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Theta Lifts of Tempered Representations for Dual Pairs $(\mathrm{Sp}_{2n}, O(V))$

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Abstract. This paper is the continuation of our previous work on the explicit determination of the structure of theta lifts for dual pairs $(\mathrm{Sp}_{2n}, O(V))$ over a non-archimedean field F of characteristic different than 2, where n is the split rank of Sp_{2n} and the dimension of the space V (over F) is even. We determine the structure of theta lifts of tempered representations in terms of theta lifts of representations in discrete series.

Introduction

This paper is the continuation of our previous work [11, 13–15] on the explicit determination of the structure of theta lifts for dual pairs $(\mathrm{Sp}_{2n}, O(V))$ over a non-archimedean field F of characteristic different than 2, where n is the split rank of Sp_{2n} and the dimension of the space V (over F) is even. In this paper we determine the structure of theta lifts of tempered representations in terms of theta lifts of representations in discrete series. Our approach follows that of [11, 13, 14] and is based on the Jacquet module technique of Bernstein–Zelevinsky and Tadić combined with Kudla's filtration of Jacquet modules of Weil representations. A different approach based on L-functions can be found in [15] for generic representations in discrete series. Now we describe our results more precisely.

Let F be a nonarchimedean field of characteristic different than 2. We look at usual towers of even-orthogonal or symplectic groups $G_n = G(V_n)$, $n \geq n_0$. (See Section 1 for the precise definition.) They are groups of isometries of F-spaces $(V_n, (\,\cdot\,))$, where $2n = \dim V_n$ and the form $(\,\cdot\,)$ is non-degenerate. Furthermore, it is skew-symmetric if the tower is symplectic and symmetric otherwise, and is built up from an anisotropic space V_{n_0}, $n_0 = 0, 1, 2$, adding $n - n_0$-hyperbolic planes. We fix one more tower of groups $G'_m = G(V'_m)$, $m \geq m_0$, that is of the form described, but that also satisfies the following: G'_m, $m \geq m_0$, are even-orthogonal groups if and only if G_n, $n \geq n_0$ are symplectic groups. Let χ_G be the character associated with the tower G_n, $n \geq n_0$ (see Section 1). It is trivial if the tower consists of symplectic groups and it is usual quadratic character of V_{n_0} if the tower consists of even-orthogonal groups. This is a convention that we follow in our papers [11, 14]. It helps to avoid the case by case analysis of [13].

The pair (see Definition 1.1) (G_n, G'_m) is a dual pair in the symplectic group $G(V_n \otimes V'_m)$ [6, 10]. We write $\omega_{n,m} = \omega^{\psi}_{n,m}$, for the smooth oscillator representation associated with that pair and a fixed non-trivial additive character ψ of F.

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For $\sigma \in \operatorname{Irr} G_n$, we write $\Theta(\sigma, m)$, for a smooth representation of G'_m, defined as a maximal σ-isotypic quotient of $\omega_{n,m}$ [10]

$$\sigma \otimes \Theta(\sigma, m) \simeq \omega_{n,m} / \bigcap_f \ker(f), \quad f \in \operatorname{Hom}_{G_n}(\omega_{n,m}|_{G_n}, \sigma).$$

$\Theta(\sigma, m)$ is a smooth representation of G'_m. More precisely, it is a zero or an admissible representation of finite length by [10, Théorème principal, p. 69]. (See [14, Corollary 3.1] for a different proof.) Let us write $m(\sigma)$ for the smallest $m \geq m_0$ such that $\Theta(\sigma, m) \neq 0$. It is called the first occurrence index of σ in the tower G'_m, $m \geq m_0$. We make the following definition (see Definition 3.6):

Definition HW Let $\sigma \in \operatorname{Irr} G_n$. We say that σ satisfies property HW if the following holds:

HW1 Every non-zero lift $\Theta(\sigma, m)$ has the unique maximal proper subrepresentation; we denote the corresponding quotient by $\sigma(m)$.

HW2 If $\Theta(\sigma, m-1) \neq 0$, then $\sigma(m) \hookrightarrow |\cdot|^{n-m+\eta_{G'}} \chi_G \rtimes \sigma(m-1)$. (Here and further $|\cdot|^{n-m+\eta_{G'}} \chi_G \rtimes \sigma(m-1)$ is a representation induced from a maximal parabolic subgroup in G'_m having the Levi $\mathrm{GL}(1, F) \times G'_{m-1}$.)

If the residue characteristic is different from 2, then σ satisfies the Howe duality conjecture, that is, HW1 holds (see [18]). (We do not use [18] in the present paper.) If σ is a supercuspidal representation, then it satisfies HW2 by a well-known result of Waldspurger [10, Théorème principal]). This explains the name of the property HW.

Now we explain the results on the lifts of representations in discrete series that we use in the present paper (see Section 4). First recall that in [11, Theorems 4.1, 4.2, 4.3] we prove that every representation of G_n, which is in discrete series satisfies HW and all its lifts and first occurrences (for various towers) can be explicitly computed in the classification of irreducible representations due to Mœglin, Tadić, Goldberg and Langlands [4, 8, 9]. On the other hand, we prove very general results on the structure of theta lifts of representations of G_n in discrete series [14]. Those proofs are based on a few simple properties of discrete series and on the validity of HW1 for representations in discrete series. (See Theorem 4.2 here for the results that we use from [14].) We remark that full HW for representations in discrete series follows from the validity of HW1 for representations in discrete series and results of [14]. This is verified in Theorem 4.2 of the present paper.

The first result that we use from [11] is that a representation $\sigma \in \operatorname{Irr} G_n$ in discrete series satisfies HW1 (see Theorem 4.1). Using this, the results of [14] (see Theorem 4.2), combined with that result (see Theorem 4.1), are enough to determine the structure of theta lifts of all tempered representations. Unfortunately, they are not enough to determine the first occurrence indices of tempered representations in all cases. We need one more result that follows from [11, Theorems 4.2, 4.3] directly. In the present form (see Theorem 6.6), it is just a reformulation of a deep result on the transfer of Jordan blocks of representations in discrete series (*cf.* [8]) under the theta correspondence obtained in [11, Theorems 4.2, 4.3].

Section 5 is devoted to the formulation of the main results of the present paper (see Theorems 5.1 and 5.2). They describe the structure and the first occurrence indices of the lifts $\Theta(\sigma, m)$ of tempered representations $\sigma \in \operatorname{Irr} G_n$ in the tower G'_m, $m \geq m_0$. Section 6 contains the proofs of the main results.

Our results depend on the work of Goldberg [4]. He stated his results in the characteristic zero, but this assumption is not necessary. In fact, all fundamental results of Harish-Chandra used there follow from [19] as it was explained to me by V. Heiermann.

1 Preliminaries

Let F be a nonarchimedean field of characteristic different than two. Let $|\cdot|$ be the (normalized as usual) absolute value of F. Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively.

Let G be an l-group [2]. Then by a representation of G we mean a pair (π, V), where V is a complex vector space and π is a homomorphism $G \to GL(V)$. We write V_∞ for the subspace of V consisting of all vectors in V having open stabilizer in G. Since G is an l-group, V_∞ is $\pi(G)$-invariant; we denote the resulting representation by (π_∞, V_∞). The representation (π, V) is smooth if $V = V_\infty$. We write $\mathcal{A}(G)$ for the category of all smooth complex representations of G. If (π, V) is a smooth representation, then we denote by $(\widetilde{\pi}, \widetilde{V})$ its smooth contragredient representation.

Let $P = MN$ be a closed subgroup of G, given as a semi-direct product of closed subgroups M and N, M normalizes N. Assume that N is a union of its open compact subgroups and G/P is compact. Then we have normalized induction and localization functors $\operatorname{Ind}_P^G\colon \mathcal{A}(M) \to \mathcal{A}(G)$ and $R_P\colon \mathcal{A}(G) \to \mathcal{A}(M)$. They are related by the Frobenius reciprocity:

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_P^G(\pi')) \simeq \operatorname{Hom}_M(R_P(\pi), \pi'),$$

$$\operatorname{Hom}_G(\operatorname{Ind}_P^G(\pi'), \pi) \simeq \operatorname{Hom}_M(\pi', \widetilde{R_P(\widetilde{\pi})})$$

(where π is an admissible representation).

Assume that G and G' are l-groups. Let V be a smooth representation of $G \times G'$. If $\rho \in \operatorname{Irr} G$ is an admissible representation, then we write $\Theta(\rho, V) \in \mathcal{A}(G')$ for the ρ-isotypic quotient of V (see [10, Ch. II, Lemme III.4]). More precisely, set $V' = \bigcap_f \ker(f)$, $f \in \operatorname{Hom}_G(V, \rho)$; then $V/V' \simeq \rho \otimes \Theta(\rho, V)$.

Next we shall describe the groups that we consider. We look at the usual towers of even-orthogonal or symplectic groups $G_n = G(V_n)$ that are groups of isometries of F-spaces $(V_n, (\cdot))$, $n \geq n_0$, where the form $(\cdot)$ is non-degenerate and is skew-symmetric if the tower is symplectic and symmetric otherwise.

The tower $(V_n, (\cdot))$, $n \geq n_0$, can be described explicitly as follows. We fix an anisotropic F-space $(V_{n_0}, (\cdot))$ of dimension $2n_0 = 0, 2, 4$. (This defines n_0.) In the case of orthogonal groups G_n, we let $\chi_G = \chi_{V_{n_0}}$ be the quadratic character of $F^\times$ associated with the quadratic space V_{n_0}. (See [5, (2.5), p. 240] or [6, Proposition 4.3].) If V_{n_0} is trivial or 4-dimensional space, then χ_G is the trivial character. In the case of symplectic groups G_n, we let χ_G be the trivial character.

Next, for any $n \in \mathbb{Z}_{\geq n_0}$, let V_n be the orthogonal direct sum of V_{n_0} with $r := n - n_0$

hyperbolic planes. We see $2n = \dim V_n$. We fix a Witt decomposition

$$V_n = V^{(1)} \oplus V_{n_0} \oplus V^{(2)}, \tag{1.1}$$

where $V^{(i)} = Fv_1^{(i)} \oplus \cdots \oplus Fv_r^{(i)}$, $i = 1, 2$, satisfying $(v_k^{(i)}, v_l^{(i)}) = 0$ and $(v_k^{(1)}, v_l^{(2)}) = \delta_{kl}$.

The decomposition (1.1) gives us the set of standard parabolic subgroups in G_n. We will describe maximal parabolic subgroups. For j, $1 \leq j \leq r$, let $V_j^{(i,n)} = Fv_{r-j+1}^{(i)} \oplus \cdots \oplus Fv_r^{(i)}$, $i = 1, 2$. Then we have the Witt decomposition

$$V_n = V_j^{(1,n)} \oplus V_{n-j} \oplus V_j^{(2,n)}.$$

Let P_j be the parabolic subgroup of G_n which stabilizes $V_j^{(1,n)}$. There is a Levi decomposition $P_j = M_j N_j$, where $M_j \simeq GL(V_j^{(1,n)}) \times G_{n-j}$. (Beware of the difference between this choice of a Levi factor and that of [5, p. 233]. There $\mathrm{GL}(V_j^{(2,n)})$ is considered instead of $\mathrm{GL}(V_j^{(1,n)})$.) Fix the isomorphism $\mathrm{GL}(j, F) \simeq \mathrm{GL}(V_j^{(1,n)})$ using the above fixed basis of $V_j^{(1,r)}$.

We end the discussion of classical groups by introducing more notation.

Definition 1.1 We fix the tower of groups $G'_m = G(V'_m)$, $m \geq m_0$, that is of the form described above but satisfying the following: G'_m, $m \geq m_0$, are even-orthogonal groups if and only if G_n, $n \geq n_0$, are symplectic groups. Throughout the paper we will write $\chi_{G'} = \chi_{V'_{m_0}}$, $\chi_G = \chi_{V_{n_0}}$ and

$$\eta_G = \begin{cases} 0 & \text{if } G_n \text{ is a symplectic group,} \\ 1 & \text{if } G_n \text{ is an even-orthogonal group.} \end{cases}$$

Similarly, we define $\eta_{G'}$.

Now we turn to the representation theory of classical groups. If $\pi \otimes \sigma$ is a smooth representation of $M_j \simeq \mathrm{GL}(V_j^{(1,n)}) \times G_{n-j}$, then we write $\pi \rtimes \sigma := \mathrm{Ind}_{P_j}^{G_n}(\pi \otimes \sigma)$, following Tadić.

Finally, in this paper, $\delta([|\det|^{-l_1}\rho, |\det|^{l_2}\rho])$ denotes the unique irreducible subrepresentation of the induced representation [20]:

$$|\det|^{l_2}\rho \times |\det|^{l_2-1}\rho \times \cdots \times |\det|^{-l_1}\rho,$$

where $\rho \in \mathrm{Irr}\,\mathrm{GL}(m_\rho, F)$ is a unitary supercuspidal representation and $l_1, l_2 \in \mathbb{R}$, $l_1 + l_2 \in \mathbb{Z}_{\geq 0}$.

2 Tempered Representations and Their Jacquet Modules

In this section we will collect some results on tempered representations. Also, we prove a fundamental, but very technical, result on tempered representations which we shall need later in the paper (see Theorem 2.7).

We start by recalling the following two results of Harish-Chandra in the connected case [19]. Mackey theory can be used to extend these results to the non-connected case that we need [7, 9].

Lemma 2.1 *Let $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$ be a tempered representation. Then there exist representations $\delta_1, \ldots, \delta_l, \sigma_d$ in discrete series such that $\sigma \hookrightarrow \delta_1 \times \cdots \times \delta_l \rtimes \sigma_d$. If $\delta_1', \ldots, \delta_{l'}', \sigma_d'$ is also a sequence of representations in discrete series such that $\sigma \hookrightarrow \delta_1' \times \cdots \times \delta_{l'}' \rtimes \sigma_d'$, then $l = l'$, $\sigma_d \simeq \sigma_d'$ and the sequence $\delta_1', \ldots, \delta_l'$ is obtained from the sequence $\delta_1, \ldots, \delta_k$ by permuting terms and replacing some of them with their contragredients. We call the multiset $\{\delta_1, \ldots, \delta_l, \widetilde{\delta}_1, \ldots, \widetilde{\delta}_l, \sigma_d\}$ the* tempered support *of σ.*

Lemma 2.2 *Assume that $\delta \in \operatorname{Irr} \mathrm{GL}(m_\delta, F)$ and $\sigma_d \in \operatorname{Irr} G_{n_d}$ are in discrete series. If $\delta \rtimes \sigma_d$ reduces, then $\widetilde{\delta} \simeq \delta$.*

Also, we need the following result of Goldberg. Mackey theory can be used to extend it to the non-connected case [7, 9].

