Instability, Chaos and Predictability
in Celestial Mechanics and
Stellar Dynamics

ISBN 1-56072-054-9

On the Measure-Preserving Mappings with Three-Dimensions

Yi-Sui Sun
Dept. of Astronomy, Najing University, P.R. of China

ABSTRACT:

We have systematically made the numerical exploration about the perturbation extension of area-preserving mappings to three-dimensional ones, in which the fixed points of area-preserving are elliptic, parabolic or hyperbolic respectively. It has been observed that: (i) the invariant manifolds in the vicinity of the fixed point generally don't exist (ii) when the invariant curve of original two-dimensional mapping exists the invariant tubes do also in the neighbourhood of the invariant curve (iii) for the perturbation extension of area-preserving mapping the invariant manifolds can only be generated in the subset of the invariant manifolds of original two-dimensional mapping, (iv) for the perturbation extension of area-preserving mappings with hyperbolic or parabolic fixed point the ordered region near and far from the invariant curve will be destroyed by perturbation more easily than the other one. This is a result different from the case with the elliptic fixed point. In the latter the ordered region near invariant curve is solid. Some of the results have been demonstrated exactly.

Finally we have discussed the Kolmogorov Entropy of the mappings and studied some applications.

INTRODUCTION

Many studies of the measure-preserving mappings with even dimension have been made in connection with various problems in physics and astronomy, because Hamiltonian systems possess always even dimension and one can use the KAM theorem. But measure-preserving mappings with odd dimension do not seem to have been considered. Following a suggestion of M.Hénon, we began to study in 1980. The first non-trivial case is dimension 3. Consider the linearized mapping in the vicinity of a fixed point, at least one eigenvalue is real and the product of all eigenvalues is equal to one, therefore, in general, there are some eigenvalues whose moduli are larger than one. Thus fixed points are generally unstable in the linear approximation and we conjectured that the mapping is chaotic in the whole domain of definition. We have systematically made the numerical exploration about the perturbed extension of area-preserving mappings to three-dimensional ones in which the fixed points of area-preserving mappings are elliptic, parabolic respectively. Some of the numerical results have been proved theoretically.

Finally we have discussed the Kolmogorov entropy of the mappings and studied some applications.

THE MAPPINGS

We study the following three-dimensional mappings which are the perturbed extensions of area-preserving mappings with elliptic, parabolic or hyperbolic fixed point respectively.

(1) The mapping T_1

$$T_1 \begin{cases} x_{n+1} = x_n + y_n + B \sin z_n, \\ y_{n+1} = y_n + A \sin x_{n+1}, \qquad (\mathrm{mod}\ 2\pi) \\ z_{n+1} = z_n + C \sin y_{n+1}, \end{cases}$$

where A is a parameter, B,C the perturbation parameters. When $B = C = 0$, the mapping T_1 will be reduced to the standard mapping, for $-4 < A < 0$ the origin is an elliptic fixed point.

(2) The mapping T_2

$$T_2 \begin{cases} x_{n+1} = x_n - Ay_n^3 + B \sin z_n, \\ y_{n+1} = x_n + y_n - Ay_n^3 + C \sin x_{n+1}, \\ z_{n+1} = z_n + D \sin y_{n+1} + E. \qquad (\mathrm{mod}\ 2\pi) \end{cases}$$

where A is a parameter, B,C,D,E, the perturbation parameters. When $B = C = D = E = 0$ the mapping T_2 will become an area-preserving mapping, for any A the fixed point (0,0) is parabolic type, the case $A = 1$ has been studied by Chirikov and Simo [1],[2].

(3) The mapping T_3

$$T_3 \begin{cases} x_{n+1} = s(x_n \cos \phi_n - y_n \sin \phi_n) + A \cos z_n, \\ y_{n+1} = s^{-1}(x_n \sin \phi_n + y_n \cos \phi_n) + B \sin z_n, \\ z_{n+1} = z_n + C \cos(x_{n+1} + y_{n+1}) + D, \qquad (\mathrm{mod}\ 2\pi) \end{cases}$$

$$\phi = (x_n^2 + y_n^2)^k,$$

where s,k are parameters, A,B,C,D the perturbation parameters. For A = B = C = D = 0, this is a hyperbolic twist mapping which has been studied by Easton [3]. The origin is a hyperbolic fixed point.

NUMERICAL RESULTS

The Mapping T_1

As we do not know any theoretical conclusions, we start with numerical exploration by LCN's (Lyapunov Characteristic Numbers) method [4]. We make a transversal exploration along the y-axis, figure 1 is the plot of LCN's $r_i^N(P_o)(i = 1,2,3,$ $N = 10^5$: the number of iterations, P_o: initial point). We can see that in almost all cases the LCN's possess approximately the same values, however we find one particular set of values $r_i^N(P_o)$ (P_o = (0.1, -2.5, 0.0), i = 1,2,3) of LCN's, whose absolute values are much smaller.

