S. E. Graversen
Nagoya Math. J.
Vol. 114 (1989), 123–133

A RIESZ DECOMPOSITION THEOREM

S. E. GRAVERSEN

Introduction

The topic of this note is the Riesz decomposition of excessive functions for a "nice" strong Markov process X. I.e. an excessive function is decomposed into a sum of a potential of a measure and a "harmonic" function. Originally such decompositions were studied by G.A. Hunt [8]. In [1] a Riesz decomposition is given assuming that the state space E is locally compact with a countable base and X is a transient standard process in strong duality with another standard process $\hat{X}$ having a strong Feller resolvent. Recently R.K. Getoor and J. Glover extended the theory to the case of transient Borel right processes in weak duality [6].

In a different direction K.L. Chung and M. Rao [2] discussed the Riesz representation and other related topics without assuming duality. Their conditions are analytic ones imposed on the potential density $u(x, y)$. To be precise, they assume that $u(x, y)$ is the potential density of a transient Hunt process and satisfies:

$u(x, y)$ is extended continuous in y for any fixed x, $u(x, y) > 0$ for any (x, y) and $u(x, y) = \infty$ if and only if $x = y$.

It is proved in [2] that the Riesz decomposition holds for any excessive function. In [9] Ming Liao extends the results of Chung and Rao under slightly weaker assumptions.

The frame for this note is a transient Borel right process X on a Lusin topological space E with potential density $u(x, y)$ with respect to a given excessive reference measure m. No duality is assumed. In Section 1—using pure potential theoretic standard H-cone technique—we construct the potential part $U\mu_s$ of the Riesz decomposition of a given excessive function s. The assumption on $u(x, y)$ needed for this construction is properness and a point separating property of the dual operator $\hat{U}$ defined by

Received October 20, 1987.

$$\hat{U}f(x) := \int f(y)u(y, x)m(dy),$$

for non-negative Borel measurable functions f. In Section 2, which is more probabilistic in nature, it is proved under further conditions on $\hat{U}$ that the function h in the decomposition $s = U\mu_s + h$ is harmonic in the following sense: $h = \tilde{P}_{K^c}h$ for every set K compact in a specified topology τ where $\tilde{P}.$ is the kernel associated with first penetration time. The main results are contained in Theorems 1.2 and 2.1, and the principal assumptions are contained in Assumptions 1.1 and 2.1.

Section 0. Notation

We shall use the standard notation ([1], [5]) of Markov processes without special mention. The following notation and minimal hypothesis will remain in force throughout.

(0.1) E is a Lusin topological space ([5]) with Borel σ-field $\mathscr{B}(E)$.

(0.2) $\mathscr{B}(E)_b = \{f\colon E \to R \mid f \text{ bounded and Borel measurable}\}$
$\mathscr{B}(E)_+ = \{f\colon E \to R \cup \{\infty\} \mid f \text{ non-negative and Borel measurable}\}$
$\mathscr{B}(E)_{b+} = \mathscr{B}(E)_b \cap \mathscr{B}(E)_+$

(0.3) $(P_t)_{t \geqq 0}$ is a Borel semigroup on $(E, \mathscr{B}(E))$ with resolvent $(U^\alpha)_{\alpha \geqq 0}$.

(0.4) $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X(t), \Theta_t, P^x)$ is a Borel right process in E with transition function (P_t), lifetime ξ, death point Δ.
(As usual Δ is adjointed to E as an isolated point.)

(0.5) m is a σ-finite excessive reference measure for X:

(0.6) q is an element of $\mathscr{B}(E)_{b+}$ so that $0 < q \leqq 1$ and $0 < Uq \leqq 1$. $(U = U^0)$.

If f, g are Borel measurable functions, $\langle f, g \rangle$ denotes the integral $\int f \cdot g \, dm$.