Lemma 2.3 *Assume that $\delta_1, \ldots, \delta_l, \sigma_d$ is a sequence of representations in discrete series. Then the induced representation $\delta_1 \times \cdots \times \delta_l \rtimes \sigma_d$ is a direct sum of mutually non-equivalent tempered representations. It has a length 2^L where L is the number of mutually non-equivalent δ_i such that $\delta_i \rtimes \sigma_d$ reduces.*

We need the following result [12, Corollary 1.1].

Lemma 2.4 *Let $\delta \in \operatorname{Irr} \mathrm{GL}(m_\delta, F)$ be a discrete series and let $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$ be a tempered representation. Then we have the following.*

(i) *If δ appears in the tempered support of σ or $\delta \rtimes \sigma_d$ is irreducible, then $\delta \rtimes \sigma$ is irreducible. (See the notation introduced in Lemma 2.1.)*
(ii) *If δ does not appear in the tempered support of σ and $\delta \rtimes \sigma_d$ is reducible, then $\delta \rtimes \sigma$ is a direct sum of two mutually non-equivalent tempered representations.*

Lemma 2.5 *Assume that $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$ is a tempered representation, $k \in (1/2)\mathbb{Z}_{\geq 0}$, and $\rho \in \operatorname{Irr} \mathrm{GL}(m_\rho, F)$ is an irreducible unitary supercuspidal representation. Let $l \in \mathbb{Z}_{>0}$. If there exists an irreducible representation σ_1 such that*

$$\begin{aligned}\sigma \hookrightarrow |\det|^k\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \\ \times |\det|^k\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \times \cdots \\ \cdots \times |\det|^k\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \rtimes \sigma_1\end{aligned}$$

(where $|\det|^k\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho$ appears l-times), then σ_1 is tempered and

$$\sigma \hookrightarrow \delta([|\det|^{-k}\rho, |\det|^k\rho]) \times \cdots \times \delta([|\det|^{-k}\rho, |\det|^k\rho]) \rtimes \sigma_1 \quad (l\text{-factors}).$$

Proof If $l = 1$, this is [12, Lemma 1.3]. In general, the first part of the proof of [12, Lemma 1.3] shows the following claim: if $\sigma \hookrightarrow |\det|^k\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \rtimes \sigma_2$, for some smooth not necessarily irreducible representation σ_2, then

$\sigma \hookrightarrow \delta([|\det|^{-k}\rho, |\det|^{k}\rho]) \rtimes \sigma_2$. We apply this to

$$\begin{aligned}\sigma_2 = |\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \\ \times |\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \times \cdots \\ \cdots \times |\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \rtimes \sigma_1\end{aligned}$$

(where $|\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho$ appears $(l-1)$-times). We obtain

$$\begin{aligned}\sigma \hookrightarrow \delta([|\det|^{-k}\rho, |\det|^{k}\rho]) \times |\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \times \cdots \\ \cdots \times |\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \rtimes \sigma_1,\end{aligned}$$

where $|\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho$ appears $(l-1)$-times. Since

$$\delta([|\det|^{-k}\rho, |\det|^{k}\rho]) \times |\det|^{i}\rho \simeq |\det|^{i}\rho \times \delta([|\det|^{-k}\rho, |\det|^{k}\rho])$$

for $-k \le i \le k$, $k - i \in \mathbb{Z}$ (see [20]), we conclude

$$\begin{aligned}\sigma \hookrightarrow |\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \times \cdots \\ \cdots \times |\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \times \delta([|\det|^{-k}\rho, |\det|^{k}\rho]) \times \sigma_1.\end{aligned}$$

Now we repeat the same argument for

$$\begin{aligned}\sigma_2 = |\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \times \cdots \times |\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \\ \cdots \times |\det|^{-k}\rho \times \delta([|\det|^{-k}\rho, |\det|^{k}\rho]) \times \sigma_1\end{aligned}$$

(where $|\det|^{k}\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho$ appears $(l-2)$-times), etc. In this way, we obtain the embedding

$$\sigma \hookrightarrow \delta([|\det|^{-k}\rho, |\det|^{k}\rho]) \times \cdots \times \delta([|\det|^{-k}\rho, |\det|^{k}\rho]) \rtimes \sigma_1.$$

It remains to prove that σ_1 is tempered. Having established the embedding, we proceed as in the second part of the proof of [12, Lemma 1.3]. Appropriately modifying the notation, the only difference is that the second equivariant morphism in [12, (1-2)] is not just a "commutation" of a segment with $[|\det|^{-k}\rho, |\det|^{k}\rho]$, but with $[|\det|^{-k}\rho, |\det|^{k}\rho]$ repeated l-times. The argument is essentially the same. We leave it to the reader to make necessary modifications. ■

The following definition will be used later in the computation of the lifts of tempered representations.

Definition 2.6 Let $\Delta = [|\det|^{-k}\rho, |\det|^{k}\rho]$, where $\rho \in \operatorname{Irr} \mathrm{GL}(m_\rho, F)$ is a unitary supercupidal representation and $k \in \mathbb{Z}_{\geq 0}$. Let $l \in \mathbb{Z}_{>0}$. Then we define the representation $\delta(\Delta, l)$ as follows:

$$\delta(\Delta, l) := \delta(\Delta) \times \cdots \times \delta(\Delta) \quad (l \text{ factors}).$$

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Clearly, the representation $\delta(\Delta, l)$ is a non-degenerate tempered representation of $\mathrm{GL}(l \cdot (2k+1) \cdot m_\rho, F)$ [19, 20].

Now we prove the main result of this section.

Theorem 2.7 *Let $\Delta = [|\det|^{-k}\rho, |\det|^{k}\rho]$, where $\rho \in \mathrm{Irr}\,\mathrm{GL}(m_\rho, F)$ is a unitary supercupidal representation and $k \in \mathbb{Z}_{\geq 0}$. Let $l \in \mathbb{Z}_{>0}$. Let $\sigma_t \in \mathrm{Irr}\, G_n$ ($n \geq n_0$) be a tempered representation such that $\delta(\Delta)$ does not appear in its tempered support (see Lemma 2.1). If $\delta(\Delta, l) \otimes \sigma_t'$ is a subquotient of $R_{P_{l\cdot(2k+1)\cdot m_\rho}}(\delta(\Delta, l) \rtimes \sigma_t)$, for some irreducible representation σ_t', then $\sigma_t' \simeq \sigma_t$.*

Proof We let $\delta(\Delta, 0) = \mathbf{1}$, where $\mathbf{1}$ is the trivial representation of the trivial group $\mathrm{GL}(0, F)$. We define $\mathbf{1} \rtimes \sigma_t = \sigma_t$. Then the claim of the lemma trivially holds for $l = 0$. The lemma is proved by induction on l. It remains to show that if the claim of the lemma holds for $l-1$, then it also holds for l. First, $\delta(\Delta, l) \rtimes \sigma_t$ is a direct sum of (at most two) irreducible representations. Therefore, if $\delta(\Delta, l) \otimes \sigma_t'$ is a subquotient of $R_{P_{l\cdot(2k+1)\cdot m_\rho}}(\delta(\Delta, l) \rtimes \sigma_t)$, then there exists an irreducible subrepresentation

$$\sigma \hookrightarrow \delta(\Delta, l) \rtimes \sigma_t,$$

such that $\delta(\Delta, l) \otimes \sigma_t'$ is a subquotient of $R_{P_{l\cdot(2k+1)\cdot m_\rho}}(\sigma)$. Since $\delta(\Delta, l) \rtimes \sigma_t$ is completely reducible, there exists an irreducible subrepresentation $\sigma_1 \hookrightarrow \delta(\Delta, l-1) \rtimes \sigma_t$, such that $\sigma \hookrightarrow \delta(\Delta) \rtimes \sigma_1$. Therefore, $\delta(\Delta, l) \otimes \sigma_t'$ is a subquotient of $R_{P_{l\cdot(2k+1)\cdot m_\rho}}(\delta(\Delta) \rtimes \sigma_1)$. The analysis of this fact requires extensive computation of Jacquet modules. We start recalling Tadić's theory of Jacquet modules. Let $R(G_n)$ be the Grothendieck group of admissible representations of G_n of finite length. Let

$$R(G) = \bigoplus_{n \geq n_0} R(G_n).$$

We will write $\geq$ or $\leq$ for the natural order on $R(G)$. In greater detail, $\pi_1 \leq \pi_2$, $\pi_1, \pi_2 \in R(G)$, if and only if $\pi_2 - \pi_1$ is a linear combination of the irreducible representations with positive coefficients. Similarly, we define

$$R(\mathrm{GL}) = \bigoplus_{n \geq 0} R(\mathrm{GL}(n, F)).$$

Let $r_1 = n_1 - n_0$. Then for every standard maximal parabolic subgroup P_j of G_{n_1}, $1 \leq j \leq r_1$, we can identify $R_{P_j}(\sigma_1)$ with its semisimplification in $R(\mathrm{GL}(j, F)) \otimes R(G_{n_1-j})$. Thus, we can consider

$$\mu^*(\sigma_1) := \mathbf{1} \otimes \sigma_1 + \sum_{j=1}^{r_1} R_{P_j}(\sigma_1) \in R(\mathrm{GL}) \otimes R(G).$$

The first term in that expression should be σ_1, but in order to avoid exceptional cases in the analysis below, we write it as $\mathbf{1} \otimes \sigma_1$, let $\mathbf{1} \times \pi := \pi$ and $\pi \times \mathbf{1} := \pi$ for every smooth representation π of some $\mathrm{GL}(m_\pi, F)$, and set $\mathbf{1} \times \mathbf{1} = \mathbf{1}$. Finally, we let $\delta(\varnothing) = \mathbf{1}$.

Now we can decompose $\mu^*(\sigma_1) = \sum_{\delta',\sigma_1} \delta' \otimes \sigma'$ into irreducible constituents in $R(G)$. Now in our case, the basic result of Tadić is the following expression (see [9] and the references therein):

$$\mu^*\big(\delta([|\det|^{-k}\rho, |\det|^{k}\rho]) \rtimes \sigma_1\big) = \sum_{\delta',\sigma'} \sum_{i=0}^{2k+1} \sum_{j=0}^{i} \delta([|\det|^{i-k}\widetilde{\rho}, |\det|^{k}\widetilde{\rho}]) \times \delta([|\det|^{k+1-j}\rho, |\det|^{k}\rho]) \times \delta' \otimes \delta([|\det|^{k+1-i}\rho, |\det|^{k-j}\rho]) \rtimes \sigma', \tag{2.1}$$

Since $\delta(\Delta, l) \otimes \sigma_t'$ is a subquotient of $R_{P_{l\cdot(2k+1)\cdot m_\rho}}(\delta(\Delta) \rtimes \sigma_1)$, we obtain

$$\mu^*\left(\delta([|\det|^{-k}\rho, |\det|^{k}\rho]) \rtimes \sigma_1\right) \geq \delta(\Delta, l) \otimes \sigma_t'.$$

To analyze this we employ (2.1). Thus, there are indices i, j, $0 \leq j \leq i \leq 2k+1$, and an irreducible constituent of $\delta' \otimes \sigma' \leq \mu^*(\sigma_1)$, such that

$$\delta([|\det|^{i-k}\widetilde{\rho}, |\det|^{k}\widetilde{\rho}]) \times \delta([|\det|^{k+1-j}\rho, |\det|^{k}\rho]) \times \delta' \geq \delta(\Delta, l) \tag{2.2}$$

and

$$\delta([|\det|^{k+1-i}\rho, |\det|^{k-j}\rho]) \rtimes \sigma' \geq \sigma_t' \tag{2.3}$$

The inequality in (2.2) shows that δ' must be non-degenerate. Therefore it is fully induced from the representations of the form $\delta(\Delta')$ (see Ze). In fact, (2.2) implies

$$\begin{aligned} \delta' &\simeq \delta([|\det|^{-k}\widetilde{\rho}, |\det|^{i-k-1}\widetilde{\rho}]) \times \delta([|\det|^{-k}\rho, |\det|^{k-j}\rho]) \times \delta(\Delta, l') \\ &\simeq \delta([|\det|^{-k}\rho, |\det|^{k-j}\rho]) \times \delta([|\det|^{-k}\widetilde{\rho}, |\det|^{i-k-1}\widetilde{\rho}]) \times \delta(\Delta, l'), \end{aligned} \tag{2.4}$$

for some $l' \in \{l-2, l-1, l\}$. We claim that

$$i-k-1, k-j \in \{-k-1, k\}. \tag{2.5}$$

If for example $i-k-1 \notin \{-k-1, k\}$, then (2.4) and $\delta' \otimes \sigma' \leq \mu^*(\sigma)$ implies that

$$\sigma_1 \hookrightarrow |\det|^{-k}\widetilde{\rho} \times |\det|^{-k+1}\widetilde{\rho} \times \cdots \times |\det|^{i-k-1}\widetilde{\rho} \rtimes \sigma_1', \tag{2.6}$$

for some irreducible representation σ_1'. Since we have the following:

$$(-k)m_\rho + (-k+1)m_\rho + \cdots + (i-k-1)m_\rho < 0,$$

(2.6) violates the temperedness criterion for σ_1. Now we consider several cases according to (2.5):

- $i-k-1 = -k-1$. Hence $i = 0$. Then $0 \leq j \leq i$ implies $j = 0$. Now (2.4) implies $\delta' \simeq \delta(\Delta, l-1)$. Since $\delta' \otimes \sigma' \leq \mu^*(\sigma)$, the inductive assumption implies $\sigma' \simeq \sigma_t$. Now (2.3) implies $\sigma_t' \simeq \sigma_t$, completing the proof of the theorem.