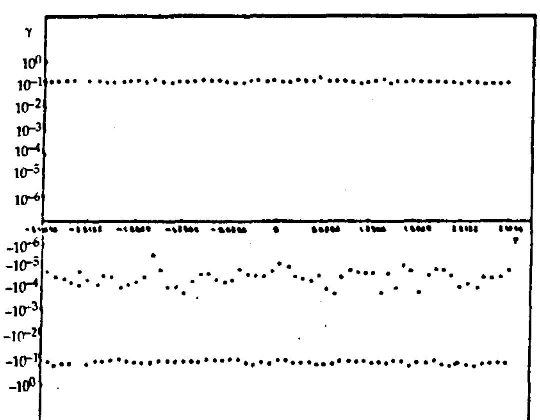

Fig.1: Variation of the $\gamma_i^{100000}(P_o)$(i=1,2,3) of mapping T_1 (A=-1.5, B=0.03) for the initial points P_o=(0.1, y, 0.0) with $-\pi \le y \le +\pi$: (+)$\gamma_1^{100000}(P_o)$, (Δ)$\gamma_2^{100000}(P_o)$. (O)$\gamma_3^{100000}(P_o)$).

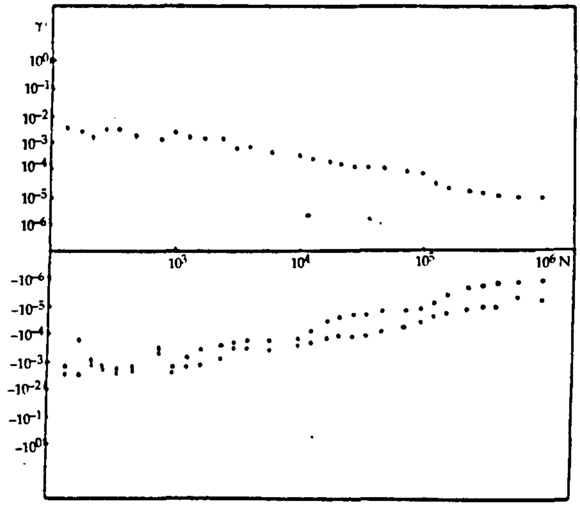

Fig. 2: Variation of the $\gamma_i'(P_o)$ of mapping T_1 (A= -1.5, B = 0.03) as functions of the number N of iterations for initial point P_n = (0.1, -2.5, 0.0): (+)$\gamma_1'(P_o)$. (Δ)$\gamma_2'(P_o)$. (O)$\gamma_3'(P_o)$

According to the usual relation between the LCN's and the ordered property of the orbit, we conjecture that the point P_o = (0.1, -2.5, 0.0) may be on an invariant manifold. In order to confirm it, we study the variations of $r_i^N(P_o)$ (i = 1,2,3) as functions of the number N of iterations upto $N=10^6$. Figure 2 shows clearly that $\lim_{N\to\infty}|r_i^N(P_o)| = 0$ (i = 1,2,3). On the other hand, we plot the slice-cutting of mapping T_1 with the same initial point P_o = (0.1, -2.5, 0.0) (see Fig. 3), which are defined by

$$|z_n - Q| < 0.01, \quad Q = (k-5)\pi/4, \quad k = 1,2,\ldots,9$$

$$-\pi \leq x_n \leq +\pi, \quad -\pi \leq y_n \leq +\pi$$

$$n = 1,2,\ldots,N$$

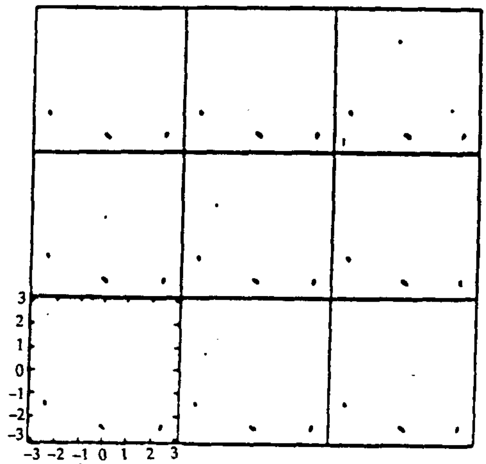

Fig. 3: Sections of mapping T_t (A = -1.5, B = 0.03) by $z = (k-5)\pi/4$, k = 1,2,...,9 with initial point P_o = (0.1, -2.5,0.0).

