*Dynamics of Populations of Planetary Systems*
*Proceedings IAU Colloquium No. 197, 2005*
*Z. Knežević and A. Milani, eds.*

DOI: 10.1017/S1743921304008464

# Dynamics of planetesimals: the role of two-body relaxation

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**Abstract.** In the standard scenario of planet formation, solid planets are formed through accretion of small bodies called planetesimals. The dynamics of planetesimals is important since it controls their growth mode and timescale. Here, I briefly explain the basic dynamics of planetesimals due to the two-body gravitational relaxation process. The important roles of two-body relaxation in a planetesimal system are viscous stirring and dynamical friction. Due to viscous stirring, the random velocities (eccentricities and inclinations) of planetesimals increase, while dynamical friction realizes the energy equipartition of the random energy. I also explain the orbital repulsion of protoplanets which is the coupling effect of two-body scattering and dynamical friction.

**Keywords.** Solar system: formation, celestial mechanics

## 1. Introduction

In the standard scenario for planet formation, terrestrial planets and the cores of Jovian planets are formed through the accretion of small bodies called planetesimals. This process is called as planetary accretion. Planetary accretion is an important stage of planet formation since it determines the timescale of planet formation and the spatial structure of the planetary system.

The dynamics of planetesimals controls the planetary accretion. For a planetesimal, the solar gravity is a dominant force. Thus, the orbit is nearly Keplerian. The deviation velocity $v$ of the planetesimal from the Keplerian velocity $v_{\rm K}$ of the local non-inclined circular orbit is called random velocity and is given by

$$v \simeq (e^2 + i^2)^{1/2} v_{\rm K}, \tag{1.1}$$

where $e$ and $i$ are the eccentricity and inclination of the planetesimal. Note that the random velocities in the radial and vertical directions are proportional to $e$ and $i$, respectively. The random velocity is an important factor that controls planet formation. For example, the growth timescale of planetesimals depends on the random velocity as $T_{\rm grow} \propto v^2$ when gravitational focusing is effective in collisions.

The orbit evolves through mutual gravitational interaction among planetesimals, i.e., two-body gravitational relaxation. In terms of stellar dynamics, a planetesimal system is a collisional system where two-body encounters play key roles in the dynamical evolution of the system. The two basic effects of two-body relaxation in a planetesimal system are viscous stirring and dynamical friction. Viscous stirring increases the random velocity of planetesimals, while dynamical friction realizes the energy equipartition of the random energy. In the present paper, I briefly demonstrate the basics of viscous stirring and dynamical friction in a planetesimal system. I also describe the orbital repulsion process in which two large bodies expand their orbital separation in a swarm of planetesimals.

For details of the velocity evolution of planetesimals, see e.g., Ida (1990), Ohtsuki (1999), and Stewart & Ida (2000).

## 2. Two-Body Relaxation

The equation of motion for a planetesimal is given as

$$\frac{d^2\boldsymbol{x}_i}{dt^2} = -GM_\odot \frac{\boldsymbol{x}_i}{|\boldsymbol{x}_i|^3} + \sum_{j=1,j\neq i}^{N} Gm_j \frac{\boldsymbol{x}_j - \boldsymbol{x}_i}{|\boldsymbol{x}_j - \boldsymbol{x}_i|^3} + \boldsymbol{f}_{\text{gas}}, \tag{2.1}$$

where the terms on the r.h.s. express the solar gravity, the mutual gravitational interaction, and the gas drag, from left to right. For the standard minimum-mass disk model, the solar gravity is dominant except for the rare cases of close encounters of planetesimals and thus the orbit of planetesimals is almost Keplerian. The mutual interaction of planetesimals is the main perturbing force in a planetesimal system.

The orbit of planetesimals gradually changes due to the mutual gravitational interaction. This process is equivalent to the two-body relaxation process in star clusters. In terms of stellar dynamics, a planetesimal system is a collisional system like globular clusters, in the sense that the system evolves through two-body encounters.