Thus, according to (2.5), we may assume $i-k-1=k$ or $i=2k+1$. It remains to consider the next two cases.

- $k-j=-k-1$. Hence $j=2k+1$. This case can be analyzed in the same way as the previous one.
- $k-j=k$. Hence $j=0$. Since $i=2k+1$, (2.4) implies $\delta' \simeq \delta(\Delta, l)$. Since $\delta' \otimes \sigma' \leq \mu^*(\sigma)$, we obtain

$$\begin{aligned}\sigma_1 \hookrightarrow |\det|^k\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \\ \times |\det|^k\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \times \cdots \\ \cdots \times |\det|^k\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho \rtimes \sigma_1'\end{aligned}$$

(where $|\det|^k\rho \times |\det|^{k-1}\rho \times \cdots \times |\det|^{-k}\rho$ appears l-times), for some irreducible representation σ_1'. Applying Lemma 2.5, this contradicts the fact that the pair $(\delta(\Delta), \widetilde{\delta(\Delta)})$ occurs exactly $l-1$-times in the tempered support of σ_t (see (2.1)). This is a contradiction. ■

3 Some Preliminary Results on Theta Correspondence

In this section we review some results about Howe correspondence and fix the notation in the paper.

The pair (see Definition 1.1) (G_n, G'_m) is a dual pair in the symplectic group $G(V_n \otimes V'_m)$ (see [6, 10]). We write $\omega_{n,m} = \omega^{\psi}_{n,m}$ for the smooth oscillator representation associated with that pair and a fixed non-trivial additive character ψ of F.

For every $\sigma \in \operatorname{Irr} G_n$, we write $\Theta(\sigma, m)$ for a smooth representation of G'_m, defined as a maximal σ-isotypic quotient of $\omega_{n,m}$ (see [10, Ch. II, Lemme III.4])

$$\sigma \otimes \Theta(\sigma, m) \simeq \omega_{n,m} / \bigcap_f \ker(f), \quad f \in \operatorname{Hom}_{G_n}(\omega_{n,m}|_{G_n}, \sigma).$$

The basic result about the Howe correspondence is the following theorem [10, Théorème principal and Remarque, p. 67].

Theorem 3.1 *Let $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$. Then the following hold:*

(i) *There exists a non-negative integer m such that $\Theta(\sigma, m) \neq 0$. We denote the smallest m such that $\Theta(\sigma, m) \neq 0$ by $m(\sigma)$. Further, for $m \geq m(\sigma)$, we have $\Theta(\sigma, m) \neq 0$. We call $m(\sigma)$ the* first occurrence index *of σ in the tower G'_m, $m \geq m_0$.*

(i) *Assume that σ is a supercuspidal representation. Then $\Theta(\sigma, m(\sigma))$ is a supercuspidal irreducible representation, and for $m \geq m(\sigma)$, $\Theta(\sigma, m)$ is an irreducible subrepresentation of*

$$\chi_G|\cdot|^{n-m+1-\eta_G} \times \cdots \times \chi_G|\cdot|^{n-m(\sigma)-\eta_G} \rtimes \Theta(\sigma, m(\sigma)).$$

The Jacquet module $R_{P'_{m-m(\sigma)}}(\Theta(\sigma, m))$ is isomorphic to

$$\chi_G|\det|^{n-\frac{m+m(\sigma)-1}{2}-\eta_G} \otimes \Theta(\sigma, m(\sigma)).$$

The next theorem that we need gives Kudla's filtration of Jacquet modules of the oscillator representation [5].

Theorem 3.2 *Let P_k ($1 \le k \le n - n_0$) be the standard maximal parabolic subgroup of G_n ($n \ge n_0$). Then $R_{P_k}(\omega_{n,m})$ has a filtration of smooth $\mathrm{GL}(k,F) \times G_{n-k} \times G'_m$-representations $0 = J_{k+1} \subset J_k \subset \cdots \subset J_0 = R_{P_k}(\omega_{n,m})$, where $J_j/J_{j+1} \simeq J_{kj}$, $0 \le j \le k$, and*

$$J_{k0} = \chi_{G'} |\det|^{m-n+\frac{k-1}{2}+\eta_G} \otimes \omega_{n-k,m} \quad \textit{(quotient)}$$

$$J_{kj} = \mathrm{Ind}_{P_{kj} \times G_{n-k} \times P'_j}^{\mathrm{GL}(k,F)\times G_{n-k}\times G'_m} (\Psi_{kj} \otimes \Sigma_j \otimes \omega_{n-k,m-j}), \quad 0 < j < k, j \le m - m_0$$

$$J_{kk} = \mathrm{Ind}_{\mathrm{GL}(k,F)\times G_{n-k}\times P'_k}^{\mathrm{GL}(k,F)\times G_{n-k}\times G'_m} (\Sigma_k \otimes \omega_{n-k,m-k}), \quad k \le m - m_0$$

$$J_{kj} = 0, \quad 1 \le j \le k, j > m - m_0.$$

Here P_{kj} is the standard parabolic subgroup of $\mathrm{GL}(k,F)$ which corresponds to the partition $(k-j, j)$, $\Psi_{kj} = \chi_{G'} |\det|^{m-n+\frac{k-j-1}{2}+\eta_G}$ is a character of $\mathrm{GL}(k-j,F)$, and Σ_j is the twist of the standard representation of $\mathrm{GL}(j,F) \times \mathrm{GL}(j,F)$ on smooth locally constant compactly supported complex valued functions $C_c^\infty(\mathrm{GL}(j,F))$:

$$\Sigma_j(g_1, g_2) f(h) = |\det g_1|^{(-1)^{\eta_G}(\eta_G \cdot n + \eta_{G'} \cdot m - \eta_G \cdot k) - \frac{j+1}{2}} \times |\det g_2|^{(-1)^{\eta_{G'}}(\eta_G \cdot n + \eta_{G'} \cdot m - \eta_G \cdot k) + \frac{j+1}{2}} \chi_G(\det g_2) \chi_{G'}(\det g_2) f(g_1^{-1} h g_2).$$

(Here the first $\mathrm{GL}(j,F)$ (resp., the second) is a part of the Levi factor of P_{kj} (resp., Levi factor of P'_j).) Finally, the representation $\Psi_{kj} \otimes \Sigma_j \otimes \omega_{n-k,m-j}$ of $\mathrm{GL}(k-j,F) \times \mathrm{GL}(j,F) \times \mathrm{GL}(j,F) \times G_{n-k} \times G'_{m-j}$ is extended to a representation of $P_{kj} \times G_{n-k} \times P'_j$ trivial over the corresponding unipotent radicals.

In order to simplify formulation of many statements and to write formulae in a uniform way, we let $G_{n_0} = G_{n_0-j}$ and $G'_{m_0} = G'_{m_0-j}$ for $j \ge 0$. Next, for $n \ge n_0$ and $m \ge m_0$, we let $P_j = M_j = \mathrm{GL}(j,F) \times G_{n-j}$ and $N_j = \{1\}$, for $j > n - n_0$, $P'_j = M'_j = G'_{m-j}$ and $N'_j = \{1\}$, for $j > m - m_0$. Finally, we let $\omega_{n,m} = 0$ if $n < n_0$ or $m < m_0$.

Let $\sigma \in \mathrm{Irr}\, G_n$ ($n \ge n_0$). Then it is clear that $\Theta(\sigma, m) = 0$ if $m < m_0$, since $\Theta(\sigma, m)$ is a σ-isotypic component of $\omega_{n,m} = 0$. In particular, if $\Theta(\sigma, m) \ne 0$, then $m \ge m_0$.

Although, P'_j ,$j > m - m_0$, is not a subgroup of G'_m ($m \ge m_0$) we let

$$\mathrm{Ind}_{P_{kj}\times G_{n-k}\times P'_j}^{\mathrm{GL}(k,F)\times G_{n-k}\times G'_m} (\Psi_{kj} \otimes \Sigma_j \otimes \omega_{n-k,m-j}) = 0$$

$$\mathrm{Ind}_{\mathrm{GL}(k,F)\times G_{n-k}\times P'_k}^{\mathrm{GL}(k,F)\times G_{n-k}\times G'_m} (\Sigma_k \otimes \omega_{n-k,m-k}) = 0.$$

Now the formula for J_{kj} is the same in all cases $0 < j \le k$. This was used implicitly in [11] and it simplifies the exposition. The same convention is used in [14].

Now we recall some results from [11]. They follow from Theorem 3.2 using some considerations based on [3, 19].

Let us introduce some more notation. For $\sigma \in \operatorname{Irr} G_n$ ($n > n_0$) and a character μ of $\mathrm{GL}(1, F)$, we shall write $R_{P_1}(\sigma)(\mu)$ for the maximal μ-isotypic quotient of $R_{P_1}(\sigma)$.

Lemma 3.3 **([11, Lemma 5.1])** *Let $\sigma \in \operatorname{Irr} G_n$ ($n \geq n_0$). Assume that*

$$R_{P_1}(\sigma)(\chi_{G'}|\cdot|^{n-m+\eta_{G'}}) = 0.$$

Then we have the following:

(i) *$\Theta(\sigma, m-1) \neq 0$ if and only if $R_{P_1'}(\Theta(\sigma, m))(\chi_G|\cdot|^{n-m+\eta_{G'}}) \neq 0$.*

(ii) *Let $\sigma(m)$ be an irreducible quotient of $\Theta(\sigma, m)$ such that*

$$R_{P_1'}(\sigma(m))(\chi_G|\cdot|^{n-m+\eta_{G'}}) \neq 0.$$

Then there exists an irreducible quotient $\sigma(m-1)$ of $\Theta(\sigma, m-1)$ such that

$$\sigma(m) \hookrightarrow \chi_G|\cdot|^{n-m+\eta_{G'}} \rtimes \sigma(m-1).$$

The assumption of Lemma 3.3 holds if σ is in discrete series (resp., tempered) representation and $n + \eta_{G'} - m \leq 0$ (resp., $n + \eta_{G'} - m < 0$) applying the criterion for square-integrability (resp., temperedness). (See [17, p. 170] for the statement of criteria in the case of symplectic groups. In the case of full-even orthogonal groups we refer to [9, §16] for the criterion for square-integrability. The criterion for temperedness can be obtained similarly from [19, Proposition III.1.1].)

The following theorem refines ([M1], Corollary 5.1).

Theorem 3.4 *Assume that $\sigma \in \operatorname{Irr} G_n$ ($n \geq n_0$) is a tempered representation. Moreover, assume the following.*

(i) *There exists $m_t \geq n + \eta_{G'}$ such that $\Theta(\sigma, m_t)$ has the unique maximal proper subrepresentation; we denote the corresponding irreducible quotient by $\sigma(m_t)$. Assume that $\sigma(m_t)$ is tempered.*

(ii) *For every $m > m_t$ and every irreducible quotient $\sigma(m)$ of $\Theta(\sigma, m)$, we have*

$$R_{P_1'}(\sigma(m))(|\ |^{n-m+\eta_{G'}}\chi_G) \neq 0.$$

Then for every $m > m_t$, $\Theta(\sigma, m)$ has the unique maximal proper subrepresentation; we denote the corresponding irreducible quotient by $\sigma(m)$. In addition, $\sigma(m)$ is the unique irreducible (Langlands) subrepresentation of

$$\chi_G|\ |^{n-m+\eta_{G'}} \times \cdots \times \chi_G|\ |^{n-m_t-1+\eta_{G'}} \rtimes \sigma(m_t).$$

Proof The fact that $\Theta(\sigma, m)$, $m \geq m_t$, has a unique irreducible quotient, say $\sigma(m)$, and its realization as a Langlands subrepresentation follow easily by induction from Lemma 3.3. It remains to prove that $\Theta(\sigma, m)$ has the unique maximal proper subrepresentation. We prove this by induction on $m \geq m_t$. If $m = m_t$, this is our assumption (i). In general, we need to prove that the space $\mathrm{Hom}_{G'_m}(\Theta(\sigma, m), \sigma(m))$ is one dimensional. Since we have the following canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_{G'_m}(\Theta(\sigma, m), \sigma(m)) &\simeq \mathrm{Hom}_{G_n\times G'_m}(\sigma \otimes \Theta(\sigma, m), \sigma \otimes \sigma(m))\\ &\simeq \mathrm{Hom}_{G_n\times G'_m}(\omega_{n,m}, \sigma \otimes \sigma(m)),\end{aligned}$$

we need to show that the last space is one dimensional. If not, then using Lemma 3.3(ii), the space

$$\mathrm{Hom}_{G_n\times G'_m}(\omega_{n,m}, \sigma \otimes |\cdot|^{n-m+\eta_{G'}}\chi_G \rtimes \sigma(m-1))$$

is at least two dimensional. By Frobenius reciprocity, the same holds for

$$\mathrm{Hom}_{G_n\times \mathrm{GL}(1,F)\times G'_{m-1}}(R_{P'_1}(\omega_{n,m}), \sigma \otimes |\cdot|^{n-m+\eta_{G'}}\chi_G \otimes \sigma(m-1)).$$

It follows from Theorem 3.2 that we have the following filtration of $R_{P'_1}(\omega_{n,m})$ as a smooth $G_n \times \mathrm{GL}(1, F) \times G'_{m-1}$-representation:

$$J_{10} = \chi_G|\cdot|^{-m+n+\eta_{G'}} \otimes \omega_{n,m-1} \quad \text{(quotient)}$$

$$J_{11} = \mathrm{Ind}_{P_1\times \mathrm{GL}(1,F)\times G'_{m-1}}^{G_n\times \mathrm{GL}(1,F)\times G'_{m-1}}(\Sigma_1 \otimes \omega_{n-1,m-1}) \quad \text{(subrepresentation).}$$

Since $m > m_t \geq n+\eta_{G'}$, $\sigma \otimes |\cdot|^{n-m+\eta_{G'}}\chi_G \otimes \sigma(m-1)$ cannot be the quotient of J_{11} (otherwise, we have $R_{P_1}(\widetilde{\sigma})(\chi_{G'}|\cdot|^{n-m+\eta_{G'}}) \neq 0$, and this contradicts the temperedness criterion for $\widetilde{\sigma}$), we see that

$$\begin{aligned}&\mathrm{Hom}_{G_n\times \mathrm{GL}(1,F)\times G'_{m-1}}(J_{10}, \sigma \otimes |\cdot|^{n-m+\eta_{G'}}\chi_G \otimes \sigma(m-1))\\ &\qquad\simeq \mathrm{Hom}_{G_n\times G'_{m-1}}(\omega_{n,m-1}, \sigma \otimes \sigma(m-1))\\ &\qquad\simeq \mathrm{Hom}_{G_n\times G'_{m-1}}(\Theta(\sigma, m-1), \sigma(m-1))\end{aligned}$$

is at least two dimensional. This contradicts the induction hypothesis. ■

The following theorem is [14, Corollary 3.2]. Its proof is based on the second adjointness functor of Bernstein [1] and Theorem 3.2.