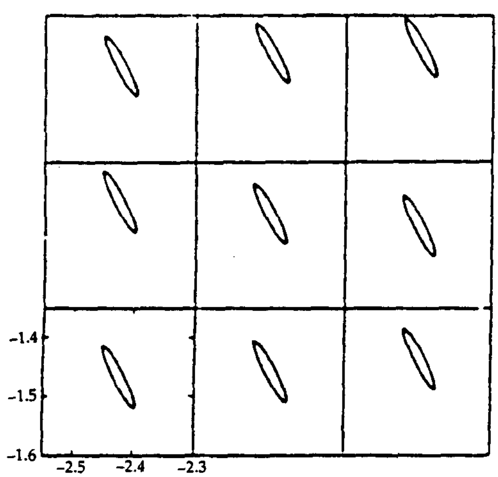

Fig. 4: Enlargement of the leftmost island-tube in Figure 3.

and we find that the orbit with initial point P_O lies indeed on invariant tubes which we shall call island-tubes. Figure 4 is the enlargement of the leftmost island-tube in Fig.3. Figure 5 displays the orbits near the island-tube studied above, it also exhibits the self-similarity structure analogous to that of the standard mapping, in the center of tube there must be an invariant curve. But for the orbit with initial point P_O = (0.1, 0.1, 0.0) which is near the origin, a very slow diffusion away from the origin appears.

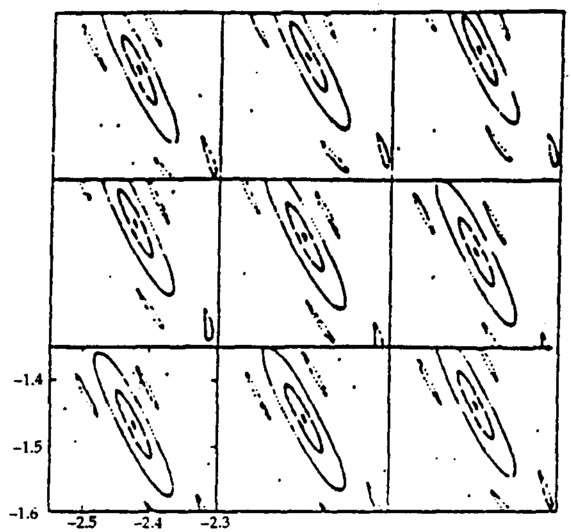

Fig. 5: Sections of mapping T_t (A = -1.5, B = 0.03).

Now a problem appears i.e. for the perturbed extension of standard mapping the large ordered region in the neighbourhood of fixed point (0,0) of the standard mapping disappears, but in the small island region far from the origin the invariant manifolds survive? We think that perhaps it is related to the variation of z. In order to confirm it, we add another parameter D to control the variation of z i.e. we study the following mapping [5].

$$T_1' \begin{cases} x_{n+1} = x_n + y_n + B \sin z_n, \\ y_{n+1} = y_n + A \sin x_{n+1}, \quad \pmod{2\pi} \\ z_{n+1} = z_n + C \sin y_{n+1} + D. \end{cases}$$

At first we analyse the existence of invariant curve. Let

$$x = F(z), y = G(z)$$

be the invariant curve of the mapping T_1' because C,D are small, we obtain approximately

$$F(z) = 0, \quad G(z) = - B \sin z$$

therefore along the invariant curve, we have

$$\Delta z = z_{n+1} - z_n \stackrel{\sim}{\sim} D-BC \sin z_n .$$

When D > BC, $\Delta z > 0$ i.e. z varies between $-\pi$ and $+\pi$, the invariant curve exists and there will exist the invariant tubes surrounding it. When D < BC, z tends to a fixed value z*, in this case the invariant curve degenerates to a point and the image points of mapping in the neighbourhood of invariant curve tend to the plane z = z*. As the mapping is measure-preserving, the diffusion along the plane z = z* will appear. Figures 6,7, which are 9 slice-cuttings as defined in(1), exhibit the above two cases respectively. Moreover, we study the perturbed extension of quadratic area-preserving mapping and obtain the analogous results [6].

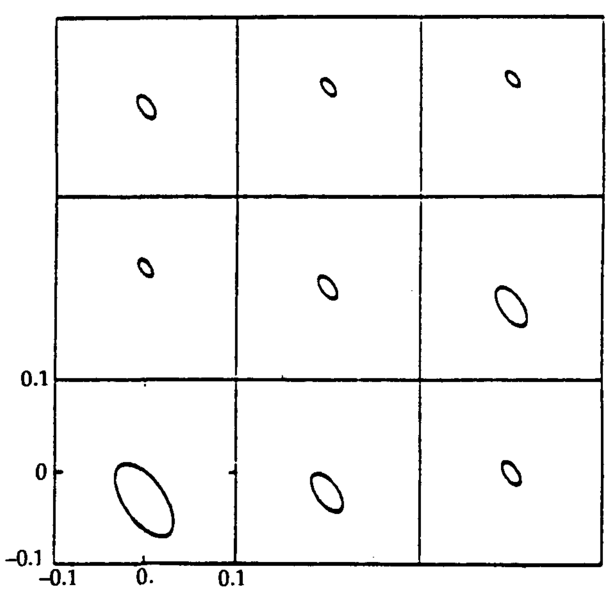

Fig.6: Sections of mapping T_1' for A= -1.5, B=C=0.03, D= 0.001 with initial point P_o = (0.00, -0.01, 0.00).

https://doi.org/10.1017/S0252921100065957 Published online by Cambridge University Press

