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Universality Under Szegő's Condition

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Abstract. This paper presents a theorem on universality on orthogonal polynomials/random matrices under a weak local condition on the weight function w. With a new inequality for polynomials and with the use of fast decreasing polynomials, it is shown that an approach of D. S. Lubinsky is applicable. The proof works at all points that are Lebesgue-points for both the weight function w and $\log w$.

1 Introduction and Results

In [6] D. Lubinsky found a simple and elementary approach to universality limits. He had a second method in [7] based on the theory of entire functions. This second, powerful method needs the verification of some preliminary estimates, which, at general points, are far from trivial. In this paper we show how those preliminary estimates can be proved under relatively light conditions, and we recapture/generalize the general results of [11] and [14] in a precise, sharpened form.

Let μ be a positive finite Borel measure with compact support Σ on the real line. We assume that Σ consists of infinitely many points, and we can then form the orthonormal polynomials $p_n(\mu;x) = \gamma_n(\mu)x^n + \cdots$ with respect to μ. Let

$$K_n(\mu;x,y) = \sum_{j=0}^{n} p_j(\mu,x)\overline{p_j(\mu,y)}$$

be the associated reproducing kernels. It is known that some universality questions in random matrix theory can be expressed in terms of orthogonal polynomials, in particular in terms of the off-diagonal behavior of the reproducing kernel; see [3, 6, 8, 9] and the references therein. When $\Sigma = [-1,1]$ and $d\mu(x) = w(x)dx$, a form of universality in random matrix theory can be stated as

$$\lim_{n\to\infty} \frac{K_n\big(x + \frac{a}{w(x)K_n(x,x)}, x + \frac{b}{w(x)K_n(x,x)}\big)}{K_n(x,x)} = \frac{\sin \pi(a-b)}{\pi(a-b)} \tag{1.1}$$

(with $K_n(x,y) = K_n(\mu;x,y)$) uniformly in a, b lying in some compact subset of the real line. This had been proved under strong conditions on w by various authors and recently by Lubinsky [6] under continuity and positivity of w. More precisely, Lubinsky proved that (1.1) holds uniformly in $x \in S$ and locally uniformly in $a, b \in \mathbf{R}$ provided μ is in the **Reg** class (see below) with support $[-1,1]$, $S \subset (-1,1)$ is a

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compact set, μ is absolutely continuous in a neighborhood of S and its density w (*i.e.*, the Radon–Nikodym derivative with respect to Lebesgue-measure) is positive and continuous on S.

Lubinsky had a second approach [7] to universality based on the theory of entire functions. This work uses this second approach about which we shall give some details in the next section.

We shall need some concepts from potential theory, in particular, the logarithmic capacity $\mathrm{cap}(\Sigma)$ and the equilibrium measure μ_Σ of a compact set $\Sigma \subset \mathbf{R}$; see the books [1,10,17]. Denote the density of the equilibrium measure μ_Σ of Σ by ω_Σ. It exits everywhere on $\mathrm{Int}(\Sigma)$ (and it is continuous – actually C^∞ – there).

We shall also need the concept of the **Reg** class. For the leading coefficients $\gamma_n(\mu)$ of $p_n(\mu; x)$, it is known ([12, Corollary 1.1.7]) that

$$\liminf_{n\to\infty} \gamma_n(\mu)^{1/n} \geq \frac{1}{\mathrm{cap}(\Sigma)},$$

and the measure μ is called to be in the **Reg** class (or is called regular from the point of view of orthogonal polynomials) if

$$\lim_{n\to\infty} \gamma_n(\mu)^{1/n} = \frac{1}{\mathrm{cap}(\Sigma)}, \tag{1.2}$$

and the right-hand side is finite. This is a rather mild assumption, and it holds under fairly general conditions on μ (see [12, Chapters 3 and 4]). For various properties of orthogonal polynomials with respect to regular measures; see [12]. In particular, if ν, μ have the same support, $\nu \geq \mu$ and μ is regular, then so is ν (since $\gamma_n(\nu) \leq \gamma_n(\mu)$).

M. Findley [4] proved a local version of (1.1) under the condition that the support of μ is $[-1,1]$, $\log w \in L^1$ in a neighborhood of x and x is a Lebesgue-point for both w and its local outer function. In [11] and [14] the limit (1.1) was verified for general measures; in particular, [14] contains the result that (1.1) is true a.e. on an interval I provided $\mu \in \mathbf{Reg}$ and $\log w \in L^1(I)$. The proof used a complicated version of the polynomial inverse image method, and it was pure luck that that method worked in this case. The main objective of this paper is to reprove and make more precise the just-mentioned result using the second approach of Lubinsky developed in [7] (see also [2]).

As before, let μ be a finite Borel measure with compact support $\Sigma \subset \mathbf{R}$. We shall always assume that μ is regular in the sense of (1.2), hence Σ is of positive capacity. If μ is absolutely continuous with respect to Lebesgue measure on an interval $I \subset \mathrm{Int}(\Sigma)$, then we call its Radon-Nikodym derivative $d\mu(x)/dx$ with respect to Lebesgue measures its *density*, and we denote it by $w(x)$.