The mutual interaction increases the random velocity of planetesimals on average. On the other hand, the gas drag damps the random velocity. In planetary accretion, there are equilibrium values of the eccentricity and inclination, at which the effects of the mutual interaction of planetesimals and the gas drag balance. In the present paper, we focus on the effect of the mutual interaction. The timescale of two-body relaxation for an equal-mass ($m$) many-body system is given by

$$T_{\text{2B}} \equiv \frac{\sigma^2}{d\sigma^2/dt} \simeq \frac{1}{n\pi {r_{\text{G}}}^2\sigma \ln\Lambda} = \frac{\sigma^3}{n\pi G^2m^2\ln\Lambda}, \tag{2.2}$$

where $n$ is the number density of constituent particles, $\sigma$ is the velocity dispersion, $r_{\text{G}}$ is the gravitational radius given by $r_{\text{G}} = Gm/\sigma^2$, and $\ln\Lambda$ is the Coulomb logarithm (e.g., Binney & Tremaine 1987).

The two major roles of two-body relaxation in a planetesimal system are viscous stirring and dynamical friction. In the following, we demonstrate both roles by showing examples of $N$-body simulations. The initial conditions of planetesimals we use in the following sections are summarized as follows: 1000 equal-mass ($m = 10^{24}$g) planetesimals are distributed in a ring of the radius $a = 1$AU with width $\Delta a = 0.07$AU. The surface density of the ring is almost consistent with the minimum-mass disk model. The initial distributions of eccentricities and inclinations are set by the Rayleigh distributions with dispersions $\sigma_e = \sigma_i = 2r_{\text{H}}/a$, where $r_{\text{H}}$ is the Hill radius of planetesimals given by

$$r_{\text{H}} = \left(\frac{2m}{3M_\odot}\right)^{1/3} a. \tag{2.3}$$

Note that $\sigma_e$ and $\sigma_i$ are proportional to the velocity dispersions in thr radial and vertical directions, respectively. For the numerical integration, the fourth-order Hermite integrator (Kokubo *et al.* 1998) is used with a special-purpose computer GRAPE-6 (Makino *et al.* 2003). Gas drag and collisions are not included for simplicity.

## 3. Viscous Stirring

Viscous stirring is the process in which velocity dispersions of planetesimals increase due to two-body encounters. Here we consider an equal-mass planetesimal system for simplicity. In the dispersion-dominated regime ($\sigma_e, \sigma_i \gtrsim r_{\text{H}}/a$) where the relative velocity

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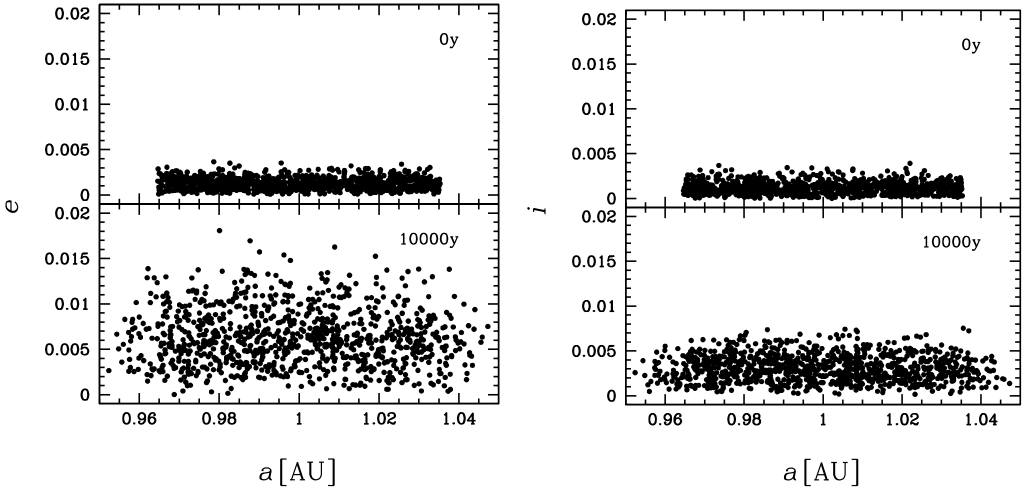


**Figure 1.** Snapshots of the planetesimal system on the $a$-$e$ (left) and $a$-$i$ (right) planes at $t = 0$ year (top) and 10000 year (bottom).