Theorem 3.5 *Let $\delta \in \mathrm{Irr}\,\mathrm{GL}(m_\delta, F)$ be an essentially square-integrable representation attached to the segment $\Delta = [\rho, |\det|^l\rho]$ (where $\rho \in \mathrm{Irr}\,\mathrm{GL}(m_\rho, F)$ is a supercuspidal representation; $l \in \mathbb{Z}_{\geq 0}$). Let $\sigma_1 \in \mathrm{Irr}\,G_{n-m_\delta}$ ($n - m_\delta \geq n_0$). Recall (§1) that $\Theta(\delta \otimes \sigma_1, R_{P_{m_\delta}}(\omega_{n,m}))$ is the maximal $\delta \otimes \sigma_1$-isotypic quotient of $R_{P_{m_\delta}}(\omega_{n,m})$. Then $\Theta(\delta \otimes \sigma_1, R_{P_{m_\delta}}(\omega_{n,m}))$ is a quotient of $\chi_G\chi_{G'}\widetilde{\delta} \rtimes \Theta(\sigma_1, m - m_\delta)$ unless $\rho = \chi_{G'}|\cdot|^{m-n+\eta_G} \in \mathrm{Irr}\,\mathrm{GL}(1, F)$, when we have the following filtration (of possibly zero) smooth G'_m-representations $0 \subset \Theta_0 \subset \Theta\left(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m})\right)$, where we have*

$$\delta([|\cdot|^{-m+n+1-\eta_G-m_\delta}\chi_G, |\cdot|^{-m+n-\eta_G}\chi_G]) \rtimes \Theta(\sigma_1, m - m_\delta) \twoheadrightarrow \Theta_0$$

and

$$\begin{aligned}&\delta([|\cdot|^{-m+n+1-\eta_G-m_\delta}\chi_G, |\cdot|^{-m+n-\eta_G-1}\chi_G]) \rtimes \Theta(\sigma_1, m - m_\delta + 1)\\ &\qquad\qquad\twoheadrightarrow \Theta(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m}))/\Theta_0.\end{aligned}$$

Note that when $m_\delta = 1$, the segment $[|\cdot|^{-m+n+1-\eta_G-m_\delta}\chi_G,\ |\cdot|^{-m+n-\eta_G-1}\chi_G]$ is an empty set and we omit it from the above formula. Thus, in this case the formula reads $\Theta(\sigma_1, m - m_\delta + 1) \twoheadrightarrow \Theta\left(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m})\right)/\Theta_0$.

The fundamental technical result used to compute the lifts of discrete series is [11, Lemma 5.2]. Its proof is based on the property of Jacquet modules of discrete series (see [11, Theorem 2.3, Definition 5.1(i)]) that fails to be true for general tempered representations. We make appropriate modifications here.

Definition 3.6 Let $\sigma \in \operatorname{Irr} G_n$ ($n \geq n_0$). We say that σ satisfies property HW if the following holds.

(i) Every non-zero lift $\Theta(\sigma, m)$ has the unique maximal proper subrepresentation; we denote the corresponding quotient by $\sigma(m)$.
(ii) If $\Theta(\sigma, m-1) \neq 0$, then $\sigma(m) \hookrightarrow |\cdot|^{n-m+\eta_{G'}}\chi_G \rtimes \sigma(m-1)$.

Example 3.7 Using the results of [11, 14], we see that if $\sigma \in \operatorname{Irr} G_n$ ($n \geq n_0$) is in discrete series, then it satisfies the property HW.

Now we are ready to state the main technical results for computing theta lifts of tempered representations.

Theorem 3.8 *Let $\sigma \in \operatorname{Irr} G_n$ ($n \geq n_0$) be a tempered representation. Assume the following.*

(i) *$\sigma \hookrightarrow \delta(\Delta, l) \rtimes \sigma_1$, where $\Delta = [|\cdot|^{-k}\rho, |\cdot|^{k}\rho]$ ($k \in \mathbb{Z}_{\geq 0}$, $\rho \in \operatorname{Irr}\operatorname{GL}(m_\rho, F)$ is a unitary and supercuspidal representation), $l \in \mathbb{Z}_{>0}$, and $\sigma_1 \in \operatorname{Irr} G_{n_1}$ is a tempered representation such that $\delta(\Delta)$ does not appear in its tempered support (see Lemma 2.1).*
(ii) *σ_1 satisfies HW.*

Then if m is such that $\Theta(\sigma, m) \neq 0$, and

$$(\rho, k) \notin \{(\chi_{G'}, m-n-\eta_{G'}), (\chi_{G'}, n-m-\eta_G)\},$$

then we have the following.

(a) *$\Theta(\sigma_1, m-n+n_1) \neq 0$.*
(b) *$\Theta(\sigma, m)$ has the unique maximal proper subrepresentation, and its irreducible quotient $\sigma(m)$ satisfies $\sigma(m) \hookrightarrow \chi_G\chi_{G'}\delta(\Delta, l) \rtimes \sigma_1(m-n+n_1)$.*

Proof This proof is similar to the proof of [11, Lemma 5.2]. We leave to the reader to formulate and prove an appropriate reformulation of [11, Remark 5.2] and then to prove Theorem 3.8 following the steps of the proof of [11, Lemma 5.2], where instead of assumptions (i) and (ii) of [11, Definition 5.1] one should use the following two:

- Theorem 2.7 instead of (i).
- $\dim_{\mathbb{C}} \operatorname{Hom}_{G_n}(\sigma, \delta(\Delta, l) \rtimes \sigma_1) = 1$ instead of (ii). This follows from the theory of R-groups (see Lemma 2.3). ∎

Theorem 3.9 *Let $\sigma \in \operatorname{Irr} G_n$ ($n \geq n_0$) be a tempered representation. Assume the following.*

(i) *σ is given by* (i) *and* (ii) *of Theorem* 3.8.
(ii) $\Theta(\sigma, m) \neq 0$ *and* $m > n + \eta_{G'}$.
(iii) $(\rho,\ k) \neq (\chi_{G'}, m - n - \eta_{G'})$.

(Conditions (ii) *and* (iii) *imply that* $\Theta(\sigma, m)$ *has a unique irreducible quotient. We write* $\sigma(m)$ *for that irreducible quotient.) Then we have the following.*

(a) *If* $m \neq n + k + \eta_{G'} + 1$ *or* $\rho \not\cong \chi_{G'}$*, then* $\Theta(\sigma_1, m - n + n_1 - 1) \neq 0$ *implies that* $\Theta(\sigma, m - 1) \neq 0$*. Moreover,* $\sigma(m) \hookrightarrow |\cdot|^{n-m+\eta_{G'}}\chi_G \rtimes \sigma(m-1)$*. (Note that* (iii) *holds for* $m - 1$ *and* $m - 1 \geq n + \eta_{G'}$ *implies that* $\Theta(\sigma, m-1)$ *has a unique maximal proper subrepresentation.)*

(b) *If* $m = n + k + \eta_{G'} + 1$ *and* $\rho = \chi_{G'}$*, then* $\Theta(\sigma_1, n_1 + k + \eta_{G'}) \neq 0$ *implies that one of the following hold:*

   (1) $\Theta(\sigma, n + k + \eta_{G'}) \neq 0$ *and* $\sigma(m) \hookrightarrow |\cdot|^{n-m+\eta_{G'}}\chi_G \rtimes \sigma(m-1)$*, for some irreducible quotient* $\sigma(m-1)$ *of* $\Theta(\sigma, m-1)$ *(*$m - 1 = n + k + \eta_{G'}$*).*

   (2) *(exceptional case)* $\sigma(m)$ *is a subrepresentation of*

$$\delta([\chi_G|\cdot|^{-k-1}, \chi_G|\cdot|^{k}]) \times \delta([\chi_G|\cdot|^{-k}, \chi_G|\cdot|^{k}], l-1) \rtimes \sigma_1(n_1 + k + \eta_{G'}),$$

   and, if $k > 0$*, then* $\Theta(\sigma_1, n_1 + k + \eta_{G'} - 1) = 0$*. (This case must hold if* $m(\sigma) = n + k + \eta_{G'} + 1$*.)*

Proof The proof of this is similar to the proof of [11, Proposition 5.1]. We leave the details to the reader. ■

4 Theta Lifts of Representations in Discrete Series

In this section we recall results from [11, 14]. We begin with the following comment. If the residue characteristic of F is different from two, then the Howe conjecture holds [18]. More precisely, let $\sigma \in \operatorname{Irr} G_n$. Then $\Theta(\sigma, m)$ is zero or it has the unique maximal proper subrepresentation; we denote the corresponding irreducible quotient by $\sigma(m)$. In general, the same is true if σ is in discrete series [11, Theorem 4.1], but using the classification of Mœglin and Tadić [8, 9] that is done under certain hypothesis. Thus, we will prove our results using the following theorem which is valid under hypothesis of [8, 9].

Theorem 4.1 *Let* $\sigma \in \operatorname{Irr} G_n$ *be a representation in discrete series. Then the lift* $\Theta(\sigma, m)$ *is zero or it has the unique maximal proper subrepresentation; we denote the corresponding irreducible quotient by* $\sigma(m)$*.*

The following theorem is proved in [14, Corollary 6.1, Theorems 6.1, 6.2].

Theorem 4.2 *Assume that* $\sigma \in \operatorname{Irr} G_n$ *(*$n \geq n_0$*) is a representation in discrete series. Let*

$$m_{\text{temp}}(\sigma) = \begin{cases} m(\sigma) & m(\sigma) > n + \eta_{G'}, \\ n + \eta_{G'} & m(\sigma) \leq n + \eta_{G'}. \end{cases}$$

Then we have the following.

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(i) *If m satisfies $m(\sigma) \leq m \leq m_{\text{temp}}(\sigma)$, then $\Theta(\sigma, m)$ is irreducible (hence, $\sigma(m) \simeq \Theta(\sigma, m)$). Moreover, $\sigma(m)$ is a tempered representation. More precisely, $\sigma(m)$ is in discrete series if one of the following holds:*

(a) $m < n + \eta_{G'}$,

(b) $m = m(\sigma) = n + \eta_{G'}$,

(c) $m = m(\sigma) > n + \eta_{G'}$ *and σ does not satisfy*

$$\sigma \hookrightarrow \delta([|\cdot|^{n-m+\eta_{G'}+1}\chi_{G'}, |\cdot|^{m-n-\eta_{G'}}\chi_{G'}]) \rtimes \sigma'',$$

for some representation $\sigma'' \in \operatorname{Irr} G_{n''}$.

If $m(\sigma) < n + \eta_{G'}$, then $\sigma(n + \eta_{G'}) \hookrightarrow \chi_G \rtimes \sigma(n + \eta_{G'} - 1)$.

(ii) *If m satisfies $m > m_{\text{temp}}(\sigma)$, then $\sigma(m)$ is a unique irreducible (Langlands) subrepresentation of*

$$|\cdot|^{n-m+\eta_{G'}}\chi_G \times |\cdot|^{n-m+\eta_{G'}+1}\chi_G \times \cdots \times |\cdot|^{n-m_{\text{temp}}(\sigma)-\eta_G}\chi_G \rtimes \sigma(m_{\text{temp}}(\sigma)).$$

(iii) *σ satisfies the property HW. (See Definition 3.6.)*

Proof Statements (i) and (ii) are proved under the assumption that the residue characteristic of F is different from 2 [14], but in that paper we use that assumption only to assure that Theorem 4.1 is valid. Now we prove (iii). Theorem 4.1 implies that σ satisfies Definition 3.6(i). It remains to prove σ satisfies Definition 3.6(ii).

If $m > n + \eta_{G'}$, the claim follows from (ii). If $m = n + \eta_{G'}$, the claim follows from the last part of (i). Assume that $m < n + \eta_{G'}$. Now since Theorem 3.2 implies that $R_{P'_1}(\omega_{n,m})$ has the quotient

$$\chi_G|\cdot|^{-m+n+\eta_{G'}} \otimes \omega_{n,m-1},$$

by the Frobenius reciprocity we see that there is a non-zero equivariant map

$$\Theta(\sigma, m) \to \chi_G|\cdot|^{-m+n+\eta_{G'}} \rtimes \Theta(\sigma, m-1).$$

Now the irreducibility of $\Theta(\sigma, m-1)$ and $\Theta(\sigma, m)$ complete the proof of (iii). ■

5 Theta Lifts of Tempered Representations

In this section we state the main results of this paper. Throughout the remainder of the paper we fix a tempered representation $\sigma \in \operatorname{Irr} G_n$ ($n \geq n_0$). Let

$$\{\delta_1, \ldots, \delta_l, \widetilde{\delta}_1, \ldots, \widetilde{\delta}_l, \sigma_d\}$$

be its tempered support (see Lemma 2.1). We define n_d by $\sigma_d \in \operatorname{Irr} G_{n_d}$. The first result is the following theorem.

Theorem 5.1 *Assume that $\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}]) \notin$ tempered support(σ), where k is defined as follows:*

$$k = \begin{cases} m(\sigma_d) - n_d - \eta_{G'} & (m(\sigma_d) \geq n + \eta_{G'}), \\ n_d - m(\sigma_d) - \eta_G & (m(\sigma_d) < n_d + \eta_{G'}). \end{cases}$$

Then we have the following.