Using (0.5) and Theorem 14, Chapter IX in [4], we can construct a family of submarkovian Borel kernels $(\hat{P}_t)$ so that for $t > 0$,

$$\langle f, P_t g \rangle = \langle \hat{P}_t f, g \rangle \qquad f, g \in \mathscr{B}(E)_+ .$$

Section 1.

Let $\mathscr{E}$ denote the cone of (P_t)-excessive functions finite $[m]$ a.e., i.e. s is in $\mathscr{E}$ if

a) $s \in \mathscr{B}(E)_+$ and $m((s = \infty)) = 0$

b) $P_t s(x) \leqq s(x),\ x \in E,\ t > 0$

c) $\sup_{t>0} P_t s(x) = s(x),\ x \in E.$

The properties of the potential density required in this section are gathered in the following assumption.

ASSUMPTION 1.1. There exists a non-negative $\mathscr{B}(E)\times\mathscr{B}(E)$ measurable function $u(x, y)$ such that

$$U(x, dy) = u(x, y)m(dy) \qquad \text{for } x \in E,$$
$$x \longrightarrow u(x, y) \text{ is excessive for } X \qquad \text{for } y \in E,$$

and there exists $\{g, g_i,\ i \geqq 1\} \subseteqq \mathscr{B}(E)_{b+}$ so that

1) $0 < g,\ 0 \leqq g_i \leqq g$ and $\langle g, 1\rangle < \infty$.

2) $0 < \hat{U}g < \infty$ everywhere.

3) $(\hat{U}g_i/\hat{U}g,\ i \geqq 1)$ separates points in E.

4) If a Borel measurable function k satisfies $\langle |k|, g_i\rangle < \infty$ and $\langle k, g_i\rangle \geqq 0$ for all i, then $k \geqq 0[m]$ a.e. and $m((k = \infty)) = 0$.

Denote by Q_+ the positive rational numbers and let A denote the smallest set of functions which is closed under finite sums, multiplication by positive constants, finite minimum and contains the following functions:

$$\hat{U}g_i\,,\quad \hat{U}g\,,\quad \hat{U}\hat{P}_t g_i \wedge \hat{U}g_i\,,\quad \hat{U}\hat{P}_t g \wedge \hat{U}g \qquad i \geqq 1,\ t \in Q_+\,.$$

LEMMA 1.1. *For all h in A, $h\cdot dm$ is an excessive measure for X and there exists a constant $\alpha(h) \geqq 0$ so that $0 \leqq h \leqq \alpha(h)\hat{U}g$.*

Proof. Since every h in A is finite everywhere, hdm is σ-finite. Denote by $\tilde{A}$ the subset of A having the desired properties. $\tilde{A}$ is then stable under finite sums, multiplication by positive constants and forming a finite minimum; therefore it suffices to prove that $\tilde{A}$ contains the set of functions listed above. Because $\hat{U}g \geqq \hat{U}g_i,\ i \geqq 1$, only the excessive property needs to be verified.

For k in $\mathscr{B}(E)_{b+}$ and all t in Q_+ we have

$$\langle k, \hat{U}\hat{P}_t g_i\rangle = \langle P_t Uk, g_i\rangle \leqq \langle k, \hat{U}g_i\rangle\,,$$

and likewise $\langle k, \hat{U}\hat{P}_t g\rangle \leqq \langle k, \hat{U}g\rangle$.

This means $\hat{U}\hat{P}_t g_i \leqq \hat{U}g_i$ and $\hat{U}\hat{P}_t g \leqq \hat{U}g[m]$ a.e. Thus we need only

https://doi.org/10.1017/S0027763000001422 Published online by Cambridge University Press

prove that $\hat{U}k\,dm$ is excessive for $k \in \mathscr{B}(E)_{b+}$ with $\hat{U}k < \infty$. But this is clear.