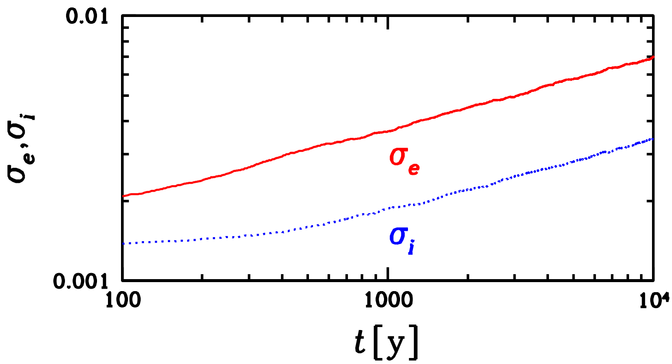


**Figure 2.** Time evolution of $\sigma_e$ (solid) and $\sigma_i$ (dashed).

of planetesimals is mainly determined by the random velocity, the rate of viscous stirring for $\sigma_e$ is given by

$$\frac{d\sigma_e^2}{dt} \sim \frac{\sigma_e^2}{T_{2\mathrm{B}}}. \tag{3.1}$$

We can obtain the rate of viscous stirring for $\sigma_i$ by replacing $\sigma_e$ with $\sigma_i$ in (3.1).

Figures 1 show the snapshots of the system at $t = 0$ year and 10000 year on the $a$-$e$ and $a$-$i$ planes. The eccentricities and inclinations of most planetesimals significantly increase in 10000 years. On average, the increase of $e$ is larger than that of $i$. The distributions of $e$ and $i$ relax into the Rayleigh distributions. We also see the diffusion of planetesimals in $a$, which is the result of random walk in $a$ due to two-body scattering. These increases of $e$ and $i$ with the diffusion in $a$ are the basics of viscous stirring.

The time evolutions of $\sigma_e$ and $\sigma_i$ are shown in figure 2. It is clearly shown that $\sigma_e$ and $\sigma_i$ increase with $t^{1/4}$, and $\sigma_e/\sigma_i \simeq 2$ that corresponds to the anisotropic velocity dispersions in radial and vertical directions. These properties of $\sigma_e$ and $\sigma_i$ are the characteristics of two-body relaxation in a disk-shaped system (Ida *et al.* 1993).

As the number density of planetesimals is inversely proportional to the scale height of the system that is proportional to the velocity dispersion, in other words, $n \propto \sigma^{-1}$, we

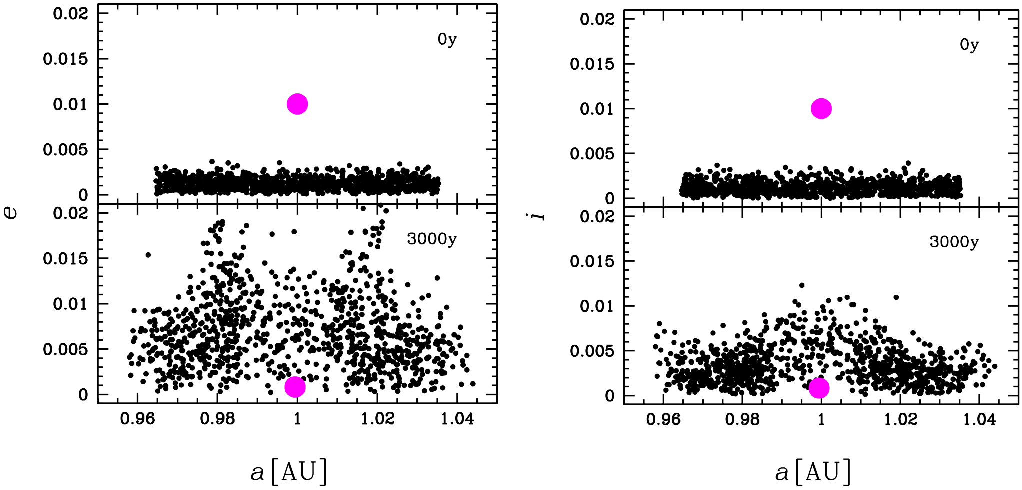


**Figure 3.** Snapshots of the planetesimal system on the $a$-$e$(left) and $a$-$i$ (right) planes at $t = 0$ year (top) and 3000 year (bottom). The large circle indicates the protoplanet.