As usual, we say that x_0 is a Lebesgue-point for w if

$$\lim_{r\to 0} \frac{1}{2r} \int_{-r}^{r} \big| w(x_0 + t) - w(x_0) \big| \, dt = 0,$$

and for a measure $\mu = \mu_{\mathrm{sing}} + \mu_a$, where $d\mu_a(x) = w(x)dx$ is its absolutely continuous part and μ_{sing} is its singular part, we call x_0 a Lebesgue-point for μ if it is a Lebesgue-point for w and

$$\lim_{r\to 0} \frac{1}{2r} \mu_{\mathrm{sing}}\big([x_0 - r, x_0 + r]\big) = 0.$$

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When w, μ are defined on a rectifiable Jordan curve (or unions of such curves), then one can similarly define the concept of Lebesgue-point with respect to arc length.

In what follows $w(x)dx$ denotes the absolutely continuous part of μ.

Theorem 1.1 *Assume that $\mu \in \mathbf{Reg}$ is a measure with compact support Σ on the real line such that $\log w \in L^1(I)$ for some interval I, and assume that $x_0 \in I$ is a Lebesgue-point for both μ and $\log w$. Then universality (1.1) holds for μ at x_0.*

As a corollary, it follows that (1.1) is true almost everywhere on I. It was observed by Levin and Lubinsky [5] that the universality in question implies fine zero spacing of orthogonal polynomials. Hence, as a second corollary, we have the following theorem for the zeros $z_{n,1} < z_{n,2} < \cdots < z_{n,n}$ of the n-th orthogonal polynomial $p_n(\mu, z)$.

Theorem 1.2 *With the assumptions of Theorem 1.1, we have*

$$\lim_{n\to\infty} n(z_{n,k+1} - z_{n,k})\omega_\Sigma(x_0) = 1$$

for $|z_{n,k} - x_0| \le L/n$ with any fixed L.

Recall that here ω_Σ is the density of the equilibrium measure of the the support Σ of μ.

In particular, if $\mu \in \mathbf{Reg}$ and w is continuous and positive on some open subinterval I of Σ, then, uniformly for x lying in any closed part of I, we have

$$\lim_{n\to\infty} n(z_{n,k+1} - z_{n,k})\omega_\Sigma(x) = 1$$

for $|z_{n,k} - x| = o(1)$; *i.e.*, the local zero spacing of the orthogonal polynomials reflect not just the global support, but also the position of the particular zero inside that support. This follows easily from the proofs below.

Theorem 1.1 follows from Lubinsky's method in [7] or directly from [2, Theorem 1] if we prove the following two results (see the next section for more details).

Theorem 1.3 *Assume that $\mu \in \mathbf{Reg}$ is a measure on the real line with compact support Σ such that $\log w \in L^1(I)$ for some interval I, and assume that $x_0 \in I$ is a Lebesgue-point for both μ and $\log w$. Let $A > 0$ be fixed. Then for all real a*

$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu; x_0 + a/n, x_0 + a/n) = \frac{\omega_\Sigma(x_0)}{w(x_0)},$$

and the convergence is uniform in $a \in [-A, A]$ for any fixed A.

Theorem 1.4 *Assume that μ is a measure on the real line for which $w, \log w \in L^1[-\delta, \delta]$ for some $\delta > 0$ and 0 is a Lebesgue-point for both w and $\log w$. Then for the corresponding reproducing kernel we have for $|z_0| \le A$ and for sufficiently large $n \ge n_A$,*

$$\frac{1}{n} K_n(z_0/n, z_0/n) \le Ce^{C|z_0|}, \tag{1.3}$$

where C is a constant independent of z_0 and A.

In this theorem (1.3) needs to be verified for complex values z_0.

2 Lubinsky's Approach to Universality

In [7] not K_n, but the kernel

$$K_n^*(\mu; x, y) = \sum_{j=0}^{n} p_j(\mu, x) p_j(\mu, y)$$

was used. This is the same as K_n for real x, y.

It was shown in [7], without the assumption $\mu \in \mathbf{Reg}$, that (1.1) holds at a point $x = x_0$ where w is continuous and positive if and only if

$$\lim_{n\to\infty} \frac{K_n^*(x_0 + a/n, x_0 + a/n)}{K_n^*(x_0, x_0)} = 1 \tag{2.1}$$

holds uniformly for a lying in compact subsets of the real line. The proof of this remarkable equivalence is along the following lines.