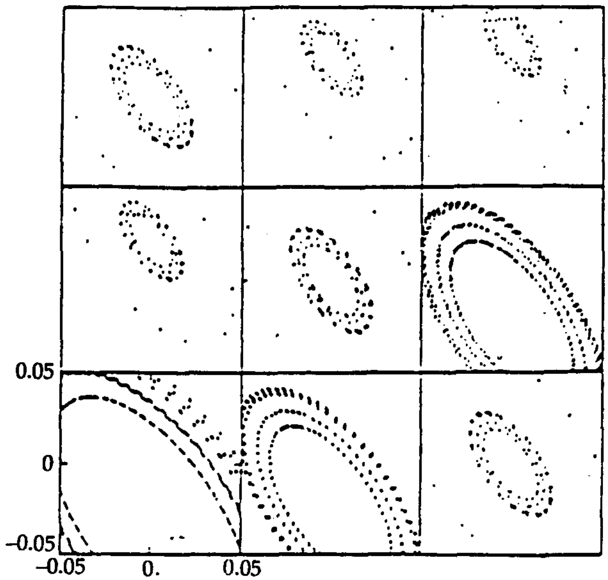

Fig.7: Sections of mapping T_1' for $A = -1.5$, $B = C = 0.03$, $D = 0.0008$ with initial point $P_o = (0.0,\ 0.0, 1.0951442)$

The Mapping T_2

Using the same methods as studied in (1), we get the criterion for the existence of invariant manifolds of mapping T_2 [7].

$$E > D(B/A)^{(1/3)}$$

When $A = 1.5$, $B = C = D = 0$, the results display the self-similarity structure and a large ordered region in the vicinity of parabolic fixed point (0,0) (see Figs. 8,9). For $A = 1.5$, $B = C = D = 0.03$, $E = 0.009$ which satisfy the criterion $E > D(B/A)^{(1/3)}$, we find the invariant tube which is generated by the invariant curve of original two-dimensional mapping (i.e. $B = C = D = D = 0$) (see Fig. 10). But for the initial point (0,0,0) we do not get the invariant curve as expected and a chaotic region appears (see Fig. 11). This result is different from the mapping T_1'. We can find the small island-tubes only if the perturbation parameters are very small (see Fig. 12).

https://doi.org/10.1017/S0252921100065957 Published online by Cambridge University Press

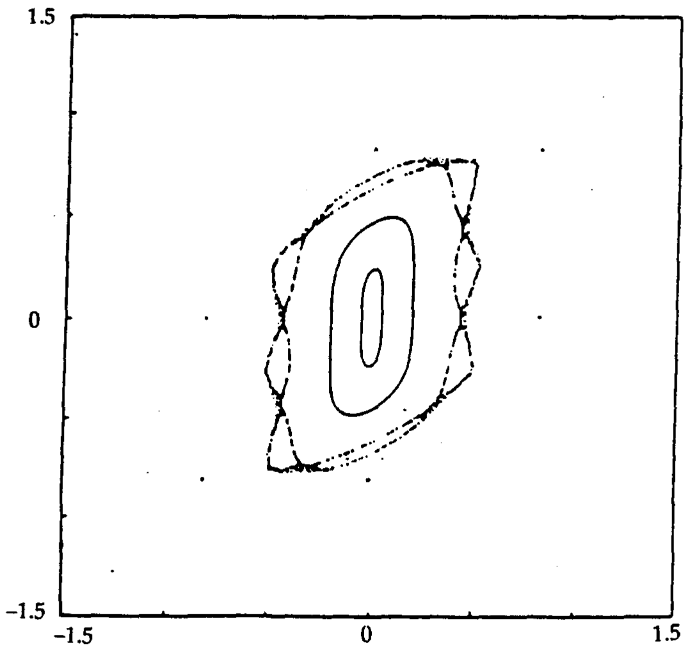

Fig.8: The mapping T_2 for A = 1.5, B = C = D = E = 0, with initial points P_o = (0.05,0.00);(0.2,0.0):(0.43,0.00):(0.815,0.000).

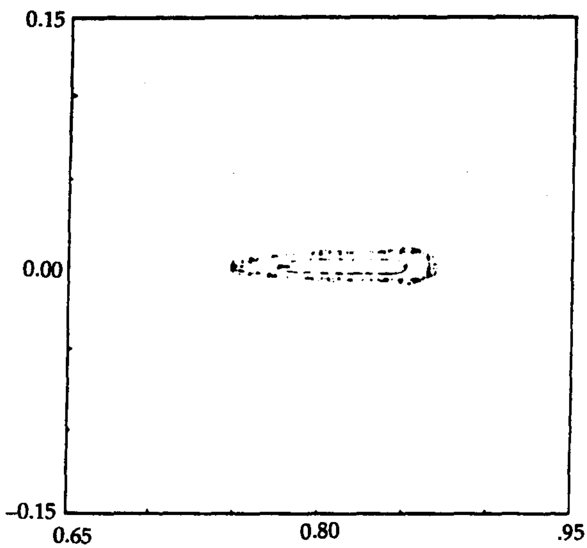

Fig.9: The mapping T_2 for A = 1.5, B = C = D = E = 0, with initial points P_o = (0.815,0.003);(0.86775,0.00000);(0.815,0.008)

https://doi.org/10.1017/S0252921100065957 Published online by Cambridge University Press