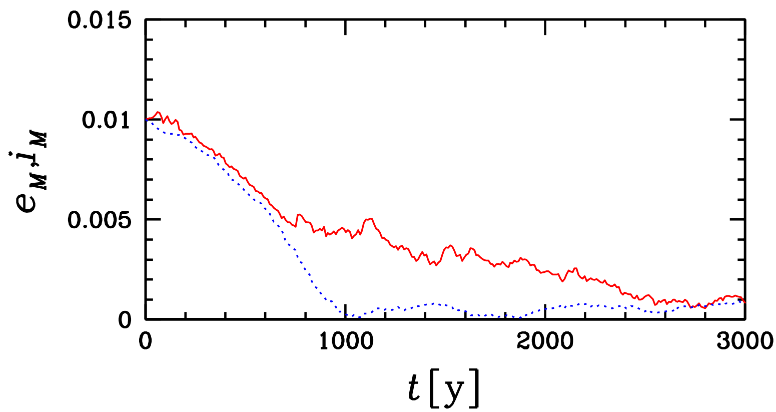


**Figure 4.** Time evolution of $e_M$ (solid) and $i_M$ (dashed) of the protoplanet.

have $T_{2B} \propto \sigma^4$. Thus, we have $\sigma \propto t^{1/4}$. The origin of the anisotropy of the velocity dispersion is the shear velocity between planetesimals with different $a$ due to the differential rotation in the Kepler potential. This type of anisotropy is also known for the galactic stellar disk (e.g., Kokubo & Ida 1992).

## 4. Dynamical Friction

Dynamical friction is the process of the equipartition of the random energy, $(1/2)mv^2$, of planetesimals. In other words, the random velocity becomes $v \propto m^{1/2}$. As an illustration of dynamical friction, we consider a simple case with a protoplanet (large planetesimal) with $M = 100\, m$ embedded in a swarm of planetesimals. We focus on the orbital evolution of the protoplanet. The initial orbital elements of the protoplanet are $a_M = 1$AU and $e_M = i_M = 0.01$.

For $M \gg m$ and $e_M > e$, the rate of dynamical friction for $e_M$ is approximated as

$$\frac{de_M^2}{dt} \sim -\frac{m}{M}\frac{e_M^2}{T_{2B}}. \tag{4.1}$$

The rate for $i_M$ is obtained by replacing $e_M$ with $i_M$ in (4.1). Figures 3 show the snapshots

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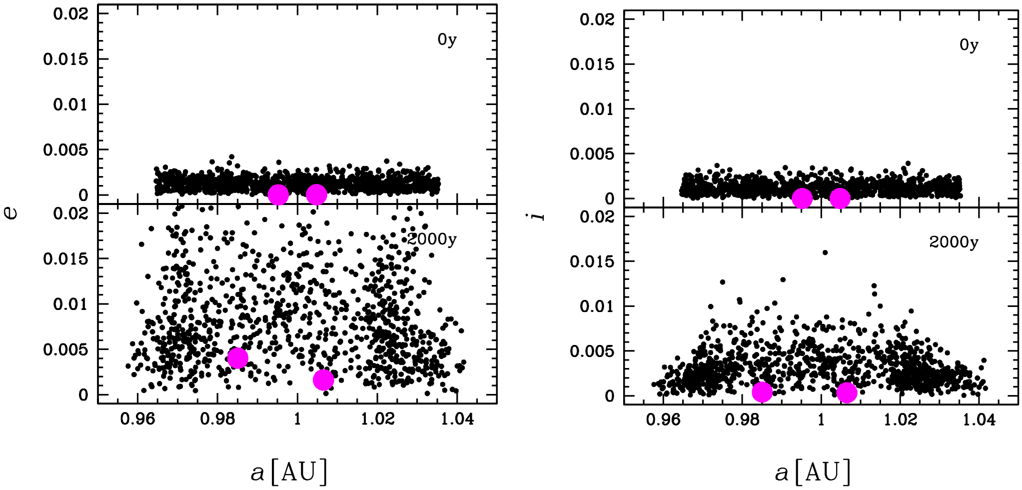


**Figure 5.** Snapshots of the planetesimal system on the $a$-$e$(left) and $a$-$i$ (right) planes at $t = 0$year (top) and 2000year (bottom).The large circles indicate the protoplanet.

of the system at $t = 0$ year and 3000 year on the $a$-$e$ and $a$-$i$ planes. We see that the eccentricity and inclination of the protoplanet decrease to almost 0 in 3000 years. However, its semimajor axis is kept almost constant. On the other hand, the eccentricities and inclinations of the neighbor planetesimals are raised by reaction. The V-like structure around the protoplanet on the $a$-$e$ plane corresponds to the constant Jacobi energy curve. This "heating of neighbor planetesimals by a protoplanet" leads to the decrease of the growth rate of the protoplanet (Ida & Makino 1993).