LEMMA 1.2. *For all h in A there is a unique σ-finite measure γ_h on $(E, \mathscr{B}(E))$ so that*

$$h\,dm = \gamma_h U \qquad \left(= \int \gamma_h(dy) U(y, dx)\right).$$

Proof. Let h in A be given. Define $\nu := h\,dm$ and $\mu := \alpha(h) g\,dm$. For all $k \in \mathscr{B}(E)_{b+}$ we have

$$\int k(y) \mu U(dy) = \int Uk(y) \mu(dy)$$
$$= \langle Uk, \alpha(h) g \rangle = \langle k, \alpha(h) \hat{U} g \rangle$$

i.e. $\mu U = \alpha(h) \hat{U} g\,dm$, and thus $\nu \leqq \mu U$. The result is now implied by Theorem 4.2 and Proposition 1.1 in [7].

Remark. Easy computations show that $\gamma_{\hat{U}g} = g\,dm$, $\gamma_{\hat{U}g_t} = g_t\,dm$, $\gamma_h = \hat{P}_t g\,dm$ if $h = \hat{U}\hat{P}_t g \wedge \hat{U}g$ and $\gamma_h = \hat{P}_t g_t\,dm$ if $h = \hat{U}\hat{P}_t g_t \wedge \hat{U} g_t$.

Define $C := \{s \in \mathscr{E} \mid \langle s, g \rangle < \infty\}$.

LEMMA 1.3. $\int s(y) \gamma_h(dy) < \infty$ *for* $s \in C$, $h \in A$.

Proof. Let $s \in C$ and $h \in A$ be given. Theorem IX T64 ([8]) implies that there exist $(\varnothing_n)_{n=1}^{\infty} \subseteqq \mathscr{B}(E)_{b+}$ so that $U\varnothing_n \uparrow s$. Therefore

$$\int s(y) \gamma_h(dy) = \sup_n \int U\varnothing_n(y) \gamma_h(dy) = \sup_n \langle \varnothing_n, h \rangle$$
$$\leqq \sup_n \langle \varnothing_n, \alpha(h) \hat{U} g \rangle = \sup_n \langle U\varnothing_n, \alpha(h) g \rangle \leqq \alpha(h) \langle s, g \rangle < \infty .$$

Define $\mathscr{H} := A - A$.

$\mathscr{H}$ is a linear space of finite real valued Borel functions stable under finite minimum and finite maximum.

For s in C, denote by $L(s, \cdot)$ the function on $\mathscr{H}$ defined by

$$L(s, h) := \int s d\gamma_{h_1} - s d\gamma_{h_2}, \tag{1.1}$$

where $h_i \in A$, $i = 1, 2$ and $h = h_1 - h_2$.

The proof of Lemma 1.3 shows that $L(s \cdot)$ is a well defined positive linear functional on $\mathscr{H}$ and that $s \to L(s, h)$ is additive for each h in $\mathscr{H}$. Likewise, if s, s_1 and s_2 are in C with $s = s_1 + s_2$, then we have

(1.2) $$L(s, h) \geqq L(s_1, h) \qquad \text{for } h \in \tilde{\mathscr{H}}_+ .$$

Define $\mathscr{H} := \{h/\hat{U}g \mid h \in \tilde{\mathscr{H}}\}$.

Then $\mathscr{H}$ is a linear space of bounded real valued Borel functions stable under finite infimum and finite maximum, containing constants and separating points in E. Furthermore, $\mathscr{H}_+$ is separable in the sup-norm. Let $\bar{E}$ denote the compactification of E by means of $\mathscr{H}_+$ ([5]). $\bar{E}$ is a compact metric space with Borel field $\mathscr{B}(\bar{E})$. The assumptions imply that $E \in \mathscr{B}(\bar{E})$ and $\mathscr{B}(E) = \mathscr{B}(\bar{E}) \cap E$. Since E is dense in $\bar{E}$, every element h in $\mathscr{H}$ has a unique continuous extension $\bar{h}$ to $\bar{E}$, and $\bar{\mathscr{H}} := \{\bar{h} \mid h \in \mathscr{H}\}$ is dense in $\mathscr{C}(\bar{E})$.