The positivity and continuity of w at x_0 easily implies that in a neighborhood $[x_0 - \delta, x_0 + \delta]$ an inequality $\frac{1}{C} \le \frac{1}{n} K_n^*(x, x) \le C$ holds, which then yields

$$\frac{1}{n} |K_n^*(\xi, t)| \le C$$

via the Cauchy–Schwarz inequality. This and the classical Bernstein–Walsh lemma for polynomials implies the bound

$$\frac{1}{n} \left| K_n^*(x_0 + a/n, x_0 + b/n) \right| \le Ce^{C(|a|+|b|)}$$

for complex a, b. Therefore, for

$$f_n(a, b) = \frac{K_n^*\big(x_0 + a/(w(x_0)K_n^*(x_0, x_0)), x + b/(w(x_0)K_n^*(x_0, x_0))\big)}{K_n^*(x_0, x_0)},$$

we also have

$$|f_n(a, b)| \le Ce^{C(|a|+|b|)} \tag{2.2}$$

with a possibly different C, which, however, is the same constant for all $|a|, |b| \le A$ for any fixed A provided n is sufficiently large (depending on A).

Hence, $\{f_n(a, b)\}_{n=1}^{\infty}$ is a normal family in both $a, b \in \mathbf{C}$, and for any (locally uniform) limit $f(a, b)$ of any subsequence of $\{f_n(a, b)\}_{n=1}^{\infty}$ we have the bound

$$|f(a, b)| \le Ce^{C(|a|+|b|)}.$$

To confirm with [7] let us mention that this last inequality, combined with the boundedness of $f(a, b)$ on the real line (which is a consequence of (2.1)), implies (see [7, Section 4, (4.4)])

$$|f(a, b)| \le Ce^{C(|\Im a|+|\Im b|)}.$$

Thus, $f(a, b)$ is an entire function of exponential type in each variable, and in [7] Lubinsky used the theory of exponential functions together with some properties of

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$K_n^*(\xi, t)$ and of some classical results for Gaussian quadrature to show that necessarily

$$f(a,b) = \frac{\sin \pi(a-b)}{\pi(a-b)}.$$

The crucial inequality (2.2) is a consequence (use Cauchy–Schwarz) of

$$\frac{1}{n}K_n(x_0 + a/n, x_0 + a/n) \le Ce^{C|a|}, \tag{2.3}$$

(here K_n and not K_n^* is used!) uniformly in $|a| \le A$ for any fixed A and sufficiently large n (say $n \ge n_A$), provided we know the behavior $K_n^*(x_0, x_0)/n \sim 1$.

Once the equivalence of (1.1) and (2.1) is established, (1.1) follows immediately at $x = x_0$ if a limit

$$\frac{1}{n}K_n^*(\mu; x_0 + a/n, x_0 + a/n) = L \tag{2.4}$$

(with a finite $L > 0$) can be established uniformly in a lying on any compact subset of the real line, and here K_n^* can be replaced by K_n. Theoretically, (2.1) could be true even if a limit like (2.4) does not hold, though so far no example has been found. Moreover, until now the limit (2.4) has been established only for measures in the **Reg** class.

As we can see from this setup, to prove (1.1) along these lines one needs two things:

(A) to prove the equivalence of (1.1) with (2.1), and
(B) to establish (2.1).

We have already mentioned that (A) has been done in [7] provided w is continuous and positive at x_0. If we drop this condition, the crucial inequality (2.3) becomes rather non-trivial, and the aim of Theorem 1.4 is to establish it under the Lebesgue-point condition stated there (*cf.* also [7, Theorem 2.1], where (A) is proved at a Lebesgue-point provided w has a positive lower bound in a neighborhood of x_0). For part (B) presently the only approach is via a limit like (2.4) using the **Reg** condition. The limit (2.4) is also less obvious in the non-continuous case, and it is the aim of Theorem 1.3 to establish (2.4) under the aforestated Lebesgue-point condition.

We emphasize, that in this paper both (A) and (B) are proved under the same Lebesgue-point condition using the same polynomial inequality to be discussed in Lemma 3.1.

Since some of the arguments sketched above are somewhat subtle in our case, we also mention that the sufficiency of Theorems 1.3 and 1.4 for Theorem 1.1 follows directly from [2, Theorem 1] by Avila, Last, and Simon. In fact, these authors used a modification of the method of Lubinsky to prove in [2, Theorem 1] that (1.1) holds at a point $x = x_0$ that is a Lebesgue-point for μ if

(a) (2.1) holds uniformly for a lying in compact subsets of the real line,
(b) $\liminf_{n\to\infty} \frac{1}{n}K_n(x_0, x_0) > 0$,
(c) for every $\varepsilon > 0$ there is a C_ε such that for any R there is an N so that for all $n > N$ and for all $z \in \mathbf{C}$ with $|z| < R$ we have

$$\frac{1}{n}K_n(x_0 + z_0/n, x_0 + z_0/n) \le C_\varepsilon \exp(\varepsilon |z_0|^2).$$

Now if x_0 is a Lebesgue-point for both μ ($\in$ **Reg**) and $\log w$, then Theorem 1.3 implies (a) and (b), while Theorem 1.4 implies (c), so it is left to prove Theorems 1.3 and 1.4.

3 Proof of Theorem 1.4

By simple scaling we can assume $\delta = 1$.