The time evolution of $e_M$ and $i_M$ of the protoplanet is shown in figure 4. In 3000 years, the eccentricity and inclination of the protoplanet are reduced to $\sim 0.001$ due to dynamical friction from small planetesimals, in other words, the orbit of the protoplanet becomes a non-inclined nearly circular orbit.

One of the important features of dynamical friction is that the rate of dynamical friction does not depend on the mass of individual particles but on the density of the system. For protoplanet-planetesimal scattering, $T_{2\mathrm{B}}$ for the protoplanet is $T_{2\mathrm{B}} \propto nM^2$ from (2.2). Therefore, (4.1) leads to $|de_M^2/dt| \propto nm = \rho$, where $\rho$ is the density of the system.

## 5. Orbital Repulsion

As a simple application of two-body relaxation, let us think the orbital evolution of two interacting protoplanets embedded in a swarm of planetesimals. Kokubo & Ida (1995) found that if the orbital separation, $b$, of the two protoplanets is smaller than a few times their mutual Hill radius, they expand their orbital separation to $b \gtrsim 5r_{\mathrm{H}}$. This phenomenon is called as orbital repulsion. In figures 5, the protoplanets with $M = 100m$ initially on non-inclined circular orbits with the orbital separation of $b = 3r_{\mathrm{H}}$ expands the orbital separation to $b \simeq 8r_{\mathrm{H}}$ keeping nearly non-inclined circular orbits in 2000 years.

By orbital repulsion, protoplanets grow keeping their orbital separation $b \gtrsim 5r_{\mathrm{H}}$. The typical orbital separation is $b \sim 10r_{\mathrm{H}}$ on the course of protoplanet growth (Kokubo & Ida 1998). This is one of the important factors that realize oligarchic growth of protoplanets.

Orbital repulsion is a coupling effect of two-body scattering and dynamical friction. Figure 6 schematically explains the two stage mechanism of orbital separation:

($a$) Scattering between two protoplanets on nearly circular non-inclined orbits increases their eccentricities and orbital separation.

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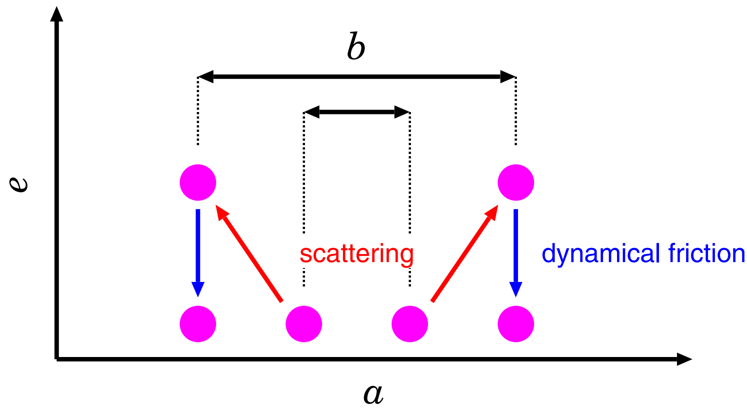


**Figure 6.** Schematic illustration of orbital repulsion on the $a$-$e$ plane.

($b$) Dynamical friction from planetesimals decreases the eccentricities of the protoplanets keeping their semimajor axes.
We can analytically show the behavior of the first stage, based on the conservation of energy and angular momentum in two-body scattering under the solar gravity.

## 6. Summary

We have demonstrated how two-body relaxation plays key roles in the velocity evolution of planetesimals by showing the examples of $N$-body simulations. The important processes are viscous stirring and dynamical friction. Viscous stirring increases $\sigma_e$ and $\sigma_i$ in proportion to $t^{1/4}$, keeping the ratio $\sigma_e/\sigma_i \simeq 2$, which is the characteristics of two-body relaxation in a disk-shape system. Dynamical friction realizes the equipartition of the random energy, in other words, $e, i \propto m^{-1/2}$. This equipartition is a sufficient condition of runaway growth of planetesimals (e.g., Kokubo & Ida 1996). Orbital repulsion of two protoplanets embedded in a swarm of planetesimals is also explained. The orbital separation of the two protoplanets is kept $b \gtrsim 5r_{\rm H}$ due to orbital repulsion. Orbital repulsion is one of the key processes for the oligarchic growth of protoplanets. All these elementary processes control the basic mode, timescale, and spatial structure of planetary accretion.

### Acknowledgements

I would like to thank Shigeru Ida and Junichiro Makino for the useful discussion on the $N$-body simulation of planetesimals.

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