Daniell's theorem implies that every positive linear functional $\bar{L}$ on $\bar{\mathscr{H}}$ is of the form

(1.3) $$\bar{L}(\bar{h}) = \int_E \bar{h}\, d\mu_{\bar{L}} + \int_{\bar{E}\setminus E} \bar{h}\, d\bar{\mu}_{\bar{L}} ,$$

where $\mu_{\bar{L}}$ and $\bar{\mu}_{\bar{L}}$ are uniquely determined bounded measures on $(\bar{E}, \mathscr{B}(\bar{E}))$ satisfying $\bar{\mu}_{\bar{L}}(E) = 0$ and $\mu_{\bar{L}}(\bar{E}\setminus E) = 0$. Thus $\mu_{\bar{L}}$ is a bounded measure on $(E, \mathscr{B}(E))$.

Every positive linear functional L on $\tilde{\mathscr{H}}$ can be regarded as a positive linear functional $\bar{L}$ on $\bar{\mathscr{H}}$ through the formula

$$\bar{L}(\bar{h}) = L(\tilde{h}) \qquad \text{where } \bar{h} = \overline{\tilde{h}/\hat{U}g} .$$

Since $\hat{U}g > 0$ everywhere, this proves that every positive linear functional L on $\tilde{\mathscr{H}}$ is of the form

(1.4) $$L(h) = \int_E h\, d\mu_L + \int_{\bar{E}\setminus E} \overline{h/\hat{U}g}\, d\bar{\mu}_L ,$$

where μ_L is a uniquely determined σ-finite measure on $(E, \mathscr{B}(E))$ and $\bar{\mu}_L$ is a bouned measure on $(\bar{E}, \mathscr{B}(\bar{E}))$ so that $\bar{\mu}_L(E) = 0$.

NOTATION. A positive linear functional L on $\tilde{\mathscr{H}}$ is called singular if $\mu_L = 0$, and it is called an integral if $\bar{\mu}_L = 0$.

Remark. Daniell's theorem implies that a positive linear functional L on $\tilde{\mathscr{H}}$ is an integral if and only if

(1.5) $$L(f_n) \downarrow 0 \quad \text{if} \quad (f_n) \subseteqq \tilde{\mathscr{H}}_+ \quad \text{and} \quad f_n \downarrow 0 .$$

A consequence of this is that L is an integral if there exists (L_L) all integrals, so that $L_n(h) \uparrow L(h)$ for all h in $\mathscr{H}_+$.

The proof of the following Lemma is immediate and therefore omitted.

LEMMA 1.4. *Let L_1 and L_2 be two positive linear functionals on $\mathscr{H}$ so that $L_1(h) \geqq L_2(h)$ for all h in $\mathscr{H}_+$. Then*

$$L_1 \text{ is an integral (singular)} \Longrightarrow L_2 \text{ is an integral (singular).}$$

Translating these results gives that for s in C

$$L(s, h) = \int_E h\, d\mu_s + \int_{\bar{E}\setminus E} \overline{h/\hat{U}g}\, d\bar{\mu}_s\,, \qquad h \in \mathscr{H}\,, \tag{1.6}$$

where μ_s is a uniquely determined σ-finite measure on $(E, \mathscr{B}(E))$ and $\bar{\mu}_s$ is a bounded measure on $(\bar{E}, \mathscr{B}(\bar{E})$ so that $\bar{\mu}_s(E) = 0$. Furthermore $s \in C$ is called singular (an integral) if $L(s, \cdot)$ is singular (an integral).
Denote by $\boldsymbol{S}(\boldsymbol{I})$ the set of singular (integral) elements in C.

THEOREM 1.1. *Let s_1 and s_2 in C be given. Assume $s_1 \gg s_2$ (strong order). Then*

$$s_1 \in \boldsymbol{S}(\boldsymbol{I}) \Longrightarrow s_2 \in \boldsymbol{S}(\boldsymbol{I})\,.$$

Proof. Use Lemma 1.4 and formula (1.2).