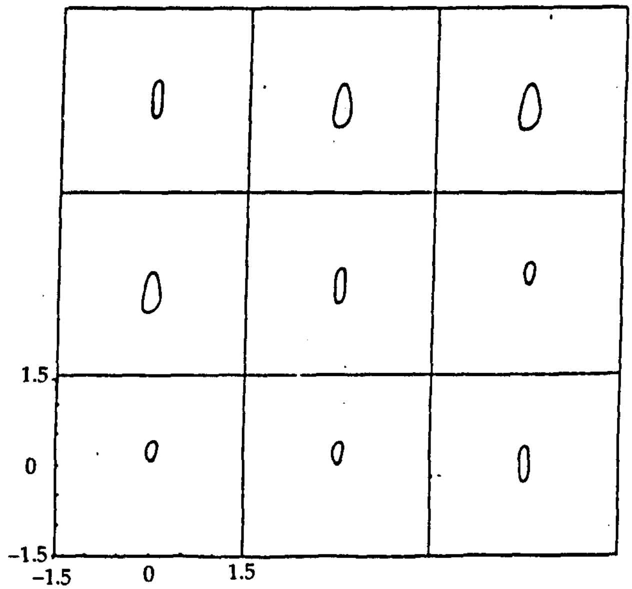

Fig. 10: Sections of mapping T_2 for A = 1.5, B = C = D = 0.03, E = 0.009 with initial point P_o = (0.08, 0.00, 0.00).

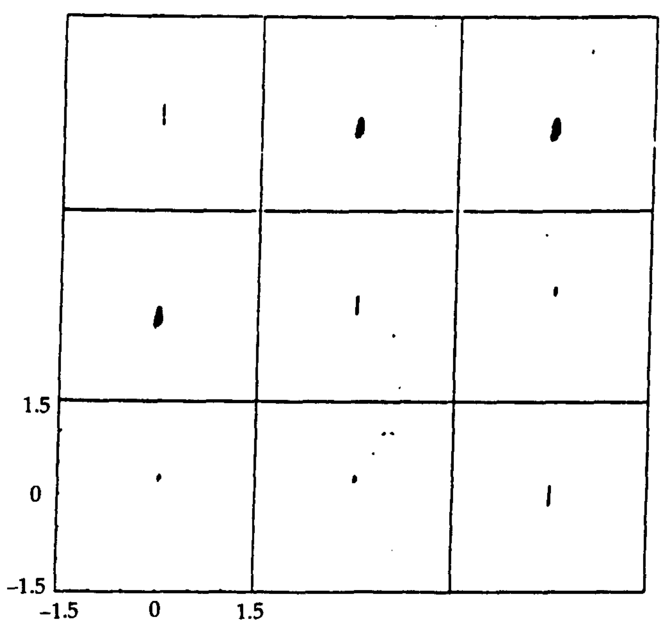

Fig.11: Sections of mapping T_2 for A = 1.5, B = C = D = 0.03, E = 0.009, with initial point P_o = (0.00, 0.00, 0.00).

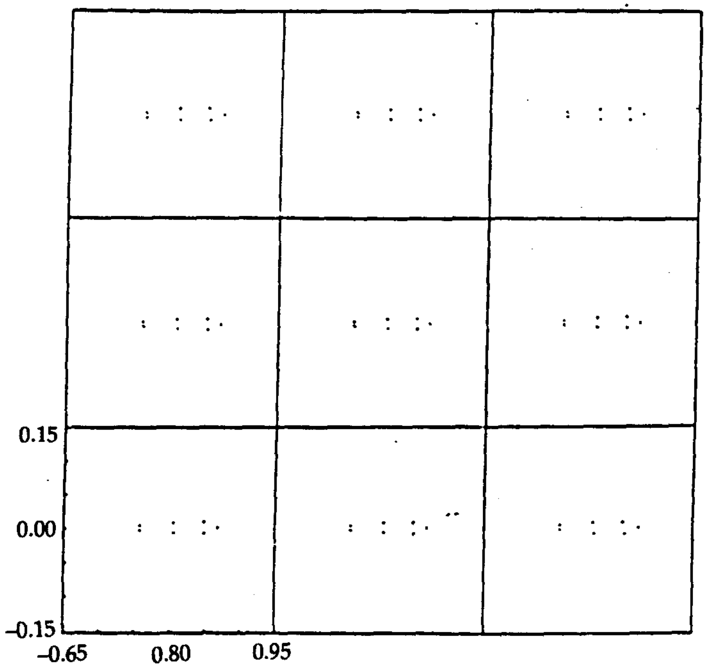

Fig.12: Sections of mapping T_2 for A = 1.5, B = C = D=0.0003, E = 0.0004 with initial point P_o = (0.86775, 0.00000, 0.00000).