For the proof we need to consider the reciprocal of the diagonal of the reproducing kernel; the Christoffel functions associated with μ are defined as

$$\lambda_n(z,\mu) = K_n(z,z)^{-1} = \Big(\sum_{k=0}^{n} |p_k(z)|^2\Big)^{-1} = \inf_{P_n(z)=1} \int |P_n|^2 d\mu,$$

where the infimum is taken for all polynomials of degree at most n that take the value 1 at z.

We shall prove Theorem 1.4 in the equivalent form

$$\lambda_n(z_0/n,\mu) \geq \frac{e^{-C|z_0|}}{Cn}. \tag{3.1}$$

Since $\lambda_n(\cdot,\mu)$ is monotone increasing in the measure μ, we can assume that the singular part μ_s of μ is zero; *i.e.*, $d\mu(x) = w(x)dx$ and μ is supported on $[-1,1]$. By symmetry, it is enough to consider $\Im z_0 \geq 0$.

The proof is based on the next lemma.

Lemma 3.1 *Let $w \geq 0$ be a function on $[-1,1]$ such that $w, \log w \in L^1[-1,1]$, and let 0 be a Lebesgue-point for $\log w$. Then there is a constant M such that for $x \in [-1,1]$ we have*

$$|P_n(x)|^2 \leq Me^{M\sqrt{n|x|}}\, n \int_{-1}^{1} |P_n|^2 w \tag{3.2}$$

for any polynomials P_n of degree at most $n = 1,2,\ldots$.

Note however, that outside $[-1,1]$ (and close to 0) nothing more than $|P_n(z)| \leq M\exp(Mn|z|)$ (more precisely $|P_n(z)| \leq M\exp(Mn|\Im z|)$) can be said (just think of the classical Chebyshev polynomials with $w \equiv 1$).

Proof of Lemma 3.1 The following version of Lemma 3.1 was proved in [16, Lemma 3].

Lemma 3.2 *Let γ be a $C^{1+\alpha}$ $(\alpha > 0)$ smooth simple Jordan curve (a homeomorphic image of the unit circle) with arc length measure s_γ, $w \geq 0$ a (s_γ-measurable) function on γ such that $w, \log w \in L^1(s_\gamma)$, and let $\zeta_0 \in \gamma$ be a Lebesgue-point for $\log w$ (with respect to s_γ). Then there is a constant M such that for $z \in \gamma$ we have*

$$|Q_n(z)|^2 \leq Me^{M\sqrt{n|z-\zeta_0|}}\, n \int_\gamma |Q_n|^2 w\, ds_\gamma \tag{3.3}$$

for any polynomials Q_n of degree at most $n = 1,2,\ldots$.

We are going to apply this with $\gamma = C_1$, the unit circle. Let

$$Q_{2n}(z) = z^n P_n\Big(\frac{1}{2}\Big(z + \frac{1}{z}\Big)\Big).$$

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This is a polynomial of degree at most $2n$ such that $|Q_{2n}(e^{it})| = |P_n(\cos t)|$. Define on C_1 the weight $W(e^{it}) = w(\cos t)||\sin t|$, for which we have

$$\begin{aligned}\int_{C_1} |Q_{2n}(z)|^2 W(z)\,ds_{C_1}(z) &= \int_{-\pi}^{\pi} |Q_{2n}(e^{it})|^2 W(e^{it})dt\\ &= \int_{-\pi}^{\pi} |P_n(\cos t)|^2 w(\cos t)|\sin t|dt\\ &= 2\int_{-1}^{1} |P_n(x)|^2 w(x)dx.\end{aligned}$$

Under the map $e^{it} \to \cos t$, the point i is mapped into 0, and it is clear that $z_0 = i$ is a Lebesgue-point for $\log W$ (with respect to arc-measure on C_1). Hence we can apply (3.3) to Q_{2n} to get

$$|Q_{2n}(e^{it})|^2 \le M e^{M\sqrt{2n|e^{it}-i|}}\, n \int_{C_1} |Q_{2n}|^2 W\, ds_{C_1}.$$

Since for $t \in [0,\pi]$ we have $|e^{it} - i| \sim |\cos t|$, estimate (3.2) follows. ∎

We shall also use the following lemma on fast decreasing polynomials, which was proved in [16, Lemma 4].