In order to state the main theorem of this section, we need to define the so-called "harmonic" elements in C.

DEFINITION. s in C is called "harmonic" iff $u \in \boldsymbol{I}$ and $s \gg u \Rightarrow u \equiv 0$.

Let $\boldsymbol{H}$ denote the set of "harmonic" elements in C. Clearly $\boldsymbol{S} \subseteqq \boldsymbol{H}$.

THEOREM 1.2. *For s in C, the following decomposition is valid,*

$$s = U\mu_s + s_1\,,$$

and $U\mu_s \in \boldsymbol{I}$ and $s_1 \in \boldsymbol{H}$. Furthermore, a decomposition of s into a sum of an integral and a "harmonic" element is unique.

The proof of Theorem 1.2 is based on the following lemmas.

LEMMA 1.5. *s in C admits at most one decomposition of the form $u + v$, where $u \in \boldsymbol{I}$ and $v \in \boldsymbol{H}$.*

Proof. Lemma 1.5 is related to the "Riesz splitting property" see

[12]. Let s in C be given and assume $s = u_i + v_i$, where $u_i \in \boldsymbol{I}$ and $v_i \in \boldsymbol{H}$ for $i = 1, 2$.

$$u_2 \ll s \Longrightarrow u_2 = u_{12} + u_{22}\,, \qquad \text{where } u_{12} \ll u_1 \text{ og } u_{22} \ll v_1\,.$$

Since $u_{22} \ll u_2$ and $u_{22} \ll v_1$, u_{22} equals 0 by Theorem 1.1. Thus $u_2 \ll u_1$ which by symmetry gives the result.

LEMMA 1.6. *$s = U\mu_s$ for all s in $\boldsymbol{I}$.*

Proof. Let s in C be given. For all $i \geqq 1$ we have

$$\begin{aligned}\langle U\mu_s, g_i\rangle &= \int \hat{U}g_i\, d\mu_s = L(s, \hat{U}g_i)\\ &= \int s\, d\gamma_{\hat{U}g_i} = \langle s, g_i\rangle < \infty\,,\end{aligned}$$

which by Assumption 1.1 provides the result.

LEMMA 1.7. *For s in C, $A_s = \{u \in C \mid s \gg u, u \in \boldsymbol{I}\}$ admits an upper bounded in the strong order.*

Proof. Let s in C be given. Since $\boldsymbol{I}$ is stable under addition, it follows that A_s is upwards filtering in the strong order. Theorem 1.5, p. 198 [1] now implies the existence of a sequence $(u_n) \subseteqq A_s$ so that $u_n \ll u_{n+1}$, $n \geqq 1$ and $\sup_n u_n = \sup\{u \mid u \in A_s\}$ (pointwise). $u_\infty := \sup_n u_n$. A simple argument shows that $u_\infty \in C$ and $u_\infty \ll s$.
Because $u_n \ll u_{n+1}$, $n \geqq 1$, there exists $(\varnothing_k^n)_{k,n\geq 1} \subseteqq \mathscr{B}(E)_{b+}$ so that

a) $\varnothing_k^n \leqq \varnothing_k^{n+1}$, $n \geqq 1$, $k \geqq 1$,

b) $U(\varnothing_k^n) \uparrow_{k\to\infty} u_n$, $n \geqq 1$.

a) and b) imply that $U(\varnothing_n^n) \uparrow_{n\to\infty} u$. Therefore

$$L(u, h) = \lim_n \langle \varnothing_n^n, h\rangle \geqq \lim_n \langle \varnothing_n^{n_0}, h\rangle = L(u_{n_0}, h)$$

for all h in $\mathscr{H}$ and all $n_0 \geqq 1$.
But by monotone convergence we have

$$\int u_n d\gamma_h \underset{n\to\infty}{\longrightarrow} \int u_\infty d\gamma_h \qquad \text{for all } h \text{ in } \mathscr{H}\,,$$

and thus $L(u_n, h)\uparrow_{n\to\infty} L(u_\infty, h)$ for $h \in \mathscr{H}_+$. u_∞ is therefore an integral. The fact that u_∞ is an upper bound for A_s is immediate.