The Mapping T_3

This is a perturbed extension of hyperbolic twist mapping. Hadjidemetriou has studied a similar three-dimensional mapping in connection with the evolution of an asteroid orbit passing through 3:1 resonance. When A = B = C = D = 0, the mapping T_3 will be reduced to the hyperbolic twist mapping, the origin is a hyperbolic fixed point and it is not analytic but belongs to C^1 class at the origin. Obviously it is analytic in an annulus surrounding the origin and is symmetric with respect to the origin. For positive k, figure 13 shows the ordered, chaotic regions and hyperbolic structure at the fixed point (0,0) and for negative k near the origin there is a large chaotic region surrounded by the invariant curves. For the perturbed extension T_3 there exists a chaotic cylinder surrounded by the invariant tori for both positive and negative k (see Figs. 14,15) [8]. This result shows that the ordered region near the fixed point (0,0) of hyperbolic twist mapping is destroyed more easily than the one distant from it, this is analogous to the mapping T_2. For the dissipative system, we study the perturbed extension of Henon map-

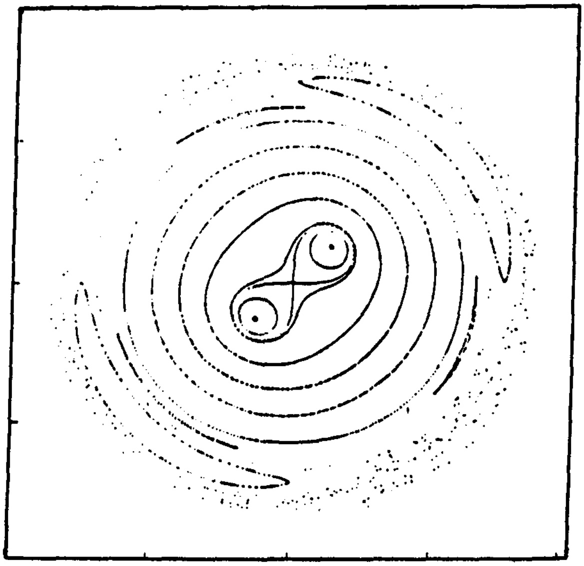

Fig.13: The mapping T_3 for k = 1.5, s = 1.05.

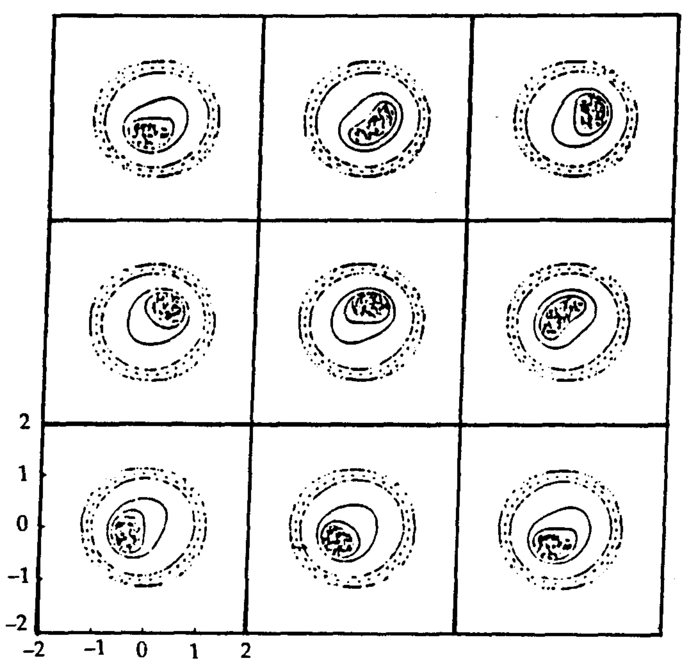

Fig.14: Sections of mapping T_3 for k = 1.5, s = 1.05, A = B = C= 0.03, D = -0.04.

https://doi.org/10.1017/S0252921100065957 Published online by Cambridge University Press