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(i) *σ satisfies HW. (See Definition* 3.6.*)*
(ii) $m(\sigma) = m(\sigma_d) + n - n_d$.
(iii) *Let*

$$m_{\text{temp}}(\sigma) = \begin{cases} m(\sigma) & m(\sigma) > n + \eta_{G'}, \\ n + \eta_{G'} & m(\sigma) \le n + \eta_{G'}. \end{cases}$$

Then for $m(\sigma) \le m \le m_{\text{temp}}(\sigma)$, the lift $\Theta(\sigma, m)$ is irreducible and tempered, and its tempered support can be described as follows:

$$\text{tempered support}(\sigma(m)) = \{\chi_G\chi_{G'}\delta_1, \dots, \chi_G\chi_{G'}\delta_l,\ \chi_G\chi_{G'}\widetilde{\delta}_1, \dots, \chi_G\chi_{G'}\widetilde{\delta}_l\} + \text{tempered support}(\sigma_d(m)).$$

(+ means the union of two multisets.) If m satisfies $m > m_{\text{temp}}(\sigma)$, then $\sigma(m)$ is the Langlands subrepresentation of

$$|\cdot|^{n-m+\eta_{G'}}\chi_G \times |\cdot|^{n-m+\eta_{G'}+1}\chi_G \times \cdots \times |\cdot|^{n-m_{\text{temp}}(\sigma)-\eta_G}\chi_G \rtimes \sigma(m_{\text{temp}}(\sigma)).$$

Now we consider the remaining case for σ. So assume that

$$\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}]) \in \text{tempered support}(\sigma).$$

Then an application of the theory of R-groups (see Lemma 2.3) shows that there is a unique tempered representation $\sigma_t \in \operatorname{Irr} G_{n_t}$ (a set of representatives of equivalence classes of irreducible smooth representations of G_{n_t}) and $h \in \mathbb{Z}_{>0}$ such that the following holds:

(i) $\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}]) \notin \text{tempered support}(\sigma_t)$,
(ii) $\sigma \hookrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}]), h) \rtimes \sigma_t$.

Note that (i) implies that the lifts of σ_t are determined by Theorem 5.1. In particular, we have $m(\sigma_t) = m(\sigma_d) + n_t - n_d$. This implies

$$k = \begin{cases} m(\sigma_t) - n_t - \eta_{G'} & (m(\sigma_t) \ge n_t + \eta_{G'}), \\ n_t - m(\sigma_t) - \eta_G & (m(\sigma_t) < n_t + \eta_{G'}). \end{cases}$$

Theorem 5.2 *Maintaining the same assumptions, we have the following.*

(i) *Assume $m(\sigma_t) \ge n_t + \eta_{G'}$. Then $\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \rtimes \sigma_t$ is a direct sum of two non-equivalent tempered representations $\sigma_1(h)$ and $\sigma_2(h)$. The representation σ is equivalent to $\sigma_1(h)$ or $\sigma_2(h)$. Next, $\delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G]), h-1) \rtimes \sigma_t(n_t + k + \eta_{G'})$ and $\delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G]), h) \rtimes \sigma_t(n_t + k + \eta_{G'})$ are irreducible. The representations $\sigma_1(h)$ and $\sigma_2(h)$ can be distinguished by their first occurrences as follows.*

(a) *$m(\sigma_1(h)) = n + k + \eta_{G'} + 1$. Then $\sigma_1(h)$ satisfies HW. Moreover, for $m \ge n + k + \eta_{G'} + 1$, $\sigma(m)$ is the Langlands subrepresentation of*

$$|\cdot|^{n-m+\eta_{G'}}\chi_G \times |\cdot|^{n-m+\eta_{G'}+1}\chi_G \times \cdots \times |\cdot|^{-1}\chi_G \times |\det|^{-1/2}\delta([|\cdot|^{-k-1/2}\chi_G, |\cdot|^{k+1/2}\chi_G])) \times \delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G]), h-1) \rtimes \sigma_t(n_t + k + \eta_{G'}).$$

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(b) *$m(\sigma_2(h)) = n + k + \eta_{G'}$ ($k > 0$) and $m(\sigma_2(h)) = n - \eta_G$ ($k = 0$). Then $\Theta(\sigma_2(h), m)$ ($m \geq n + k + \eta_{G'}$) has the unique maximal proper subrepresentation, the corresponding quotient is denoted by $\sigma_2(h)(m)$, while for $k = 0$ all irreducible quotients of $\Theta(\sigma_2(h), n - \eta_G)$ are isomorphic to*

$$\delta([\chi_G], h - 1) \rtimes \sigma_t(n + \eta_{G'}).$$

If we let $\sigma_2(h)(n - \eta_G)$ be that induced representation, then HW2 holds. Next we have

$$\Theta(\sigma_2(h), n + k + \eta_{G'}) \simeq \delta([|\cdot|^{-k}\chi_G, |\cdot|^k\chi_G], h) \rtimes \sigma_t(n_t + k + \eta_{G'}),$$

and, for $m > n+k+\eta_{G'}$, we have that $\sigma(m)$ is the Langlands subrepresentation of

$$|\cdot|^{n-m+\eta_{G'}}\chi_G \times |\cdot|^{n-m+\eta_{G'}+1}\chi_G \times \cdots \times |\cdot|^{-k-1}\chi_G \rtimes \sigma_2(h)(n + k + \eta_{G'}).$$

(ii) *Assume $m(\sigma_t) < n_t + \eta_{G'}$. Then $\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \rtimes \sigma_t$ is irreducible and isomorphic to σ. Next, $\delta([|\,|^{-k}\chi_G, |\cdot|^k\chi_G]), h) \rtimes \sigma_t(n_t - k - \eta_G)$ is a direct sum of two non-isomorphic tempered representations. Exactly one of them is an irreducible quotient of $\Theta(\sigma, n-k-\eta_G)$. In particular, $m(\sigma) = n-k-\eta_G$. Next,* HW1 *holds for $m > n - k - \eta_G$, while* HW2 *holds for $m > n - k + \eta_{G'}$. $\Theta(\sigma, m)$ is irreducible and tempered for $n+\eta_{G'} \geq m > n-k-\eta_G$. Moreover, we have the following:*

$$\text{tempered support}(\sigma(m)) = \{2h \cdot \delta([|\cdot|^{-k}\chi_G, |\cdot|^k\chi_G])\} + \text{tempered support}(\sigma_t(m - n + n_t)).$$

If $m > n + \eta_{G'}$, then $\sigma(m)$ is the Langlands subrepresentation of

$$|\cdot|^{n-m+\eta_{G'}}\chi_G \times |\cdot|^{n-m+\eta_{G'}+1}\chi_G \times \cdots \times |\cdot|^{-1}\chi_G \rtimes \sigma(n + \eta_{G'}).$$

6 Proof of the Main Results

In this section we prove Theorems 5.1 and 5.2. We start with some general observations. Assume the following:

(i) $\sigma \hookrightarrow \delta(\Delta, l) \rtimes \sigma_1$, where $\Delta = [|\det|^{-k'}\rho, |\det|^{k'}\rho]$ ($k' \in \mathbb{Z}_{\geq 0}$ and $\rho \in \operatorname{Irr} GL(m_\rho, F)$ is a unitary and supercupidal representation), $l \in \mathbb{Z}_{>0}$, and $\sigma_1 \in \operatorname{Irr} G_{n_1}$ is a tempered representation such that $\delta(\Delta)$ does not appear in its tempered support (see Lemma 2.1).

(ii) σ_1 satisfies HW

We start with the following lemma.

Lemma 6.1 *Let $m \in \mathbb{Z}_{\geq m_0}$. If $(k', \rho) \neq (n-m-\eta_G, \chi_{G'})$, then we have the following:*

$$\chi_G\chi_{G'}\widetilde{\delta(\Delta, l)} \rtimes \Theta(\sigma_1, m - n + n_1) \twoheadrightarrow \Theta(\sigma, m).$$

Proof By (i) above, there are tempered representations $\sigma_i \in \operatorname{Irr} G_{n_i}$, $i = 2, \ldots, l+1$, such that $\sigma = \sigma_{l+1}$ and $\sigma_{i+1} \hookrightarrow \delta(\Delta) \rtimes \sigma_i$ $(i = 1, \ldots, l)$. By the Frobenius reciprocity, we obtain $R_{P_{n_{i+1}-n_i}}(\sigma_{i+1}) \twoheadrightarrow \delta(\Delta) \otimes \sigma_i$ $(i = 1, \ldots, l)$. Put $m_i = n_i - n + m$, $i = 1, \ldots, l+1$. Then $\omega_{n_{i+1}, m_{i+1}} \twoheadrightarrow \sigma_{i+1} \otimes \Theta(\sigma_{i+1}, m_{i+1})$ and the exactness of Jacquet functor imply

$$R_{P_{n_{i+1}-n_i}}(\omega_{n_{i+1}, m_{i+1}}) \twoheadrightarrow \delta(\Delta) \otimes \sigma_i \otimes \Theta(\sigma_{i+1}, m_{i+1}) \quad (i = 1, \ldots, l).$$

By the definition of an isotypic component, we have

$$\Theta(\delta(\Delta) \otimes \sigma_i, R_{P_{n_{i+1}-n_i}}(\omega_{n_{i+1}, m_{i+1}})) \twoheadrightarrow \Theta(\sigma_{i+1}, m_{i+1}). \tag{6.1}$$

Then our assumption implies

$$|\det|^{-k'} \rho \not\simeq |\det|^{m_{i+1} - n_{i+1} + \eta_G} \chi_{G'}. \tag{6.2}$$

Now (6.1), (6.2), and Theorem 3.5 imply

$$\chi_G \chi_{G'} \widetilde{\delta(\Delta)} \rtimes \Theta(\sigma_i, m_i) \twoheadrightarrow \Theta(\sigma_{i+1}, m_{i+1}), \quad (i = 1, \ldots, l).$$

The lemma follows by an easy induction. ∎

We continue by the following lemma.

Lemma 6.2 *If $(k', \rho) \neq (n_1 - m(\sigma_1) + \eta_{G'}, \chi_{G'})$, then we have $m(\sigma) \geq n - n_1 + m(\sigma_1)$.*

Proof Let $m = n - n_1 + m(\sigma_1) - 1$. If the claim of the lemma does not hold, then $\Theta(\sigma, m) \neq 0$. Then since $\eta_G + \eta_{G'} = 1$, we obtain

$$(k', \rho) \neq (n_1 - m(\sigma_1) + 1 - \eta_G, \chi_{G'}) = (n - m - \eta_G, \chi_{G'}).$$

Now Lemma 6.1 implies $\Theta(\sigma_1, m(\sigma_1) - 1) \neq 0$. This is a contradiction. ∎

Lemma 6.3 *Let $\rho = \chi_{G'}$. If $m(\sigma_1) > n_1 + k' + \eta_{G'}$, then $m(\sigma) = n - n_1 + m(\sigma_1)$. Moreover, σ satisfies* HW *and $\sigma(m(\sigma)) \hookrightarrow \chi_G \chi_{G'} \delta(\Delta, l) \rtimes \sigma_1(m(\sigma_1))$.*

Proof By Lemma 6.2, $m(\sigma) \geq n - n_1 + m(\sigma_1)$. Hence $m(\sigma) > n + k' + \eta_{G'}$. Applying Theorem 3.8, $m \geq m(\sigma)$, $\Theta(\sigma, m)$ has the unique maximal proper subrepresentation, and its irreducible quotient $\sigma(m)$ satisfies

$$\sigma(m) \hookrightarrow \chi_G \chi_{G'} \delta(\Delta, l) \rtimes \sigma_1(m - n + n_1).$$

In addition, the direct application of Lemma 3.3(ii) show that σ satisfies HW. Taking $m = m(\sigma)$, we obtain the displayed embedding. Finally, we need to compute $m(\sigma)$. This follows directly from Theorem 3.9. The exceptional case of type (b) does not show up. ∎

The next two lemmas have the similar proofs. The details are left to the reader.

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Lemma 6.4 *Let $\rho = \chi_{G'}$. Assume $n_1 + k' + \eta_{G'} > m(\sigma_1) \geq n_1 + \eta_{G'}$. Then $m(\sigma) = n - n_1 + m(\sigma_1)$. Moreover, σ satisfies* HW *and*

$$\sigma(m(\sigma)) \hookrightarrow \chi_G\chi_{G'}\delta(\Delta, l) \rtimes \sigma_1(m(\sigma_1)).$$

Lemma 6.5 *Let $\rho \not\simeq \chi_{G'}$. Assume $m(\sigma_1) \geq n_1 + \eta_{G'}$. Then $m(\sigma) = n - n_1 + m(\sigma_1)$. Moreover, σ satisfies* HW *and $\sigma(m(\sigma)) \hookrightarrow \chi_G\chi_{G'}\delta(\Delta, l) \rtimes \sigma_1(m(\sigma_1))$.*

Now we prove Theorem 5.1. First we assume that $m(\sigma_d) \geq n+\eta_{G'}$. Then Theorem 5.1(i) and (ii) are proved by the induction on the number of elements in the tempered support of σ. The base of this induction is the case $\sigma = \sigma_d$. Then Theorem 5.1(i) is Theorem 4.2(iii) and Theorem 5.1(ii) trivially holds. Assume that Theorem 5.1(i) and (ii) hold for σ_1. Then they hold for σ applying Lemmas 6.3, 6.4, and 6.5. Part (iii) related to the irreducibility of the lifts $\Theta(\sigma, m)$ ($m(\sigma) \leq m \leq m_{temp}(\sigma)$) and the temperedness of $\sigma(m)$ follows again by the same type of induction. If $\sigma = \sigma_d$, then the claims follow from Theorem 4.2. Assume that they hold for σ_1. Then the structure of tempered support of σ follows from the embedding displayed in Lemmas 6.3, 6.4, and 6.5. Also, for $m(\sigma) \leq m \leq m_{\text{temp}}(\sigma)$, by the inductive assumption $\sigma_1(m - n + n_1) \simeq \Theta(\sigma_1, m - n + n_1)$ is a tempered representation. Now Lemma 6.1 implies

$$\chi_G\chi_{G'}\widetilde{\delta(\Delta)} \rtimes \sigma_1(m - n + n_1) \twoheadrightarrow \Theta(\sigma, m).$$

Since the induced representation on the left-hand side is completely irreducible, we conclude that $\Theta(\sigma, m)$ is completely reducible. This implies its irreducibility since $\Theta(\sigma, m)$ has the unique irreducible quotient (by Theorem 5.1(i)). This completes the proof of Theorem 5.1 for $m(\sigma_d) \geq n + \eta_{G'}$.

