A PURELY ANALYTIC CRITERION FOR A DECOMPOSABLE OPERATOR

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In [**3**] E. Bishop introduced the notion of an operator with a "duality theory of type 3" and gave a certain sufficient condition for an operator to have a duality theory of type 3. In this note we show that in fact Bishop's sufficient condition implies that a given operator is decomposable [**4**]. Moreover, this condition characterizes a decomposable operator.

Throughout this paper X denotes a reflexive complex Banach space, and T denotes a bounded linear operator on X. According to Bishop [**3**], Definition 5, T has a duality theory of type 3 if for each open cover $\{G_1, G_2, \ldots, G_n\}$ of the complex plane $\mathbf{C}$ there are invariant subspaces $M_1, \ldots, M_n$ which span X such that $\sigma(T \mid M_i) \subset \bar{G}_i$ $(i = 1, \ldots, n)$. The above mentioned sufficient condition that T have a duality theory of type 3 is that T and its adjoint T' both have the following property β ([**3**], p. 394).

β. If $f_n : D \to X$ is a sequence of analytic functions such that $(\lambda - T)f_n(\lambda) \to 0$ uniformly on D, then $\{f_n\}$ is uniformly bounded on compact subsets of D.

Decomposable operators are due to Foias [**4**] and may be defined as follows. First, the operator T is said to have the single-valued extension property (SVEP) if zero is the only analytic function $f : D \to X$ for which $(\lambda - T)f(\lambda) = 0$ for all $\lambda \in D$. In this case the spectral manifold $X_T(F)$ is defined for $F \subset \mathbf{C}$ as the set of $x \in X$ such that $x = (\lambda - T)f(\lambda)$ for f analytic on $\mathbf{C} \backslash F$. Now T is said to be decomposable if it has the SVEP and for each cover $\{G_1, \ldots, G_n\}$ of $\mathbf{C}$ the manifolds $X_T(\bar{G}_i)$ are closed $(i = 1, \ldots, n)$ and $X = X_T(\bar{G}_1) + \ldots + X_T(\bar{G}_n)$. Moreover, $\sigma(T \mid X_T(\bar{G}_i)) \subset \bar{G}_i$ for each i. Thus a decomposable operator has a duality theory of type 3, but the converse is false [**1**] (at least on nonreflexive spaces).

To prove the desired result we require two lemmas.

LEMMA 1. *If T has property β, then T has the SVEP.*

Proof. Let $f : D \to X$ be analytic such that $(\lambda - T)f(\lambda) = 0$ for $\lambda \in D$. Put $f_n(\lambda) = nf(\lambda)$, $n = 1, 2, \ldots$, and note that for $\lambda \in D$ fixed $\|nf(\lambda)\| \leq R$ for $R > 0$ by β. Hence $f(\lambda) = 0$ and T has the SVEP.

By Lemma 1 the conclusion of the next lemma makes sense.

LEMMA 2. *Let X' be the dual space of X, and let T' denote the adjoint of T. If T and T' both have property β, then $X_T(G)^\perp = X'_{T'}(\mathbf{C} \backslash G)$ for each open set G. In particular, $X'_{T'}(F)$ (dually, $X_T(F)$) is closed for F closed.*

Proof. First let H, K be arbitrary disjoint sets in $\mathbf{C}$. For $x \in X_T(H)$ and $u \in X'_{T'}(K)$ it follows by a straightforward application of Liouville's theorem that $\langle x, u \rangle$ (evaluation of u at x) is 0. Hence for any $G \subset \mathbf{C}$, we obtain $X_T(G)^\perp \supset X'_{T'}(\mathbf{C} \backslash G)$.

We next prove the reverse inclusion. Let G be open, and let H and K be two open sets such that $\bar{H}\subset G$ and such that $\{G, K\}$ and $\{H, K\}$ separately cover $\mathbf{C}$. By [3], Theorems 3 and 4, p. 394, and Definition 3, p. 381, and the evident fact that $X_T(\bar{H})\subset X_T(G)$, we obtain the inclusions $X_T(G)^\perp\subset X_T(\bar{H})^\perp\subset X'_{T'}(\bar{K})$. Now let $\{K_j\}$ be a sequence of open sets such that $\{G, K_j\}$ covers $\mathbf{C}$ for $j=1, 2, \ldots$, and $\mathbf{C}\backslash G=\cap\bar{K}_j$. By the last inclusion $X_T(G)^\perp\subset\cap X'_{T'}(\bar{K}_j)=X'_{T'}(\cap\bar{K}_j)=X'_{T'}(\mathbf{C}\backslash G)$, since $X_T(\ \ )$ preserves intersections. Thus $X_T(G)^\perp=X'_{T'}(\mathbf{C}\backslash G)$.

THEOREM. *Let X be reflexive with dual X', and let T be an operator on X with adjoint T'. Then T is decomposable if and only if T and T' both have property β.*

Proof. Suppose T and T' both have property β. By a recent result of Radjabalipour [7], it is enough to prove that T is 2-decomposable, i.e. $X=X_T(\bar{G}_1)+X_T(\bar{G}_2)$ and $X_T(\bar{G}_i)$ are closed whenever G_1, G_2 cover $\mathbf{C}$. Let $\{G_1, G_2\}$ be such a cover, so that $H_i=\mathbf{C}\backslash\bar{G}_i$ $(i=1,2)$ have disjoint closures. By [2], Lemma 2.3, and Lemma 2, $X'_{T'}(\bar{H}_1)+X'_{T'}(\bar{H}_2)$ is a direct sum, hence $X'_{T'}(H_1)^-+X'_{T'}(H_2)^-$ is also direct. It is not hard to prove that $X'_{T'}(H_1)^\perp+X'_{T'}(H_2)^\perp=X$ (see [6, p. 1057]). By Lemma 2 (applied to T') $X=X_T(\bar{G}_1)+X_T(\bar{G}_2)$ and the latter are closed. Hence T is decomposable.

Conversely, let T be decomposable. Then T has property β by [5], and T' is 2-decomposable by [6], Theorem 2; hence T' also has property β by [5], final remark This completes the proof.

REFERENCES

1. E. Albrecht, An example of a weakly decomposable operator which is not decomposable *Rev. Roumaine Math. Pures Appl.* **20** (1975), 855–861.

2. C. Apostol, Roots of decomposable operator-valued analytic function, *Rev. Roumain Math. Pures Appl.* **13** (1968), 147–150.

3. E. Bishop, A duality theory for an arbitrary operator, *Pacific J. Math.* **9** (1959), 379–397.

4. C. Foias, Spectral maximal spaces and decomposable operators in Banach spaces, *Arch Math. (Basel)* **14** (1963), 341–349.

5. C. Foias, On the maximal spectral spaces of a decomposable operator, *Rev. Roumain Math. Pures. Appl.* **15** (1970), 1599–1606.

6. S. Frunza, A duality theorem for decomposable operators, *Rev. Roumaine Math. Pure Appl.* **16** (1971), 1055–1058.

7. M. Radjabalipour, Equivalence of decomposable and 2-decomposable operators, *Pacific J Math.*, **28** (1978), 243–247

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