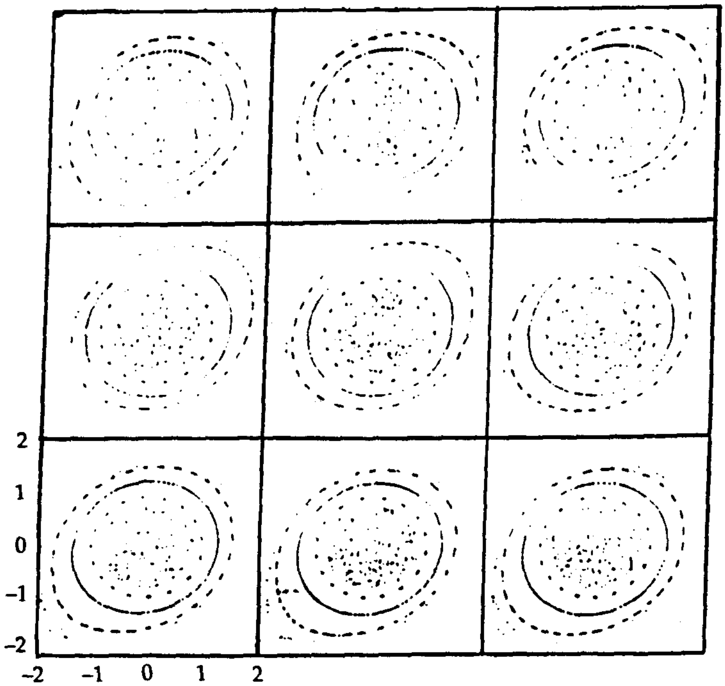

Fig. 15: Sections of mapping T_3 for k = -1.5, s = 1.05, A = B = C = 0.03, D = -0.04.

ping, it seems that the strange attractor in Henon mapping is destroyed by perturbed extension more easily than the invariant manifold in standard mapping [9], this case is similar a bit to that in the mappings T_2 and T_3.

In addition, we also discuss the variation of ordered region with the parameters for the above three kinds of mappings. The conclusion is that the ordered region decreases when the perturbation parameters increase.

From the above discussions we come to the following main conclusions:

(i) For the perturbed extension the invariant manifolds in the vicinity of fixed point generally do not exist.

(ii) When the invariant curve of perturbed three-dimensional extension exists the invariant tubes do also in the neighbourhood of it.

(iii) For the perturbed extension of area-preserving mappings the invariant manifolds can only be generated in the subset of invariant manifolds of original two-dimensional mappings.

(iv) For the perturbed extension of area-preserving mappings with parabolic or hyperbolic fixed point the ordered region near and far from the invariant curve will be destroyed by perturbation more easily than the other one. This result is different from the case with elliptic fixed point. In the latter, the ordered-region near invariant curve is solid.

Some Theoretical Results

In the above numerical explorations, an important result is that when the invariant curve of perturbed three-dimensional extension exists, the invariant tubes do also in the neighbourhood of it, this conclusion is similar to the Moser's existence theorem of invariant curves. Now we shall only give a simple description of theoretical proof, the details of proof can be referred in [10],[11].

We assume that there is an invariant curve for the three dimensional analytical measure-preserving M. We consider a family of mappings near the curve, which is of the form

$$M_1: \begin{cases} X_1 = A(S)X + F_o(S,X) \\ S_1 = S+B(S) + G_o(S,X) \quad (S \bmod 2\pi,\ X \in \mathbf{R}^2) \end{cases}$$

where A, B, G_o, G_o are all of them 2π-periodic functions in S.

$$F_o(S,X) = O(\|X\|^2),\quad G_o(S,X) = O(\|X\|)$$

where $\|.\|$ denotes the Euclidean norm.

Analogous to the Birkhoff normalization of area-preserving mapping, we can prove that the mapping M_t can be transformed into the following form

$$M_1': \begin{cases} \psi_1 = \psi + f(r) + \Psi(r,\theta,\psi) \\ \theta_1 = \theta + g(r) + \Theta(r,\theta,\psi) \\ r_1 = r + R(r,\theta,\psi) \end{cases}$$

Ψ, Θ, R are analytical functions on the domain

$$|\mathrm{Im}\ \psi| \leq \rho,\ |\mathrm{Im}\ \theta| \leq \rho,\ |r-r_o| \leq s,\ \forall r_o \in [a,b]$$

$$|\Psi| + |\Theta| + |R| \leq d \text{ and } f,g \text{ are differentiable}$$

functions on the interval [a,b]. There is $r_o \in [a,b]$ so that $g(r_o) = m/n$, m,n: integers and $f(r_o)$ satisfies inequalities

$$|k \times f(r_o) + 2n\pi| \geq c_o|k|^{-\mu} (c_o > 0, \mu > 1) \ \forall (k,n) \in Z^2/(0.0)$$

and $f'(r_o)$, $g'(r_o) \neq 0$. Then we can prove the existence of periodically invariant curves in the vicinity of the invariant curve.

For the existence of invariant tori in the neighbourhood of the invariant curve, as the number of the fast variables or angles is less than the number of slow variables or actions, we can not compensate the change of frequencies, due to the average part of the perturbations, by correcting the initial conditions in every approximation. This difficult implies the intrinsic difference between the systems in which the number of angles and actions are equal and the systems in which there are more angles than actions. In order to overcome this difficult, at first we prove that there is a Cantor set with positive Lebesque measure in the frequency interval such that two frequency of angles satisfy still irrational inequality in every approximation. Py simulating the proof method of Moser's existence theorem of invariant curves, we complete the proof of existence of invariant tori.