Now we prove Theorem 5.1 when $m(\sigma_d) < n + \eta_{G'}$. This is done by induction on the number of elements in the tempered support of σ. If $\sigma = \sigma_d$, the claim follows from Theorem 4.2. Assume that the claim holds for σ_1 (see (i) and (ii) above). We show that the first occurrence is given by $m(\sigma) = m(\sigma_1) + n - n_1$. To accomplish this, we observe that because of the symmetry of Θ-correspondence we have proved the analogue result (the part of Theorem 5.1 already proved), also for lifting from the tower G'_m, $m \geq m_0$, to the tower G_n, $n \geq n_0$. In more detail, we look at the following induced representation:

$$\chi_G\chi_{G'}\widetilde{\delta(\Delta, l)} \rtimes \sigma_1(m(\sigma_1)) \quad \text{of } G'_{m(\sigma_1)+n-n_1}. \tag{6.3}$$

We have (see the notation introduced in Lemma 2.1) $(\sigma_1(m(\sigma_1)))_d = \sigma_d(m(\sigma_d))$ by the inductive assumption and Theorem 4.2. Now describe the first occurrence of $\sigma_d(m(\sigma_d))$. First by our assumption we have $n_d + \eta_{G'} > m(\sigma_d)$. Since $\eta_G + \eta_{G'} = 1$, we have

$$m(\sigma_d) + \eta_G \leq (n_d + \eta_{G'} - 1) + \eta_G = n_d. \tag{6.4}$$

Thus, since its lift to G_{n_d} is σ_d, a representation in discrete series, by Theorem 4.2 we obtain

$$m(\sigma_d(m(\sigma_d))) = n_d. \tag{6.5}$$

Also, we have the following

$$m(\sigma_d(m(\sigma_d))) - m(\sigma_d) - \eta_G = n_d - m(\sigma_d) - \eta_G \neq k. \tag{6.6}$$

Now (6.4), (6.5), and (6.6) show that the analogue result (a part of Theorem 5.1 already proved), except for lifting from the tower G'_m, $m \geq m_0$ to the tower G_n, $n \geq n_0$, can be applied to (tempered) subrepresentations of the induced representations in (6.3). Furthermore, by that part of Theorem 5.1, their full lifts to G_n are irreducible and are subrepresentations of

$$\delta(\Delta, l) \rtimes \sigma_1. \tag{6.7}$$

We would like to show that one of them lifts to σ. At this point we need the theory of R-groups. In more detail, Lemma 2.4 shows that the induced representation in (6.3) (resp., (6.7)) is reducible (and, consequently, the direct sum of two non-equivalent tempered representations) if and only if $\delta(\Delta) \rtimes \sigma_d(m(\sigma_d))$ (resp., $\delta(\Delta) \rtimes \sigma_d$) reduces. Now we apply the following result that follows from [11, Theorems 4.2, 4.3].

Theorem 6.6 *Let $\delta \in \operatorname{Irr} \mathrm{GL}(m_\delta, F)$ be a square-integrable representation. Let $\sigma \in \operatorname{Irr} G_n$ be in discrete series. Assume that $m(\sigma) < n + \eta_{G'}$. (Theorem 4.2 implies $\sigma(m)$ is in discrete series for $m(\sigma) \leq m < n + \eta_{G'}$.) Let $k = n - m - \eta_G$. (Note $k \geq 0$.) Then if $\delta \not\simeq \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}])$, then $\delta \rtimes \sigma$ is reducible if and only if $\chi_G\chi_{G'}\delta \rtimes \sigma(m)$ is reducible. If $\delta \simeq \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}])$, then $\delta \rtimes \sigma$ is irreducible, while $\chi_G\chi_{G'}\delta \rtimes \sigma(m)$ is reducible.*

We note that this theorem uses almost all parts of the classification of discrete series [8, 9]. It depends on the hypothesis of [8, 9]. We do not know how to prove this deep theorem directly.

Now the proof of other claims of Theorem 5.1, in the case $m(\sigma_d) < n_d + \eta_{G'}$, follows the same lines as the one for $m(\sigma_d) \geq n_d + \eta_{G'}$. We leave the details to the reader. For example HW2 can be established as in the case of discrete series. (See the proof of Theorem 4.2.)

Now we begin the proof of Theorem 5.2. We recall the assumption stated before Theorem 5.2 for σ. So assume that $\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}]) \in$ tempered support(σ). Then an application of the theory of R-groups (see Lemma 2.3), shows that there is the unique tempered representation $\sigma_t \in \operatorname{Irr} G_{n_t}$ and $h \in \mathbb{Z}_{>0}$ such that the following holds

(i) $\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}]) \notin$ tempered support(σ_t).
(ii) $\sigma \hookrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}]), h) \rtimes \sigma_t$.

Note that (i) implies that the lifts of σ_t are determined by Theorem 5.1. In particular, we have $m(\sigma_t) = m(\sigma_d) + n_t - n_d$. This implies

$$k = \begin{cases} m(\sigma_t) - n_t - \eta_{G'} & \text{if } m\sigma_t) \geq n_t + \eta_{G'}, \\ n_t - m(\sigma_t) - \eta_G & \text{if } m(\sigma_t) < n_t + \eta_{G'}. \end{cases} \tag{6.8}$$

We start with the following lemma.

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Lemma 6.7 *Let σ and h be as in Theorem* 5.2*, that is,* (i) *and* (ii) *above hold). Assume $m(\sigma_t) \geq n_t + \eta_{G'}$. (See* (6.8)*.) Then $\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \rtimes \sigma_t$ is a direct sum of two non-equivalent tempered representations $\sigma_1(h)$ and $\sigma_2(h)$ satisfying*

$$\sigma_i(h) \simeq \delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G], h) \rtimes \sigma_i(h-1), \quad h > 1, i = 1, 2.$$

Moreover, $\delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G], h) \rtimes \sigma_t(n_t + k + \eta_{G'})$ is irreducible

Proof Let $\tau_d = \sigma_d(n_t + k + \eta_{G'})$. Since it lifts to $\sigma_d \in \operatorname{Irr} G_{n_d}$, we have

$$m(\tau_d) \leq n_d < n_d + k + 1 = (n_d + k + \eta_{G'}) + \eta_G.$$

This shows that we can apply Theorem 6.6 to the lift of τ_d to G_n. Since

$$(n_d + k + \eta_{G'}) - n - \eta_{G'} = k,$$

we obtain that $\delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G]) \rtimes \tau_d$ is irreducible and $\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_G]) \rtimes \sigma_d$. Now Lemmas 2.3 and 2.4 complete the proof. ■

Lemma 6.8 *We maintain the assumptions of Lemma* 6.7*. We have the following:*

(i) *$m(\sigma) = n + k + \eta_{G'} + 1$ holds. Then σ satisfies* HW *and*

$$\sigma(n + k + \eta_{G'} + 1) \hookrightarrow \delta([|\cdot|^{-k-1}\chi_G, |\cdot|^{k}\chi_G]) \times \delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G], h-1) \rtimes \sigma_t(n_t + k + \eta_{G'}). \tag{6.9}$$

(Note that Lemma 2.4 *and Lemma* 6.7 *imply that $\delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G], h-1) \rtimes \sigma_t(n_t + k + \eta_{G'})$ is irreducible.)*

(ii) *$m(\sigma) \leq n + k + \eta_{G'}$ holds. Then $\Theta(\sigma, m)$ has the unique maximal proper subrepresentation and $\sigma(m) \hookrightarrow |\cdot|^{n-m+\eta_{G'}}\chi_G \rtimes \sigma(m-1)$ $(m \geq n + k + \eta_{G'} + 1)$, where, for $m = n + k + \eta_{G'} + 1$, $\sigma(m-1)$ is some irreducible quotient of $\Theta(\sigma, m-1)$. Moreover, every irreducible quotient $\sigma(n + k + \eta_{G'})$ of $\Theta(\sigma, n + k + \eta_{G'})$ satisfies*

$$\sigma(n + k + \eta_{G'}) \hookrightarrow \delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G], h) \rtimes \sigma_t(n_t + k + \eta_{G'}).$$

If $k > 0$, then $m(\sigma) = n + k + \eta_{G'}$.

Proof We argue as in Lemma 6.3. In particular, Theorem 3.9 shows $m(\sigma) \leq n + k + \eta_{G'} + 1$. For $m = n + k + \eta_{G'} + 1$, Theorem 3.9 (b) is applicable. If Theorem 3.9 (b)(1) holds, then we obtain (ii), where for the description of all irreducible quotients of $\Theta(\sigma, n + k + \eta_{G'})$ we use Lemma 6.1 and the fact that $\Theta(\sigma_t, m(\sigma_t))$ is irreducible and tempered. (If $k > 0$, then Lemma 6.2 implies that $m(\sigma) \geq n + k + \eta_{G'}$.) If Theorem 3.9(b)(1) holds, then we obtain all claims in (i) except the $m(\sigma) = n + k + \eta_{G'} + 1$. If we would have $m(\sigma) \leq n + k + \eta_{G'}$, then we obtain that (i) holds. In particular, all irreducible quotients of $\Theta(\sigma, n + k + \eta_{G'})$ are tempered. Now Theorem 3.4 implies $\sigma(n + k + \eta_{G'} + 1) \hookrightarrow |\cdot|^{-k-1}\chi_G \rtimes \sigma(n + k + \eta_{G'})$, for some irreducible quotient $\sigma(n + k + \eta_{G'})$ of $\Theta(\sigma, n + k + \eta_{G'})$ giving the realization of the same representation $\sigma(n + k + \eta_{G'} + 1)$ in the Langlands classification different than (6.9). This is a contradiction. ■

Lemma 6.9 *We maintain the assumptions of Lemma* 6.7*. If* $m(\sigma) \leq n+k+\eta_{G'}$*, then* $\Theta(\sigma, n+k+\eta_{G'})$ *is irreducible.*

Proof First Lemma 6.1 and the irreducibility of $\Theta(\sigma_t, n_t+k+\eta_{G'})$ imply

$$\widetilde{\delta}([|\cdot|^{-k}\chi_G, |\cdot|^k\chi_G], h) \rtimes \sigma_t(n_t+k+\eta_{G'}) \twoheadrightarrow \Theta(\sigma, n+k+\eta_{G'}).$$

Now Lemma 6.7 completes the proof. ■

Lemma 6.10 *We maintain the assumptions of Lemma* 6.7*. The representations* $\sigma_i(h)$*,* $i=1,2$*, can be distinguished as follows:*

When $m(\sigma_1(h)) = n+k+\eta_{G'}+1$*,*

(6.10)
$$\sigma_1(h)(n+k+\eta_{G'}+1) \hookrightarrow \delta([|\cdot|^{-k-1}\chi_G, |\cdot|^k\chi_G]) \times \delta([|\cdot|^{-k}\chi_G, |\cdot|^k\chi_G], h-1) \rtimes \sigma_t(n_t+k+\eta_{G'}),$$

When $m(\sigma_2(h)) \leq n+k+\eta_{G'}$*,*

$$\sigma_2(h)(n+k+\eta_{G'}) \simeq \delta([|\cdot|^{-k}\chi_G, |\cdot|^k\chi_G], h) \rtimes \sigma_t(n_t+k+\eta_{G'}), \tag{6.11}$$

Proof Let us remark that this follows immediately from Lemma 6.8 if the Howe duality holds [18], a result that we do not use directly in this paper.

Now we begin the proof. Using Lemma 6.8, it is enough to show that both representations $\sigma_i(h)$, $i=1,2$, cannot satisfy one of the following:

$$\begin{aligned} m(\sigma_1(h)) = m(\sigma_2(h)) &= n+k+\eta_{G'}+1; \\ m(\sigma_1(h)), m(\sigma_2(h)) &\leq n+k+\eta_{G'}. \end{aligned} \tag{6.12}$$

Assume that the first system of equalities in (6.12) holds. Then

$$\begin{aligned} \sigma_i(h)(n+k+\eta_{G'}+1) \hookrightarrow{} & \delta([|\cdot|^{-k-1}\chi_G, |\cdot|^k\chi_G]) \\ & \times \delta([|\cdot|^{-k}\chi_G, |\cdot|^k\chi_G], h-1) \rtimes \sigma_t(n_t+k+\eta_{G'}), \quad i=1,2. \end{aligned} \tag{6.13}$$

The Langlands classification implies that

$$\delta([|\cdot|^{-k-1}\chi_G, |\cdot|^k\chi_G]) \times \delta([|\cdot|^{-k}\chi_G, |\cdot|^k\chi_G], h-1) \rtimes \sigma_t(n_t+k+\eta_{G'}).$$

It has a unique irreducible subrepresentation which we denote by τ. Hence, (6.13) implies $\sigma_i(h)(n+k+\eta_{G'}+1) \simeq \tau$, $i=1,2$. Now Proposition 6.11(i) below implies

$$\omega_{n,n+k+\eta_{G'}+1} \twoheadrightarrow (\sigma_1(h) \oplus \sigma_2(h)) \otimes \tau. \tag{6.14}$$

Since $\sigma_i(h) \hookrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \rtimes \sigma_t$, the Frobenius reciprocity implies

$$R_{P_{h\cdot(2k+1)}}(\sigma_i(h)) \twoheadrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \otimes \sigma_t. \tag{6.15}$$

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Thus, applying $R_{P_{h\cdot(2k+1)}}(\cdot)$ to (6.14), we obtain

$$R_{P_{h\cdot(2k+1)}}(\omega_{n,n+k+\eta_{G'}+1}) \twoheadrightarrow (R_{P_{h\cdot(2k+1)}}(\sigma_1(h)) \oplus R_{P_{h\cdot(2k+1)}}(\sigma_i(h))) \otimes \tau.$$