Lemma 3.3 *Let K be a compact subset of the plane, Ω the unbounded component of its complement, and $Z \in \partial\Omega$ a point on the outer boundary of K. Assume that there is a disk in Ω that contains Z on its boundary. Then for every $\beta < 1$ there are constants $c_1, C_1 > 0$, and for every $n = 1, 2, \dots$ polynomials S_n of degree at most n, such that $S_n(Z) = 1$, $|P_n(z)| \le 1$ for $z \in K$ and*

$$|S_n(z)| \le C_1 e^{-c_1(n|z-Z|)^{\beta}}, \qquad z \in K.$$

(The constants C_1, c_1 depend on β.) We shall apply this lemma to a K, say bounded by a smooth Jordan curve, which contains the segment $[-2,2]$ on its boundary and contains all the segments $[-2,2] - i\rho$ with $0 < \rho \le 1$ in its interior. If $|z_0| \le A$, $\Im z_0 \ge 0$ and n is sufficiently large, then we shall set $Z = 0$, $\beta = 2/3$ in Lemma 3.3 and consider with the S_n from that lemma the polynomials $S_n^*(z) = S_n(z - z_0/n)$. For it we have $S_n^*(z_0/n) = 1$, and for $x \in [-1,1]$ (in which case $z := x - z_0/n$ lies in K)

$$|S_n^*(x)| \le C_1 e^{-c_1(n|x-z_0/n|)^{2/3}} \le C_1 e^{c_1|z_0|^{2/3}} e^{-c_1(n|x|)^{2/3}} \tag{3.4}$$

with some absolute constants $c_1, C_1 > 0$.

Now we are ready for the proof of (3.1).

Proof of (3.1) Recall that $d\mu(x) = w(x)dx$, $x \in [-1,1]$ and $w, \log w \in L^1[-1,1]$. We have to estimate $\lambda_n(z_0/n,\mu)$ from below for $|z_0| \le A$. Let P_n be a polynomial of degree at most n such that $P_n(z_0/n) = 1$ and

$$\lambda_n(z_0/n,\mu) = \int |P_n|^2 w.$$

If

$$\int |P_n|^2 w \ge \frac{1}{n}$$

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then there is nothing to prove, otherwise we obtain from Lemma 3.1 that

(3.5) $$|P_n(x)|^2 \le Me^{M\sqrt{n|x|}}, \qquad x \in [-1,1].$$

With the S_n^* from (3.4), let

(3.6) $$R_n(z) = P_n(z)S_n^*(z).$$

This has degree at most $2n$, it has value 1 at z_0/n, and we estimate its square integral with respect to w on $[-1,1]$ as follows.

The Lebesgue-point property of w at 0 means that for every $\varepsilon > 0$ there is a $\rho > 0$ such that if $0 \le \tau \le \rho$, then

(3.7) $$\int_{|\zeta|\le\tau} |w(\zeta) - w(0)|d\zeta \le \varepsilon\tau.$$

We define the measure ν as $d\nu(x) = w(0)dx$ on $[-1,1]$. We shall compare the values $\lambda_n(z_0/n,\mu)$ and $\lambda_{2n}(z_0/n,\nu)$ of the Christoffel functions associated with μ and ν, respectively. From that comparison (3.1) will follow using the following facts. Since the measure ν is just a constant multiple of the Lebesgue-measure, for it we have (see *e.g.*, [13, Theorem 1])

$$\lambda_n(x,\nu) \sim \frac{1}{n}$$

uniformly on $[-1/2,1/2]$, hence there is a constant C_0 such that

(3.8) $$\sum_{j=0}^{n} q_j(x)^2 \le C_0 n, \qquad x \in [-1/2,1/2],$$

where q_j denote the orthonormal polynomials with respect to ν (they are a constant multiple of the classical Legendre polynomials). Let $z_0 \in \mathbf{C}$ be arbitrary. There are constants $|\varepsilon_j| = 1$ such that

$$\sum_{j=0}^{n} |q_j(z_0/n)|^2 = \sum_{j=0}^{n} \varepsilon_j q_j(z_0/n)^2.$$

For the polynomial $Q(z) = \sum_{j=0}^{n} \varepsilon_j q_j(z)^2$ we then have $Q(z_0/n) = \sum_{j=0}^{n} |q_j(z_0/n)|^2$, and at the same time for all $x \in [-1/2,1/2]$ the inequality $|Q(x)| \le C_0 n$ holds (because of (3.8)). Therefore, by the Bernstein–Walsh lemma [18, p. 77] if

$$g(z) = \log\left|2z + \sqrt{(2z)^2 - 1}\right|$$

denotes the Green's function of $\overline{\mathbf{C}} \setminus [-1/2,1/2]$, then

$$\sum_{j=0}^{n} |q_j(z_0/n)|^2 = |Q(z_0/n)| \le e^{2ng(z_0/n)} C_0 n \le e^{C_2|z_0|} C_0 n,$$

where we used that g is Lip 1 in a neighborhood of the origin. This inequality proves

(3.9) $$\lambda_{2n}(z_0/n,\nu) \ge w(0)c_2 e^{-C_2|z_0|}/n$$

with some constants $c_2, C_2 > 0$ that are independent of n and z_0, and then, as we shall see, a similar inequality holds for $\lambda_n(z_0/n,\mu)$.

Clearly,

$$\lambda_{2n}(z_0/n,\nu) \le \int_{-1}^{1} |R_n|^2 w(0)$$

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with the polynomial R_n from (3.6), and we compare the right-hand side with the square integral of R_n against w. It follows from (3.4) and (3.5) that

$$|R_n(x)| \le \sqrt{M} C_1 e^{c_1|z_0|^{2/3}} \exp\big(M\sqrt{n|x|}/2 - c_1(n|x|)^{2/3} \big), \qquad x \in [-1,1],$$

and hence

$$|R_n(x)| \le M_1 e^{c_1|z_0|^{2/3}} \exp\big(-(c_1/2)(n|x|)^{2/3} \big), \qquad x \in [-1,1] \tag{3.10}$$

with some constant M_1.

