ON THE NATURE OF THE SPECTRUM OF SINGULAR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

E. A. CODDINGTON AND N. LEVINSON

LET $p(x) > 0$, $q(x)$ be two real-valued continuous functions on $0 \leqslant x < \infty$. Suppose that the differential equation with the real parameter λ

$$(py')' + (\lambda - q)y = 0 \tag{1}$$

does not possess two linearly independent solutions of class $L^2(0, \infty)$ for some λ. According to the Weyl classification [6] equation (1) is then said to be of the *limit-point* type. In this case (1) together with a boundary condition

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0 \tag{2}$$

determine an eigenvalue problem.

For every α in (2) there corresponds a spectrum $S(\alpha)$ which includes a (possibly empty) point spectrum $P(\alpha)$, the latter set consisting of those λ for which there exists a real solution of (1) satisfying (2) and is of class $L^2(0, \infty)$. The derivative $S'(\alpha)$ of $S(\alpha)$ consists of the continuous spectrum and the cluster points of the point spectrum. Using the theory of bounded quadratic forms whose differences are completely continuous, and the idea of a Hellinger integral, Weyl proved [5, p. 378; 6, p. 251] that the set $S'(\alpha)$ does not depend on α, and hence can be denoted simply by S'.

Recently one of the authors [3] gave a simplified proof of the Parseval relation corresponding to the problem (1) and (2). This relation was obtained by a natural limiting process on the Parseval equality which holds for the corresponding Sturm-Liouville problem on a bounded interval $0 \leqslant x \leqslant b < \infty$. It turns out that the formulae (but not the end result) developed in this proof (which were obtained by Titchmarsh [4] using other methods[1]) can be used to obtain a direct proof of the invariance of the set S'. Also the oscillation and separation theorems due to Hartman and Wintner [1], [2] can be obtained in a similar way (see (II), (III) below).

Denote by $S^*(\alpha)$ the complement of the set $S'(\alpha)$ with respect to the set $-\infty < \lambda < \infty$. Since $S'(\alpha)$ is closed, $S^*(\alpha)$ is open. In this notation we prove:

(I) *The set of cluster points $S'(\alpha)$ of the spectrum $S(\alpha)$ for the problem* (1) *and* (2) *is independent of α, and hence can be denoted by S'.*

Received March 24, 1950.

[1]K. Kodaira, *American Journal of Mathematics*, vol. 71 (1949), pp. 921-945, obtains Titchmarsh's results and also a formula for $m(\lambda)$ using the spectral representation of a self-adjoint operator in Hilbert space.

(II) *If* $\lambda \in S^*$, *(S^* being the complement of S') then* $\lambda \in P(\alpha)$ *for some* $\alpha = \alpha(\lambda)$.

(III) *The function* $\alpha(\lambda)$ *for which* $\lambda \in P(\alpha)$ *is regular, monotone increasing on* S^*.

Proof of I. We need some known facts. Let $\theta(x, \lambda)$ and $\phi(x, \lambda)$ be solutions of (1) which satisfy

$$\begin{array}{ll} p(0)\theta(0, \lambda) = \cos \alpha, & p(0)\theta'(0, \lambda) = \sin \alpha \\ \phi(0, \lambda) = \sin \alpha, & \phi'(0, \lambda) = -\cos \alpha. \end{array} \tag{3}$$

For $\lambda = u + iv$, $v \neq 0$, consider the solution of (1):

$$\psi_b(x, \lambda) = \theta(x, \lambda) + l_b(\lambda)\, \phi(x, \lambda)$$

which satisfies a real boundary condition at $x = b$,

$$\psi_b(b, \lambda) \cos \beta + \psi'_b(b, \lambda) \sin \beta = 0.$$

For each b, as β varies, $l_b(\lambda)$ describes a circle C_b in the complex plane, and it can be shown that as $b \to \infty$, the circles C_b converge either to a limit-circle or to a limit-point. Since we are assuming the limit-point case, let us denote this point by $m(\lambda) = m(\lambda, \alpha)$. For any $v \neq 0$ it is true that, if

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\, \phi(x, \lambda), \tag{4}$$

then

$$\int_0^\infty |\psi(x, \lambda)|^2 dx \leqslant -\frac{\Im(m)}{v}. \tag{5}$$

Also the function $m(\lambda)$ is analytic in either half-plane $v > 0$, $v < 0$. It was shown in [3] that there exists a monotone non-decreasing function $\rho(\sigma) = \rho(\sigma, \alpha)$ on $-\infty < \sigma < \infty$ and a real constant c such that

$$\frac{-\Im(m)}{v} = \int_{-\infty}^{\infty} \frac{d\rho(\sigma)}{(u - \sigma)^2 + v^2} + c. \tag{6}$$

It follows from the proof of (6) that c is a non-negative constant, and as a matter of fact this can be easily seen from (6) and (5) by letting $v \to \infty$.