Hence, using (6.15), we obtain

$$\varphi\colon R_{P_{h\cdot(2k+1)}}(\omega_{n,n+k+\eta_{G'}+1}) \twoheadrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \otimes \sigma_t \otimes (\tau \oplus \tau).$$

We show that φ induces an epimorphism

$$\text{(6.16)} \quad \operatorname{Ind}_{\mathrm{GL}(h\cdot(2k+1),F)\times G_{n_t}\times P'_{h\cdot(2k+1)}}^{\mathrm{GL}(h\cdot(2k+1),F)\times G_{n_t}\times G'_{n+k+\eta_{G'}+1}}(\Sigma_{h\cdot(2k+1)} \otimes \omega_{n_t,n_t+k+\eta_{G'}+1}) \twoheadrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \otimes \sigma_t \otimes (\tau \oplus \tau).$$

To show (6.16) we use the filtration of $R_{P_{h\cdot(2k+1)}}(\omega_{n,n+k+\eta_{G'}+1})$ given by Theorem 3.2:

$$0 = J_{h\cdot(2k+1)+1} \subset J_{h\cdot(2k+1)} \subset \cdots \subset J_0 = R_{P_{h\cdot(2k+1)}}(\omega_{n,n+k+\eta_{G'}+1}).$$

It is enough to show that the restriction $\varphi|_{J_{h\cdot(2k+1)}}$ is an epimorphism. If not, then the cokernel of $\varphi|_{J_{h\cdot(2k+1)}}$ must have a quotient isomorphic to $\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \otimes \sigma_t \otimes \tau$. We write p for the composition of quotient maps

$$\begin{aligned}\delta([|\cdot|^{-k}\chi_{G'},|\cdot|^k\chi_{G'}], h) &\otimes \sigma_t \otimes (\tau \oplus \tau)\\ &\twoheadrightarrow \big(\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \otimes \sigma_t \otimes (\tau \oplus \tau)\big) \,/\, \operatorname{Im}\varphi|_{J_{h\cdot(2k+1)}}\\ &\twoheadrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \otimes \sigma_t \otimes \tau.\end{aligned}$$

Let $\psi = p \circ \varphi$. Then

$$\psi\colon R_{P_{h\cdot(2k+1)}}(\omega_{n,n+k+\eta_{G'}+1}) \twoheadrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \otimes \sigma_t \otimes \tau, \psi|_{J_{h\cdot(2k+1)}} = 0.$$

Now let $0 \le j < h\cdot(2k+1)$ be defined by $\psi|_{J_{j+1}} = 0$ and $\psi|_{J_j} \neq 0$. Then ψ induces an epimorphism $J_j/J_{j+1} \twoheadrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \otimes \sigma_t \otimes \tau$. Now as in the proof of [11, Remark 5.1, p. 121], the Frobenius reciprocity and the fact that $\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h)$ is non-degenerate [20] imply that $j = h\cdot(2k+1) - 1$ and there exists an irreducible representation δ' (perhaps, the trivial representation **1** of $\mathrm{GL}(0, F)$, see the conventions introduced in the proof of Theorem 2.7) such that the following holds:

$$\Psi_{h\cdot(2k+1)h\cdot(2k+1)-1} \times \delta' \twoheadrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h),$$

where, in our case, we have $\Psi_{h\cdot(2k+1)h\cdot(2k+1)-1} = \chi_{G'}|\;|^{k+2}$. Taking contragredients, we obtain the following:

$$\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \hookrightarrow \chi_{G'}|\cdot|^{-k-2} \times \widetilde{\delta}'.$$

Now the Frobenius reciprocity shows that the Jacquet module of

$$\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h)$$

with respect to appropriate maximal parabolic subgroup contains $\chi_{G'}|\cdot|^{-k-2} \otimes \widetilde{\delta}'$. On the other hand, using [20], it is easy to compute all Jacquet modules of

$$\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h),$$

showing that such a term cannot exist. This is a contradiction, which proves (6.16).

Hence, applying the Frobenius reciprocity to (6.16), we obtain a non-zero equivariant map:

$$\Sigma_{h\cdot(2k+1)} \otimes \omega_{n_t, n_t+k+\eta_{G'}+1} \longrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \otimes \sigma_t \otimes (\widetilde{R}_{P'_{h\cdot(2k+1)}}(\widetilde{\tau}) \oplus \widetilde{R}_{P'_{h\cdot(2k+1)}}(\widetilde{\tau})).$$

By Proposition 6.11(ii), it descends to the equivariant map

$$\Sigma_{h\cdot(2k+1)} \otimes \sigma_t \otimes \Theta(\sigma_t, n_t + k + \eta_{G'} + 1) \longrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \otimes \sigma_t \otimes (\widetilde{R}_{P'_{h\cdot(2k+1)}}(\widetilde{\tau}) \oplus \widetilde{R}_{P'_{h\cdot(2k+1)}}(\widetilde{\tau})).$$

Further, by Proposition 6.11(ii), it descends to the equivariant map

$$(6.17)\quad \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \otimes \Theta(\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h), \Sigma_{h\cdot(2k+1)}) \otimes \sigma_t \otimes \Theta(\sigma_t, n_t + k + \eta_{G'} + 1) \longrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \otimes \sigma_t \otimes (\widetilde{R}_{P'_{h\cdot(2k+1)}}(\widetilde{\tau}) \oplus \widetilde{R}_{P'_{h\cdot(2k+1)}}(\widetilde{\tau})).$$

Since

$$\Theta(\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h), \Sigma_{h\cdot(2k+1)}) \simeq \delta([|\cdot|^{-k}\chi_{G}, |\cdot|^{k}\chi_{G}], h),$$

(6.17) implies

$$(6.18)\quad \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \otimes \delta([|\cdot|^{-k}\chi_{G}, |\cdot|^{k}\chi_{G}], h) \otimes \sigma_t \otimes \Theta(\sigma_t, n_t + k + \eta_{G'} + 1) \longrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \otimes \sigma_t \otimes (\widetilde{R}_{P'_{h\cdot(2k+1)}}(\widetilde{\tau}) \oplus \widetilde{R}_{P'_{h\cdot(2k+1)}}(\widetilde{\tau})).$$

Composing the equivariant map in (6.18) with projections, corresponding to the direct sum, we obtain two non-zero equivariant maps. Their images must be of the form

$$\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \otimes \sigma_t \otimes \delta([|\cdot|^{-k}\chi_{G}, |\cdot|^{k}\chi_{G}], h) \otimes \tau_i$$

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for the i-th component. Thus, Proposition 6.11(iii) results in an epimorphism:

$$\Theta(\sigma_t, n_t + k + \eta_{G'} + 1) \twoheadrightarrow \tau_1 \oplus \tau_2.$$

This contradicts HW for σ_t. This shows that the first line in (6.12) cannot hold. Similarly, we can prove that the second one cannot hold; we sketch a short proof, which we believe is of some interest. If the second line in (6.12) holds, then Lemma 6.7 and Lemma 6.8(ii) imply

$$\sigma_i(h)(n + k + \eta_{G'}) \simeq \delta([|\cdot|^{-k}\chi_G, |\cdot|^k\chi_G], h) \rtimes \sigma_t(n_t + k + \eta_{G'}) =: \tau, \quad i = 1, 2.$$

Now Proposition 6.11(i) implies

$$\omega_{n,n+k+\eta_{G'}+1} \twoheadrightarrow (\sigma_1(h) \oplus \sigma_2(h)) \otimes \tau. \tag{6.19}$$

Applying $R_{P_{h\cdot(2k+1)}}(\,\cdot\,)$ to (6.19), we obtain

$$R_{P_{h\cdot(2k+1)}}(\omega_{n,n+k+\eta_{G'}+1}) \twoheadrightarrow (R_{P_{h\cdot(2k+1)}}(\sigma_1(h)) \oplus R_{P_{h\cdot(2k+1)}}(\sigma_i(h))) \otimes \tau.$$

Hence, using (6.15), we obtain

$$R_{P_{h\cdot(2k+1)}}(\omega_{n,n+k+\eta_{G'}}) \twoheadrightarrow \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \otimes \sigma_t \otimes (\tau \oplus \tau).$$

As in (6.16), we use Theorem 3.2 to show that the above epimorphism descends to

$$\mathrm{Ind}_{\mathrm{GL}(h\cdot(2k+1),F)\times G_{n_t}\times P'_{h\cdot(2k+1)}}^{\mathrm{GL}(h\cdot(2k+1),F)\times G_{n_t}\times G'_{n+k+\eta_{G'}}}(\Sigma_{h\cdot(2k+1)} \otimes \omega_{n_t,n_t+k+\eta_{G'}}) \twoheadrightarrow \\ \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^k\chi_{G'}], h) \otimes \sigma_t \otimes (\tau \oplus \tau).$$

Now [14, Lemma 3.3] implies

$$\delta([|\cdot|^{-k}\chi_G, |\cdot|^k\chi_G], h) \rtimes \Theta(\sigma_t, n_t + k + \eta_{G'}) \twoheadrightarrow \tau \oplus \tau.$$

This is a contradiction, since $\sigma_t(n_t + k + \eta_{G'}) \simeq \Theta(\sigma_t, n_t + k + \eta_{G'})$ and

$$\delta([|\cdot|^{-k}\chi_G, |\cdot|^k\chi_G], h) \rtimes \sigma_t(n_t + k + \eta_{G'})$$

is irreducible. ∎

The next proposition collects some general results used in the proof of the previous lemma.

Proposition 6.11 (i) *Let G be an l-group. Assume that (π, V) is a smooth representation of G and (ρ_i, V_i), $i = 1, 2$, two non-equivalent irreducible admissible representations of G. If there are non-zero equivariant maps $V \to V_i$, $i = 1, 2$, then their direct sum is an equivariant epimorphism $V \to V_1 \oplus V_2$.*

(ii) *Assume that G_i, H_i, $i = 1, 2$, are l-groups. Assume that (π_i, V_i) is a smooth representation of $G_i \times H_i$ and (ρ_i, W_i) is an irreducible admissible representation of G_i, $i = 1, 2$. Then we consider a smooth representation $V_1 \otimes V_2$ of $G_1 \times H_1 \times G_2 \times H_2$. If (σ, W) is an admissible representation of $H_1 \times H_2$ of finite length and if we have a non-zero $G_1 \times H_1 \times G_2 \times H_2$-equivariant map $V_1 \otimes V_2 \to W_1 \otimes W_2 \otimes W$, then it factors through the cannonical map $V_1 \otimes V_2 \to W_1 \otimes \Theta(W_1, V_1) \otimes V_2$, to $G_1 \times H_1 \times G_2 \times H_2$-equivariant map*

$$W_1 \otimes \Theta(W_1, V_1) \otimes V_2 \to W_1 \otimes W_2 \otimes W.$$

Finally, this map factors through the canonical map

$$W_1 \otimes \Theta(W_1, V_1) \otimes V_2 \to W_1 \otimes \Theta(W_1, V_1) \otimes W_2 \otimes \Theta(W_2, V_2),$$

resulting in a non-zero $G_1 \times H_1 \times G_2 \times H_2$-equivariant map

$$W_1 \otimes \Theta(W_1, V_1) \otimes W_2 \otimes \Theta(W_2, V_2) \to W_1 \otimes W_2 \otimes W.$$

Moreover, we have the following isomorphism of smooth representations of $H_1 \times H_2$:

$$\Theta(W_1 \otimes W_2, V_1 \otimes V_2) \simeq \Theta(W_1, V_1) \otimes \Theta(W_2, V_2).$$

(iii) *Assume that G, H are l-groups. Let (ρ, U) be an irreducible admissible representation of G. Assume that (π, V), (σ, W) are smooth representations of H. If we have a $G \times H$-equivariant epimorphism $U \otimes V \twoheadrightarrow U \otimes W$, then there is an H-equivariant epimorphism $V \twoheadrightarrow W$.*

Proof (i) is elementary. It is obvious that the equivariant map in (ii) factors through the canonical maps. Then the last isomorphism follows by definition of an isotypic component. We leave the details to the reader. We prove (iii), which follows from the next result:

$$\mathrm{Hom}_H(V, W) \simeq \mathrm{Hom}_{G\times H}(U \otimes V, U \otimes W), \quad \psi \mapsto \mathrm{id}_U \otimes \psi,$$

which we prove now. It is obvious that $\psi \mapsto \mathrm{id}_U \otimes \psi$ is injective. We prove that it is surjective. Let $\varphi \in \mathrm{Hom}_{G\times H}(U \otimes V,\ U \otimes W)$. Let us write $(\widetilde{\rho}, \widetilde{U})$ for the contragredient representation of (ρ, U). Let $\langle\ ,\ \rangle\colon \widetilde{U} \times U \to \mathbb{C}$ be the canonical pairing. It induces a $G \times G$-equivariant map $p\colon \widetilde{U} \otimes U \to \mathbb{C}$. We claim that there is $\psi \in \mathrm{Hom}_H(V, W)$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{U} \otimes U \otimes V & \xrightarrow{\mathrm{id}_{\widetilde{U}} \otimes \varphi} & \widetilde{U} \otimes U \otimes W \\ \Big\downarrow{\scriptstyle p \otimes \mathrm{id}_V} & & \Big\downarrow{\scriptstyle p \otimes \mathrm{id}_W} \\ V & \xrightarrow{\quad\psi\quad} & W. \end{array}$$

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It is enough to prove

$$(p\otimes \mathrm{id}_V)(x)=0 \implies \big((p\otimes \mathrm{id}_W)\circ(\mathrm{id}_{\widetilde{U}}\otimes\varphi)\big)(x)=0.$$

Let us write $x=\sum_{i=1}^{l}\widetilde{u}_i\otimes u_i\otimes v_i$, with the minimal possible number of v_i's. Then $v_1,\dots,v_l$ are linearly independent, and

$$0=(p\otimes \mathrm{id}_V)(x)=\sum_{i=1}^{l}\langle\widetilde{u}_i,u_i\rangle v_i.$$