It follows from (3.7) for $2^k/n < \rho/2$, $k = 1, 2, \dots$, that

$$\int_{2^k/n \le |x| \le 2^{k+1}/n} |R_n(x)|^2 |w(x) - w(0)| dx \le M_1^2 e^{2c_1|z_0|^{2/3}} \varepsilon \frac{2^{k+1}}{n} \exp\big(-(c_1/2) 2^{2k/3} \big),$$

and also

$$\int_{|x| \le 2/n} |R_n(x)|^2 |w(x) - w(0)| dx \le M_1^2 e^{2c_1|z_0|^{2/3}} \varepsilon \frac{2}{n}.$$

For the integral over $|x| \ge \rho/2$, we write (see (3.10))

$$\int_{\rho/2 \le |x| \le 1} |R_n(x)|^2 |w(x) - w(0)| dx \le C_3 M_1^2 e^{2c_1|z_0|^{2/3}} \exp\big(-(c_1/2)(n\rho/2)^{2/3} \big).$$

Summing these up we obtain

$$\int_{[-1,1]} |R_n|^2 d\nu - \int_{[-1,1]} |R_n|^2 d\mu \le C_4 M_1^2 e^{2c_1|z_0|^{2/3}} \frac{\varepsilon}{n} + o(1/n)$$

with a constant C_4 that depends only on w. Hence, in view of $|R_n(\zeta)| \le |P_n(\zeta)|$, it follows that

$$\lambda_{2n}(z_0/n, \nu) \le \lambda_n(z_0/n, \mu) + C_4 M_1^2 e^{2c_1|z_0|^{2/3}} \frac{\varepsilon}{n} + o(1/n).$$

Given A (recall that $|z_0| \le A$), choose $\varepsilon > 0$ so that

$$C_4 M_1^2 e^{2c_1|A|^{2/3}} \varepsilon \le \frac{w(0) c_2 e^{-C_2 A}}{4}$$

(*cf.* (3.9)), and then with this $\varepsilon > 0$ for sufficiently large n, say for $n \ge n_A$, we get from the previous estimate,

$$\lambda_{2n}(z_0/n, \nu) \le \lambda_n(z_0/n, \mu) + \frac{w(0) c_2 e^{-C_2 A}}{2n}.$$

This gives, in view of (3.9),

$$\lambda_n(z_0/n, \mu) \ge \frac{w(0) c_2 e^{-C_2|z_0|}}{2n},$$

and (3.1) has been verified. ∎

4 Proof of Theorem 1.3

We prove the theorem in the equivalent form

$$\lim_{n\to\infty} n\lambda_n(x_0 + a/n, \mu) = \frac{w(x_0)}{\omega_\Sigma(x_0)}.$$

We use the method of [16].

Without loss of generality we can assume that $x_0 = 0$ and the support of μ is contained in $[-1/2, 1/2]$.

We need to prove that, under the assumption that the point 0 is a Lebesgue-point for both μ and $\log w$, we have

$$\limsup_{n\to\infty} n\lambda_n(a/n, \mu) \le \frac{w(0)}{\omega_\Sigma(0)} \tag{4.1}$$

and

$$\liminf_{n\to\infty} n\lambda_n(a/n, \mu) \ge \frac{w(0)}{\omega_\Sigma(0)}. \tag{4.2}$$

Recall that the *Lebesgue-point property* of μ at 0 means that for every $\varepsilon > 0$ there is a $\rho > 0$ such that if $0 \le \tau \le \rho$, then (3.7) as well as

$$\mu_{\text{sing}}(\{x \mid |x| \le \tau\}) \le \varepsilon\tau \tag{4.3}$$

hold.

We define the measure ν as $d\nu(t) = w(0)dt$ in a small neighborhood of 0 and $\nu = \mu$ outside of that neighborhood. It easily follows from the localization theorem [12, Theorem 5.3.3] that ν is also in the **Reg** class with support equal to the support of μ. We shall compare the values $\lambda_n(a/n, \mu)$ and $\lambda_n(a/n, \nu)$ of the Christoffel functions associated with μ and ν, respectively. Since the density ν is constant ($= w(0)$) in a neighborhood of 0, in this neighborhood we have (see [13] and also [15, Section 8])

$$\lim_{n\to\infty} n\lambda_n(x, \nu) = \frac{w(0)}{\omega_\Sigma(0)} \tag{4.4}$$

locally uniformly (recall that Σ is the support of μ and ω_Σ is the density of the equilibrium measure of Σ). In particular,

$$\lim_{n\to\infty} n\lambda_n(a/n, \nu) = \frac{w(0)}{\omega_\Sigma(0)}$$

uniformly in $|a| \le A$ for any given $A > 0$.

