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ON FIXED POINT THEOREMS FOR MAPPINGS IN A SEPARATED LOCALLY CONVEX SPACE

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The Banach contraction principle has been generalized by Tan [6] to the mappings in separated locally convex spaces. We show that the result of Sehgal [5] and also of Holmes [3] can be generalized in the same way.

Throughout this note, we let X be a separated locally convex space, U a base for the closed absolutely convex neighborhoods of the origin O in X, K a nonempty subset of X, and T a mapping from K to K. For each $u \in U$, we denote P_u the gauge of u defined by

$$P_u = \inf\{\lambda > 0 : x \in \lambda u\} \quad \text{for each } x \in X.$$

We refer to [4] for the concept of gauge functions.

Theorem 1 is similar to the result in [5] but we do not assume the continuity of T (cf. [2]). This is due to the referee, to whom the author expresses many thanks.

THEOREM 1. *Let K be sequentially complete. Suppose that for each $x \in K$ there is a positive integer $N(x)$, and for each $u \in U$ there is a constant λ_u with $0 \leq \lambda_u < 1$ such that*

$$P_u(T^{N(x)}(x) - T^{N(x)}(y)) \leq \lambda_u P_u(x-y)$$

for all $x, y \in K$ and for all $u \in U$. Then T has a unique fixed point ξ (in K) and $\lim_n T^n(x) = \xi$ *for each $x \in K$.*

Proof. Let $x_0 \in K$ and $x_{n+1} = T^{N(x_n)}(x_n)$ for $n \geq 0$. Then since P_u is a seminorm, it follows as in [5] that $\{x_n\}$ is a Cauchy sequence in the seminormed space (X, P_u), $u \in U$, and hence $\{x_n\}$ is Cauchy in K. As K is sequentially complete, $x_n \to \xi \in K$. Then by the hypothesis, $T^{N(\xi)}(x_n) \to T^{N(\xi)}(\xi)$. Since for any u, P_u is continuous,

$$P_u(T^{N(\xi)}(\xi) - \xi) = \lim_n P_u(T^{N(\xi)}(x_n) - x_n) = 0,$$

i.e. $T^{N(\xi)}(\xi) = \xi$. It follows that ξ is the unique fixed point for $T^{N(\xi)}$, and therefore $T(\xi) = \xi$ is unique fixed point of T. The proof of $T^n(x_0) \to \xi$ follows again as in [5].

In case that K is not sequentially complete, following Holmes [3] using a modified condition due to Bailey [1], one can prove

THEOREM 2. *Let T be continuous. Suppose that for each pair $x, y \in K$, there is a positive integer $N(x, y)$ and for each $u \in U$, there is a constant λ_u with $0 \leq \lambda_u < 1$ such that*

$$P_u(T^{N(x,y)+t}(x) - T^{N(x,y)+t}(y)) \leq \lambda_u P_u(x-y)$$

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for each pair x, $y \in K$ and for each $t=0, 1, 2, 3, \ldots$. Furthermore, suppose that there is an $x_0 \in K$ such that the sequence $\{T^n(x_0)\}$ contains a subsequence converging to $\xi \in K$. Then ξ is the unique fixed point (in K) of T, and $\{T^n(y)\}$ converges to ξ for each $y \in K$.

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