. Denote the set of integers where θ has critical points by C. Denote the set of non-critical integers by R, i.e. $R=\{1,2,3,\ldots,2^n\}\setminus C$. Let $\eta=\theta^*\circ\rho_{i_1}\circ\rho_{i_2}\circ\cdots\circ\rho_{i_{2k-1}}$. If c belongs to C then η has exactly one critical point in $\{2c,2c-1\}$. If r, where $1<r<2^n$, belongs to R then one of η or $\eta\circ\rho_r\circ\rho_c$ has exactly two critical points in $\{2r,2r-1\}$ and the other permutation has none. Notice that both η and $\eta\circ\rho_r\circ\rho_c$ belong to Sim (2^{n+1}).

If s is either 1 or 2^n, then one of η, $\eta\circ\rho_s\circ\rho_c$ has exactly one critical point in $\{2s,2s-1\}$ and the other permutation has none. Again, both η and $\eta\circ\rho_s\circ\rho_c$ belong to Sim (2^{n+1}).

Thus it can be seen that given any subset S of R it is possible to construct an element η_s of Sim (2^{n+1}) such that the following hold:

(i) η_s has two critical points in $\{2k,2k-1\}$ if k belongs to S and $1<k<2^n$;

(ii) η_s has one critical point in $\{2k,2k-1\}$ if k belongs to S and k is either 1 or 2^n;

(iii) η_s has 1 critical point in $\{2k,2k-1\}$ if k belongs to C;

(iv) η_s has no other critical points.

In general a subset S does not define a unique element.

Let $\varnothing$ denote the empty set then $\eta_\varnothing$ has $\gamma(\theta)$ critical points. Choosing $S=R$ gives an element η_R that has $2^{n+1}-2-\gamma(\theta)$ critical points. Given m satisfying $\gamma(\theta)\leq m\leq 2^{n+1}-2-\gamma(\theta)$ it is clear that there exists an element η_s with $\gamma(\eta_s)=m$ for some S contained in R □

An immediate corollary is the following.

LEMMA 3.3. *If θ belongs to* Sim (2^n) *then* $\gamma(\theta_*)\leq\gamma(\theta)\leq 2^n-2-\gamma(\theta_*)$.

LEMMA 3.4. *If θ belongs to* Sim (2^n) *then* $\theta^{2^{n-1}+1}$ *is simple and*

$$\gamma(\theta)+\gamma(\theta^{2^{n-1}+1})=2^n-2.$$

Proof. The proof follows from lemma 2.2 and the proof of lemma 3.2 after noting the following fact. If $\theta=\phi^*\circ\rho_{i_1}\circ\cdots\circ\rho_{i_{2k-1}}$ then

$$\theta^{2^{n-1}+1}=\phi^*\circ\rho_{j_1}\circ\cdots\circ\rho_{j_{2m-1}},$$

where the two sets $\{i_1,\ldots i_{2k-1}\}$ and $\{j_1,\ldots,j_{2m-1}\}$ have empty intersection and their union is $\{n\in\mathbb{Z}|1\leq n\leq 2^n\}$.

LEMMA 3.5. *There exist only two elements of* Sim (2^n) *that have only one critical point, for $n\geq 2$*

Proof. This will be proved by induction.

When $n=2$ there are only two elements of Sim (2^2); these are

$$\begin{pmatrix}1&2&3&4\\3&4&2&1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix},$$

both of which only have one critical point.

Suppose θ belongs to Sim (2^r) and $\gamma(\theta)=1$. Then there exists a unique permutation η belonging to Sim (2^{r+1}) with $\gamma(\eta)=1$ such that $\eta\lhd\theta$. This is the unique permutation defined by taking S to be the empty set, where S is defined in the proof of lemma 3.2. It is unique because C contains a single element. □

Remark. Of these two permuations that have only one critical point one has a relative maximum and the other has a relative minimum.

COROLLARY 3.6. *There exist exactly two elements of* Sim (2^n), $n \geq 2$ *that have* $2^n - 3$ *critical points.*

Proof. The proof is an immediate consequence of lemmas 3.4 and 3.5. □

Remark. It is interesting to note that if θ belongs to Sim (2^n) and $\gamma(\theta) = 2^n - 3$ then $\gamma(\theta_*) = 1$.

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