Hence $\langle\widetilde{u}_i,u_i\rangle=0$, $1\le i\le l$. Let $w^*\in W^*$ (the space of all linear functionals on W). Then the mapping

$$(\widetilde{u},u)\mapsto\Big\langle\big((p\otimes \mathrm{id}_W)\circ(\mathrm{id}_{\widetilde{U}}\otimes\varphi)\big)(\widetilde{u}\otimes u\otimes v),w^*\Big\rangle,$$

is a G-invariant bilinear form $\widetilde{U}\times U\to\mathbb{C}$. Since (ρ,U) is irreducible and admissible, Schur's lemma implies that there is bilinear form $c\colon V\times W^*\to\mathbb{C}$ such that

$$\Big\langle\big((p\otimes \mathrm{id}_W)\circ(id_{\widetilde{U}}\otimes\varphi)\big)(\widetilde{u}\otimes u\otimes v),w^*\Big\rangle=\langle\widetilde{u},u\rangle c(v,w^*),$$

for all $u\in U,\widetilde{u}\in\widetilde{U},v\in V,w^*\in W^*$. Hence, for $w^*\in W^*$, we obtain

$$\sum_{i=1}^{l}\Big\langle\big((p\otimes id_W)\circ(\mathrm{id}_{\widetilde{U}}\otimes\varphi)\big)(\widetilde{u}_i\otimes u_i\otimes v_i),w^*\Big\rangle=\sum_{i=1}^{l}\langle\widetilde{u}_i,u_i\rangle c(v_i,w^*)=0,$$

proving the claim. Finally, we prove $\varphi=\mathrm{id}_U\otimes\psi$. First, it is clear that if we put $\mathrm{id}_U\otimes\psi$ in the above diagram instead of φ, the diagram would commute. Thus, it is enough to show that if $\psi=0$, then $\varphi=0$. First, the commutative diagram implies $((p\otimes \mathrm{id}_W)\circ(\mathrm{id}_{\widetilde{U}}\otimes\varphi))=0$. If $\varphi\neq 0$, then there are $u\in U$, $v\in V$, such that $\varphi(u\otimes v)\neq 0$. We write $\varphi(u\otimes v)=\sum_{i=1}^{l}u_i\otimes w_i$, with a minimal possible number of w_i's. Then $w_1,\dots,w_l$ are linearly independent, and, for $\widetilde{u}\in\widetilde{U}$,

$$0=(p\otimes \mathrm{id}_W)(\widetilde{u}\otimes\varphi(u\otimes v))=\sum_{i=1}^{l}\langle\widetilde{u},u_i\rangle w_i.$$

This implies $\langle\widetilde{u},u_i\rangle=0$, $1\le i\le l$, for all $\widetilde{u}\in\widetilde{U}$. This is a contradiction. ■

The next lemma refines Lemma 6.8(ii) and Lemma 6.10 (see (6.11)) for $k=0$. Also, it completes the proof of Theorem 5.2(i). (Combining Lemmas 3.3, 6.8, 6.9, and 6.10, $\sigma_2(h)$ satisfies HW for $k>0$. The case $k=0$ is discussed in the next lemma.)

Lemma 6.12 *We maintain the assumptions of Lemma* 6.10*. Let* $k=0$*. Then* $m(\sigma_2(h))=n-\eta_G$*. Moreover, every irreducible subquotient of* $\Theta(\sigma_2(h),n-\eta_G)$ *is of the form* $\delta([\chi_G],h-1)\rtimes\sigma_t(n+\eta_{G'})$*. (Note that this induced representation is irreducible by Lemma* 6.7*.)*

Proof First we prove $m(\sigma) \geq n - \eta_G$. If not, then $\Theta(\sigma_2(h), n - \eta_G - 1) \neq 0$. Moreover, Lemma 6.1 implies

$$\widetilde{\delta}([\chi_G], h) \rtimes \Theta(\sigma_t, n_t - \eta_G - 1) \twoheadrightarrow \Theta(\sigma_2(h), n - \eta_G - 1).$$

Hence, $\Theta(\sigma_2(h), n - \eta_G - 1) \neq 0$. This is a contradiction. The non-vanishing $\Theta(\sigma_2(h), n - \eta_G) \neq 0$ can be proved using induction on h using [11, Lemma 9.1(ii)] (the proof is based on Theorem 3.2 only). We leave the details to the reader. Now we prove the rest of the lemma. Let $\sigma_2(0) = \sigma_t(n + \eta_{G'})$. Now Lemma 6.7 implies $\sigma_2(h) \hookrightarrow \chi_{G'} \rtimes \sigma_2(h-1)$, $h \geq 1$. Hence, the Frobenius reciprocity implies $R_{P_1}(\sigma_2(h)) \twoheadrightarrow \chi_{G'} \otimes \sigma_2(h-1)$. Thus, applying $R_{P_1}(\,\cdot\,)$ to $\omega_{n,n-\eta_G} \twoheadrightarrow \sigma_2(h) \otimes \Theta(\sigma_2(h), n - \eta_G)$, we obtain

$$R_{P_1}(\omega_{n,n-\eta_G}) \twoheadrightarrow \chi_{G'} \otimes \sigma_2(h-1) \otimes \Theta(\sigma_2(h), n - \eta_G).$$

(The argument is similar to the proof of Lemma 6.10.) Applying Theorem 3.5 we see that $\Theta(\sigma_2(h), n - \eta_G)$ has a filtration of the form $0 \subset \Theta \subset \Theta(\sigma_2(h), n - \eta_G)$, where

$$\chi_G \rtimes \Theta(\sigma_2(h-1), (n-1) - \eta_G) \twoheadrightarrow \Theta,$$
$$\Theta(\sigma_2(h), n - \eta_G)/\Theta \twoheadrightarrow \Theta(\sigma_2(h-1), (n-1) + \eta_{G'}).$$

Lemma 6.9 implies $\Theta(\sigma_2(h-1), (n-1) + \eta_{G'}) \simeq \sigma_2(h-1)((n-1) + \eta_{G'})$. Now the lemma follows by induction on h using (6.11). ■

Now we prove Theorem 5.2(ii).

Lemma 6.13 *Assume that* (a) *and* (b) *of Theorem* 5.2(i) *hold. Assume* $m(\sigma_t) < n_t + \eta_{G'}$. *(See* (6.8)*.) Then we have the following:*

(i) $n + \eta_{G'} \geq m(\sigma) \geq n - k - \eta_G$.

(ii) $\Theta(\sigma, m)$ *is irreducible and tempered for* $n + \eta_{G'} \geq m \geq m(\sigma)$, $m > n - k - \eta_G$. *Moreover, we have the following:*

$$\text{tempered support}(\sigma(m)) =$$
$$\{2h \cdot \delta([|\ |^{-k}\chi_G, |\cdot|^k\chi_G])\} + \text{tempered support}(\sigma_t(m - n + n_t)).$$

(iii) *If* $m > n + \eta_{G'}$, *then* $\sigma(m)$ *is a unique irreducible (Langlands) subrepresentation of*

$$|\cdot|^{n-m+\eta_{G'}}\chi_G \times |\cdot|^{n-m+\eta_{G'}+1}\chi_G \times \cdots \times |\cdot|^{-1}\chi_G \rtimes \sigma(n + \eta_{G'}).$$

(iv) *Let* $m > n - k - \eta_G$, $m \geq m(\sigma)$. *Then* $\Theta(\sigma, m)$ *has a unique maximal proper subrepresentation; we denote the corresponding irreducible quotient by* $\sigma(m)$, *and* $\sigma(m+1) \hookrightarrow \chi_G|\cdot|^{n-m+1+\eta_{G'}} \rtimes \sigma(m)$.

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Proof We prove (i). The inequality $m(\sigma) \geq n - k - \eta_G$ follows directly from Lemma 6.2. The other follows from Theorem 3.5. The argument needed to prove (ii) is already explained in the proof of Theorem 5.1. (iii) follows from Lemma 3.3 and Theorem 3.4. Finally, (iv) follows using the argument used in the proof of Theorem 4.2(iii). ∎

The next lemma completes the proof of Theorem 5.2(ii).

Lemma 6.14 *We maintain the assumptions of the previous lemma. Then*

$$\delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \rtimes \sigma_t$$

is irreducible. In particular, $\sigma \simeq \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \rtimes \sigma_t$. *Moreover,*

$$\delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G], h) \rtimes \sigma_t(n_t - k - \eta_G)$$

is a direct sum of two non-equivalent tempered representations. Exactly one of them is an irreducible quotient of $\Theta(\sigma, n - k - \eta_G)$*. Finally,* $m(\sigma) = n - k - \eta_G$*.*

Proof Note that $m(\sigma_d) < n_d - k - \eta_G + 1 = n_d - k + \eta_{G'} \leq n_d + \eta_{G'}$. Hence, Theorem 4.2 implies that $\tau_d = \sigma_d(m(\sigma_d)) \in \operatorname{Irr} G'_{m(\sigma_d)}$ is in a discrete series. We compute its first occurrence index. First, obviously we have $m(\tau_d) \geq n_d$, since it is lifted to σ. Next, since τ_d is a representation of $G'_{m(\sigma_d)}$, we see that

$$m(\tau_d) \geq n_d = m(\sigma_d) + \eta_G + k > m(\sigma_d) + \eta_G.$$

Now a direct application of Theorem 4.2 shows that $m(\tau_d) = n_d$, since their lift σ_d to G_{n_d} is in discrete series. Next we remark that

$$k = n_d - m(\sigma_d) - \eta_G = m(\tau_d) - m(\sigma_d) - \eta_G. \tag{6.20}$$

Thus, Theorem 5.1 applied to σ_t gives the description of the tempered support for $\tau_t := \sigma_t(n_t - k - \eta_G)$. This shows that Theorem 5.1 is applicable to τ_t. In particular, using (6.20), we obtain $m(\tau_t) = n_t, \Theta(\tau_t, n_t) \simeq \sigma_t$. Now Theorem 5.2(i) describes the lifts of irreducible components (see Lemma 6.7)

$$\tau_1(h) \oplus \tau_2(h) \simeq \delta([|\cdot|^{-k}\chi_G, |\cdot|^{k}\chi_G]), h) \rtimes \tau_t.$$

Applying, Lemma 6.10, we obtain

$$\tau_2(h)(n) \simeq \delta([|\cdot|^{-k}\chi_{G'}, |\cdot|^{k}\chi_{G'}], h) \rtimes \sigma_t \simeq \sigma, \quad m(\tau_2(h)) \leq n.$$

This proves that $\tau_2(h)$ is a quotient of $\Theta(\sigma, n - k - \eta_G)$. Hence, Lemma 6.13 implies $m(\sigma) = n - k - \eta_G$. Since Lemma 6.10 implies $m(\tau_1(h)) = n + 1$, the proof of the lemma is complete. ∎

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References

[1] J. Bernstein, *Second adjointness for representations of p-adic reductive groups.* Preprint (Harvard 1987).

[2] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive $\mathfrak{p}$-adic groups. I.* Ann. Sci. École Norm Sup. **10**(1977), no. 4, 441–472.

[3] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups.* Preprint, www.math.ubc.ca/~cass/research/p-adic-book-dvi.

[4] D. Goldberg, *Reducibility of induced representations for* $\mathrm{Sp}(2n)$ *and* $\mathrm{SO}(n)$. Amer. J. Math. **116**(1994), no. 5, 1101—1151.

[5] S. S. Kudla, *On the local theta-correspondence.* Invent. Math. **83**(1986), no. 2, 229–255.

[6] ———, *On the Theta Correspondence.* (lectures at European School of Group Theory, Beilngries, 1996), www.math.toronto.ca/~skudla/castle/pdf.

[7] E. Lapid, G. Muić, and M. Tadić, *On the generic unitary dual of quasi-split classical groups.* Int. Math. Res. Not. **2004**, no. 26, 1335–1354.

[8] C. Mœglin, *Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité.* J. Eur. Math. Soc. (JEMS) **4**(2002), no. 2, 143–200.

[9] C. Mœglin and M. Tadić, *Construction of discrete series for classical p-adic groups.* J. Amer. Math. Soc **15**(2005), no. 3, 715–786.

[10] C. Moeglin, M.-F. Vignéras, and J.L. Waldspurger, *Correspondence de Howe sur un corps p-adique.* Lecture Notes in Mathematics 1291, Springer-Verlag, Berlin, 1987.

[11] G. Muić, *Howe correspondence for discrete series representations; the case of* $(\mathrm{Sp}(n), O(V))$. J. Reine Angew. Math. **567**(2004), 99–150.

[12] ———, *Reducibility of standard representations.* Pacific J. Math. **222**(2005), no. 1, 133–168.

[13] ———, *On the structure of the full lift for the Howe correspondence of* $(\mathrm{Sp}(n), \mathrm{O}(V))$ *for rank-one reducibilities.* Canad. Math. Bull. **49**(2006), no. 4, 578–591.

[14] ———, *On the structure of theta lifts of discrete series for dual pairs* $(\mathrm{Sp}(n), O(V))$. Israel J. Math. **164**(2008), 87–124.

[15] G. Muić and G. Savin, *Symplectic-orthogonal theta lifts of generic discrete series.* Duke Math. J. **101**(2000), no. 2, 317–333.

[16] S. Rallis, *On the Howe duality conjecture.* Compositio Math. **51**(1984), no. 3, 333–399.

[17] M. Tadić, *On regular square integrable representations of p-adic groups.* Amer. J. Math **120**(1998), no. 1, 159–210.

[18] J.-L. Waldspurger, *Démonstration d'une conjecture de duality de Howe dans le case p-adiques, $p \neq 2$.* Festschrift in Honor of I.I. Piatetski-Shapiro, Part II, Israel Math. Conf. Proc. 2, Weizmann, Jerusalem, 1990, pp. 267–324.

[19] ———, *La formule de Plancherel pour les groupes p-adiques, d'après Harish-Chandra.* J. Inst. Math. Jussieu **2**(2003), no. 2, 235–333.

[20] A. V. Zelevinsky, *Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of* $\mathrm{GL}(n)$. Ann. Sci. Ecole Norm. Sup. **13**(1980), no. 2, 165–210.

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