An immediate consequence of (6) is that at points of continuity $\sigma = \sigma_1$, $\sigma = \sigma_2$ of $\rho(\sigma)$

$$\rho(\sigma_2) - \rho(\sigma_1) = \lim_{v \to 0} \frac{1}{\pi} \int_{\sigma_1}^{\sigma_2} -\Im(m(u + iv))du. \tag{7}$$

In proving (6) it was of course shown that the integral

$$\int_{-\infty}^{\infty} \frac{d\rho(\sigma)}{1 + \sigma^2} \tag{8}$$

was convergent. If one considers the function of λ given by

$$\int_{-\infty}^{\infty} \left[\frac{1}{\lambda - \sigma} + \frac{1}{\sigma + i} \right] d\rho(\sigma) - c\lambda + d$$

where $c > 0$ is the constant in (6) and d is a (complex) constant, then the integral obviously exists by virtue of the convergence of (8). Also the imaginary part of this function is identical with $\Im(m)$ (see (6)), if

$$\Im(d) = \int_{-\infty}^{\infty} \frac{d\rho(\sigma)}{1 + \sigma^2}.$$

Therefore, since $m(\lambda)$ and this function are regular on $v > 0$, the latter coincides with $m(\lambda)$ except for a real constant, which may be incorporated into d. Hence

$$m(\lambda) = \int_{-\infty}^{\infty} \left[\frac{1}{\lambda - \sigma} + \frac{1}{\sigma + i} \right] d\rho(\sigma) - c\lambda + d. \tag{9}$$

For a given α, the spectrum $S(\alpha)$ is the σ set which is the complement of the set of points in the neighbourhood of which $\rho(\sigma, \alpha)$ is constant. The jumps of $\rho(\sigma, \alpha)$ correspond to the point spectrum $P(\alpha)$. Clearly $\rho(\sigma, \alpha)$, considered as a function on $S^*(\alpha)$, is constant except for isolated jumps. Thus $m(\lambda, \alpha)$ by (9) is analytic on $S^*(\alpha)$ except at the isolated discontinuities of $\rho(\sigma, \alpha)$ where $m(\lambda, \alpha)$ has simple poles. Also $m(u, \alpha)$ is real on $S^*(\alpha)$.

Consider the boundary condition (2) corresponding to α_1, α_2 where $\alpha_1 \not\equiv \alpha_2$ (mod π), and define the solutions $\theta(x, \lambda, \alpha_i)$, $\phi(x, \lambda, \alpha_i)$, $\psi(x, \lambda, \alpha_i)$, $(i = 1, 2)$ by (3) and (4). From these relations it is clear that if $\gamma = \alpha_2 - \alpha_1$, then

$$\phi(x, \lambda, \alpha_1) = \cos\gamma\, \phi(x, \lambda, \alpha_2) - p(0) \sin\gamma\, \theta(x, \lambda, \alpha_2)$$

$$p(0)\theta(x, \lambda, \alpha_1) = \sin\gamma\, \phi(x, \lambda, \alpha_2) + p(0) \cos\gamma\, \theta(x, \lambda, \alpha_2).$$

Consequently

$$p(0)\psi(x, \lambda, \alpha_1) =$$
$$[\cos\gamma - p(0)m(\lambda, \alpha_1) \sin\gamma] p(0)\, \theta(x, \lambda, \alpha_2) + [\sin\gamma + p(0)m(\lambda, \alpha_1) \cos\gamma] \phi(x, \lambda, \alpha_2),$$

and since $\psi(x, \lambda, \alpha_1)$, $\psi(x, \lambda, \alpha_2)$ are both of class $L^2(0, \infty)$ one is a constant multiple of the other by virtue of the limit-point assumption. This implies that[2]

$$p(0)m(\lambda, \alpha_2) = \frac{\sin\gamma + p(0)m(\lambda, \alpha_1) \cos\gamma}{\cos\gamma - p(0)m(\lambda, \alpha_1) \sin\gamma}, \gamma = \alpha_2 - \alpha_1. \tag{10}$$