THE KOLMOGOROV ENTROPY

It is well known that the Kolmogorov entropy is a good indicator of chaos. Pesin's formula has given the relation between the LCN's and the Kolmogorov entropy of a dynamical system. Let $\rho(P)$ denote the sum of all positive LCN's . The formula states that the total entropy is given by

$$h = \int_V \rho(P) d\mu$$

where μ is the Lebesque measure v.

At first, we study the Kolmogorov entropy of the mapping T_1' and obtain

$$h \approx \lim_{i\to\infty} \frac{1}{i} \ln \prod_{k=1}^{i} |ABC \cos x_{k+1} \cos y_{k+1} \cos z_k|$$

For the large B,C, the mapping T_1' is ergodic, according to the ergodic theorem, we can approximately replace

$$h \approx \lim_{i\to\infty} \frac{1}{i} \ln \prod_{k=1}^{i} |ABC \cos x_{k+1} \cos y_{k+1} \cos z_k|$$

by $\frac{1}{8\pi^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \ln|ABC \cos x \cos y \cos z| dxdydz$

then we have

$$h \approx \ln \frac{|ABC|}{8}$$

Figure 16 shows the good agreement between numerical and analytical computations [12].

Moreover, we compute the Kolmogorov entropy of the mapping T_3. The results show that the hyperbolicity of the fixed point (0,0) i.e. the parameter s and the negative k play more important role in generating the chaos than the other parameters [13].

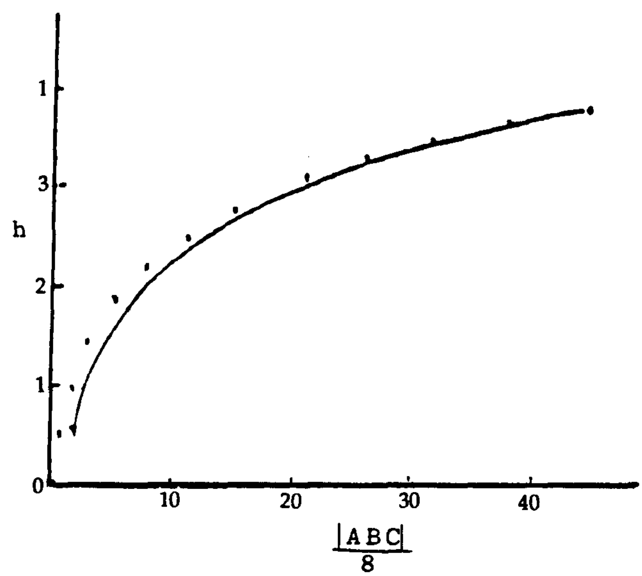

Fig. 16: Variation of the entropy h with the parameters B,C. The crosses show the numerical estimations. The curve gives the analytical results.

Finally, we apply the results of three-dimensional measure-preserving mappings to the Couette-Taylor system in fluid dynamics. Because the flow of incompressible fluid can be related to the measure-preserving mapping and the flow between two rotating cylinders with an infinite length is circular direction, it will correspond to a three-dimensional measure-preserving mapping which is the composition of a two-dimensional twist mapping and a one-dimensional one. The numerical simulations display some experimental results. Feingold, Kadanoff and Piro discuss the dynamics of a medium-sized particle (passive scalar) suspended in a general time-periodic imcompressible fluid flow which can be described by three-dimensional measure-preserving mappings. For mappings with only one action they find strong evidence for the existence of invariant surfaces that survive the nonlinear perturbation in a KAM-like way. For the two-action case the motion is confined to invariant lines that break for arbitrary small size of the nonlinearity [14].

REFERENCES

[1] Chirikov, B.D. and Izraelev, F.M. (1973), Colloque International du CNRS, 229, 409.

[2] Simo, C. (1980), Lecture Notes in Mathematics,819,418.

[3] Easton, R.W. (1979), Instability in Dynamical Systems, D.Reidel Pub.Co. p.41.

[4] Sun, Y.S. (1983), Celest. Mech.,30,7.

[5] Sun, Y.S. (1984), Celest. Mech.,33,111.

[6] Sin, Y.S. (1984), Scientia Sinica (A),27,174.

[7] Zhang, T.L. and Sun, Y.S. (1989), J. of Nanjing Univ., 25,187.

[8] Sun, Y.S. and Yan, Z.M. (1988), Celest, Mech., 42,369.

[9] Sun, Y.S. (1985), Celest. Mech. 37,171.

[10] Cheng, C.Q. and Sun, Y.S.(1989), Celest. Mech., 47,293.

[11] Cheng, C.Q. and Sun, Y.S. (1990),Celest. Mech., 47,275.

[12] Sun, Y.S. (1983), Acta Astronomica Sinica, 24,128.

[13] Yan, Z.M. and Sun,Y.S., On Kolmogorov Entropy of Some Measure-Preserving Mappings (to appear in Scientia Sinica (A)).

[14] Feingold, M., Kadnoff, L.P. and Piro, O.(1988),J.Stat. Phy., 50, 529.