We can assume that ρ in (3.7) and (4.3) is so small that in $[-\rho, \rho]$ we have $d\nu(x) = w(0)dx$.

Proof of (4.1) It follows from the proof of (4.4) in [15] that there are polynomials Q_n of degree at most n such that $Q_n(a/n) = 1$, $|Q_n(z)| \le 1$ for all $z \in \Sigma$ and

$$\lim_{n\to\infty} n \int |Q_n|^2 d\nu = \frac{w(0)}{\omega_\Sigma(0)}. \tag{4.5}$$

With $\beta = 2/3$ and some small $\delta > 0$, let $S_{\delta n}$ be the polynomials of degree δn from Lemma 3.3 for $K = [-1, 1]$ and $Z = 0$, and set $R_n(x) = Q_n(x)S_{\delta n}(x - a/n)$. This

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is a polynomial of degree at most $n(1+\delta)$ with $R_n(a/n) = 1$, $|R_n(x)| \le |Q_n(x)| \le 1$ $(x \in \Sigma)$, and this will be our test polynomial to get an upper bound for $\lambda_{n(1+\delta)}(a/n, \mu)$.

We estimate the integral of $|R_n|^2$ against μ using the Lebesgue-point properties (3.7) and (4.3). Since for fixed A and for $|a| \le A$

$$|R_n(t)| \le C_1 \exp\big(-c_1(n\delta|t - a/n|)^{2/3}\big) \le C_A \exp\big(-c_1(n\delta|t|)^{2/3}\big), \, t \in [-1/2, 1/2]$$

with some c_1, C_1, C_A (where C_A may depend on A), it follows for $2^k/n\delta < \rho/2$, $k = 1, 2, \dots$, that (see (3.7))

$$\int_{2^k/n\delta \le |t| \le 2^{k+1}/n\delta} |R_n(t)|^2 |w(t) - w(0)| dt \le C_A \varepsilon \frac{2^{k+1}}{n\delta} \exp\big(-c_1 2^{2k/3}\big)$$

and also

$$\int_{|t| \le 2/n\delta} |R_n(t)|^2 |w(t) - w(0)| dt \le \varepsilon \frac{2}{n\delta}.$$

On the other hand, for the integral over $|t| \ge \rho/2$, we write

$$\int_{\rho/2 \le |t|, \ t \in \Sigma} |R_n(t)|^2 |w(t) - w(0)| dt \le C \exp\big(-c_1(n\delta\rho/2)^{2/3}\big).$$

Summing these up we obtain

$$\int_\Sigma |R_n|^2 w - \int_\Sigma |R_n|^2 w(0) \le C \frac{\varepsilon}{\delta n} + o(1/n),$$

where C may depend on A but not on ε, δ, or n.

Similar reasoning based on (4.3) rather than (3.7) gives

$$\int_\Sigma |R_n|^2 d\mu_{\text{sing}} \le C \frac{\varepsilon}{\delta n} + o(1/n).$$

From these (as well as from the estimates leading to these inequalities) and from the fact that $\nu = \mu$ outside the interval where $d\nu(x) = w(0)dx$, we infer

$$\int |R_n|^2 d\mu - \int |R_n|^2 d\nu \le C \frac{\varepsilon}{\delta n} + o(1/n).$$

Hence, we obtain from (4.5)

$$\begin{aligned}\limsup_{n\to\infty} n(1+\delta)\lambda_{n(1+\delta)}(a/n, \mu) &\le \limsup_{n\to\infty} n(1+\delta) \int |R_n|^2 d\mu \\ &\le \limsup_{n\to\infty} n(1+\delta) \int |Q_n|^2 d\nu + C_2 \frac{\varepsilon}{\delta}(1+\delta) \\ &= (1+\delta)\frac{w(0)}{\omega_\Sigma(0)} + C_2 \frac{\varepsilon}{\delta}(1+\delta)\end{aligned}$$

with some constant C_2 that depends only on A. Then the monotonicity of λ_n in n implies that for the whole sequence of natural numbers

$$\limsup_{n\to\infty} n\lambda_n(a/n, \mu) \le (1+\delta)\frac{w(0)}{\omega_\Sigma(0)} + C_2 \frac{\varepsilon}{\delta}(1+\delta).$$

Letting $\varepsilon \to 0$ and then $\delta \to 0$, we obtain (4.1) ∎

Proof of (4.2) Assume again that $0 \in \Sigma$ is a Lebesgue-point for both μ (see (3.7), (4.3)) and $\log w$, and select ρ so that (3.7), (4.3) is true for all $\tau \le \rho$.