Since $m(\lambda, \alpha_1)$ is meromorphic on $S^*(\alpha_1)$, so is $m(\lambda, \alpha_2)$ and because $m(u, \alpha_1)$ is real on $S^*(\alpha_1)$, it follows that for values of u on this set $\Im(m(\lambda, \alpha_2)) \to 0$, $v \to +0$, except at isolated poles of $m(\lambda, \alpha_2)$. From this fact we see from (7) that $\rho(\sigma, \alpha_2)$ is constant on $S^*(\alpha_1)$ except for jumps at isolated poles of $m(\lambda, \alpha_2)$. This proves that $S^*(\alpha_2) \supset S^*(\alpha_1)$. Since the roles of α_1 and α_2 can be interchanged we get $S^*(\alpha_2) = S^*(\alpha_1) = S^*$, and hence $S'(\alpha_2) = S'(\alpha_1)$, thus proving (I).

[2]The boundary condition (2) is often replaced by $y(0) \cos\alpha + p(0)y'(0) \sin\alpha = 0$, which has the effect of eliminating $p(0)$ in (10).

Proof of (II) *and* (III). From (9) it is clear that, except for poles of $m(\lambda, \alpha_1)$, on S^*

$$\frac{\partial m(u, \alpha_1)}{\partial u} = -\int_{-\infty}^{\infty} \frac{d\rho(\sigma)}{(u - \sigma)^2} - c$$

and hence $\frac{\partial m(u, \alpha_1)}{\partial u} < 0$. We see therefore that on S^*, $m(u, \alpha_1)$ is a regular monotone decreasing function of u, except for poles of $m(\lambda, \alpha_1)$.

Let $\lambda_1 \in P(\alpha_1)$ and suppose $\lambda \in S^*$ ($= S^*(\alpha_1)$) is in a sufficiently small open interval about λ_1 which contains no other points of $P(\alpha_1)$. Then it follows from (10) that $m(\lambda, \alpha_2)$ will have a pole for some $\alpha_2 \not\equiv \alpha_1 \pmod{\pi}$ determined by the relation $p(0)m(\lambda, \alpha_1) = \cot\gamma$, that is, there exists an $\alpha_2 = \alpha_2(\lambda)$ for which $\lambda \in P(\alpha_2)$. Every $\lambda \in S^*$ can be brought within an open interval containing at most one point of $P(\alpha_1)$, so that $\alpha_2 = \alpha_2(\lambda)$ exists for all $\lambda \in S^*$, even at the points λ_1 where $\alpha_2 \equiv \alpha_1 (\text{mod } \pi)$ and $\lambda_1 \in P(\alpha_1)$. This proves (II).

From the relation $p(0)m(\lambda, \alpha_1) = \cot\gamma = \cot(\alpha_2 - \alpha_1)$ it follows that

$$\alpha_2(\lambda) = \alpha_1 + \cot^{-1}[p(0)m(\lambda, \alpha_1)]. \tag{11}$$

Moreover, from the discussion above, since $m(\lambda, \alpha_1)$ is regular on an interval about λ_1, $\alpha_2(\lambda)$ is a regular function of λ on such an interval and

$$\frac{d\alpha_2(\lambda)}{d\lambda} = \frac{-p(0)}{[1 + (p(0)m(\lambda, \alpha_1))^2]} \frac{\partial m(\lambda, \alpha_1)}{\partial \lambda}. \tag{12}$$

But $p(0) > 0$, $\frac{\partial m(\lambda, \alpha_1)}{\partial \lambda} < 0$ on S^* (except for poles of $m(\lambda, \alpha_1)$, and therefore $\frac{d\alpha_2(\lambda)}{d\lambda} > 0$ on an open interval about λ_1, that is, $\alpha_2(\lambda)$ is increasing on S^*. This completes the proof of (III).

References

[1] P. Hartman and A. Wintner, *An oscillation theorem for continuous spectra*, Proc. Nat. Acad. Sci., vol. 33 (1947), 376-379.

[2] ———, *A separation theorem for continuous spectra*, Amer. J. Math., vol. 71 (1949), 650-662.

[3] N. Levinson, *A simplified proof of the expansion theorem for singular second order linear differential equations*, Duke Math. J. vol. 18 (1951), 57-71.

[4] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, (Oxford, 1946).

[5] H. Weyl, *Ueber beschränkte quadratische Formen, deren Differenz vollstetig ist*, Rend. Circ. Palermo, vol. 27 (1909), 373-392.

[6] H. Weyl, *Ueber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann., vol. 68 (1910), 222-269.

Massachusetts Institute of Technology