Assume to the contrary that there is an $\alpha < 1$ and an infinite sequence $\mathcal{N} \subseteq \mathbf{N}$ of the natural numbers such that for every $n \in \mathcal{N}$ there are polynomials Q_n of degree at most n with the properties $Q_n(a/n) = 1$ and

$$\int |Q_n|^2 d\mu \le \alpha \frac{w(0)}{\omega_\Sigma(0)} \frac{1}{n}. \tag{4.6}$$

In particular,

$$\int_\Sigma |Q_n|^2 w \le \alpha \frac{w(0)}{\omega_\Sigma(0)} \frac{1}{n}.$$

Let $\Delta > 0$ be such that $\log w \in L^1[-\Delta, \Delta]$. Recall that ν was equal to μ outside a small neighborhood of 0, and it does not matter what neighborhood we take, so we can assume that μ and ν coincide outside $[-\Delta, \Delta]$.

Lemma 3.1, transformed from $[-1,1]$ onto $[-\Delta, \Delta]$, gives

$$|Q_n(t)| \le M \exp(M\sqrt{n|t|}), \qquad t \in [-\Delta, \Delta],$$

with some constant M (recall that 0 is a Lebesgue-point of $\log w$, so Lemma 3.1 is applicable).

With $\beta = 2/3$ and some $\delta > 0$ consider again the polynomials $S_{\delta n}$ of degree δn from Lemma 3.3 for $K = [-1,1]$ and for the point $Z = 0$, and set

$$R_n(x) = Q_n(x) S_{\delta n}(x - a/n).$$

This is a polynomial of degree at most $n(1+\delta)$ with

$$R_n(a/n) = 1, \quad |R_n(t)| \le |Q_n(t)| \quad (t \in \Sigma),$$

and this will be our test polynomial to get an upper bound for $\lambda_{n(1+\delta)}(1, \nu)$, $n \in \mathcal{N}$. Since, as before,

$$|S_{\delta n}(t - a/n)| \le C_A \exp\big(-c_1(n\delta|t|)^{2/3}\big), \qquad t \in [-1/2, 1/2],\ |a| \le A,$$

it immediately follows that

$$|R_n(t)| \le MC_A \exp\big(M\sqrt{n|t|} - c_1(n\delta|t|)^{2/3}\big), \qquad t \in [-\Delta, \Delta],$$

and hence

$$|R_n(t)| \le C_A M_\delta \exp\big(-(c_1/2)(n\delta|t|)^{2/3}\big), \qquad t \in [-\Delta, \Delta] \tag{4.7}$$

with an M_δ depending on δ.

It follows from (3.7) and (4.7) for $2^k/n\delta < \rho/2 (\le \Delta)$, $k = 1, 2, \dots$, that

$$\int_{2^k/n\delta \le |t| \le 2^{k+1}/n\delta} |R_n(t)|^2 |w(t) - w(0)| dt \le C_A^2 M_\delta^2 \varepsilon \frac{2^{k+1}}{n\delta} \exp\big(-(c_1/2)2^{2k/3}\big),$$

and also

$$\int_{|t| \le 2/n\delta} |R_n(t)|^2 |w(t) - w(0)| dt \le C_A^2 M_\delta^2 \varepsilon \frac{2}{n\delta}.$$

For the integral over $\Delta \geq |t| \geq \rho/2$, we have

$$\int_{\rho/2\leq|t|\leq\Delta} |R_n(t)|^2|w(t)-w(0)|dt \leq C_A^2 C M_\delta^2 \exp\big(-(c_1/2)(n\delta\rho/2)^{2/3}\big),$$

where C is the integral of $|w(t)-w(0)|$ over $[-\Delta,\Delta]$. Summing these up we obtain

$$\int_{[-\Delta,\Delta]} |R_n|^2 d\nu - \int_{[-\Delta,\Delta]} |R_n|^2 w ds \leq C_A^2 C M_\delta^2 \frac{\varepsilon}{\delta n} + o(1/n).$$

These yield again (as $\nu = \mu$ outside $[-\Delta,\Delta]$)

$$\int |R_n|^2 d\nu \leq \int |R_n|^2 d\mu + C_A^2 C M_\delta^2 \frac{\varepsilon}{\delta n} + o(1/n).$$

Hence, in view of $|R_n(t)| \leq |Q_n(t)|$, it follows from (4.6)

$$\begin{aligned}\limsup_{n\in\mathcal{N}} n(1+\delta)\lambda_{n(1+\delta)}(a/n,\nu) &\leq \limsup_{n\in\mathcal{N}} n(1+\delta)\int |R_n|^2 d\nu\\ &\leq \limsup_{n\in\mathcal{N}} n(1+\delta)\int |R_n|^2 d\mu + C_A^2 C M_\delta^2 \frac{\varepsilon}{\delta}(1+\delta)\\ &\leq (1+\delta)\alpha\frac{w(0)}{\omega_\Sigma(0)} + C_A^2 C M_\delta^2 \frac{\varepsilon}{\delta}(1+\delta),\end{aligned}$$

and here C_A and C are independent of ε and δ. But for $(1+\delta)\alpha < 1$ (and we can make this happen by selecting a small δ) and small ε, this contradicts (4.4). This contradiction proves the lower estimate in (4.2) and the proof is complete. ∎

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