Proof of Theorem 1.2. Only the existence of a decomposition remains to be shown. Let s in C be given and let u be the upper bound for A_s

constructed in Lemma 1.7,

$$u \ll s \Longrightarrow \exists v \in C\colon s = u + v\,.$$

If v does not belong to $\boldsymbol{H}$, there exists $v_1 \in \boldsymbol{I}\setminus\{0\}$ so that $v \gg v_1$, but this leads to a contradiction because in that case $u + v_1$ will be an element of A_s. The proof of Theorem 1.2 is thus complete.

Section 2.

Throughout the rest of this paper we will assume, together with Assumption 1.1, the following

ASSUMPTION 2.1. There exists a second countable metric topology τ on E so that

a) $\mathscr{B}(\tau) = \mathscr{B}(E)$.

b) Every function in $\mathscr{H}$ is τ-continuous.

c) τ is finer than the given topology on E.

d) There exists a Borel semipolar set B so that $\tau|_{B^c}$ is coarser than the fine topology restricted to B^c.

The aim of this section is to show that the elements in $\boldsymbol{H}$ deserve the name "harmonic" elements.

DEFINITION. For every $F \in \mathscr{B}(E)$, let $\tilde{T}_F$ denote the penetration time into F, i.e.

$$\tilde{T}_F(\omega) := \inf\{t \geqq 0 \,|\, [0, t] \cap \{u \geqq 0 \,|\, X_u(\omega) \in F\} \text{ is uncountable}\}$$

$\tilde{T}_F$ is a stopping time and exact terminal time satisfying $\tilde{T}_F \cdot \Theta_{\tilde{T}_F} = 0$ on $\{\tilde{T}_F < \infty\}$ [3]. For all s in $\mathscr{E}$ and F in $\mathscr{B}(E)$, denote by $\tilde{P}_F s(x)$ the value $E_x(s(X_{\tilde{T}_F}))$. The properties of $\tilde{T}_F$ imply that $x \to \tilde{P}_F s(x)$ is again an element of $\mathscr{E}$.

LEMMA 2.1. *Let O denote a τ-open set. There exists a finely open set V so that $O \cap B^c = V \cap B^c$ and*

$$\tilde{P}_O s(x) = P_V s(x) = E_x(s(X(T_V)))$$

for $x \in E$ and $s \in \mathscr{E}$, where T_V is the first hitting time to V.

Proof. Let $s \in \mathscr{E}$ be given. Using Assumption 2.1 d), there exists a finely open set V so that $O \cap B^c = V \cap B^c$. Since B is semipolar and thus only visited countably often, we have $\tilde{T}_O = \tilde{T}_{O\cap B^c}$ and $\tilde{T}_V = \tilde{T}_{V\cap B^c}$ a.s.

P^x for $x \in E$. But since V is a finely open set, $T_V = \tilde{T}_V$ a.s. P^x for $x \in E$. Thus

$$\tilde{T}_O = T_V \quad \text{a.s.} \quad \text{for } x \in E\,,$$

so that $\tilde{P}_O s = P_V s$.

LEMMA 2.2. *Let $s \in \boldsymbol{H}$ and K a τ-compact set be given. There exists a τ-open set O containing K so that $s \neq \tilde{P}_O s$.*

Proof. Since $s \notin \boldsymbol{I}$, there exists h, $\{h_n\}_n$ in $\mathscr{H}_+$, so that $h_n \uparrow h$ and $L(s, h) - \sup_n L(s, h_n) = \beta > 0$. $\hat{U}g$ is strictly positive, so we can choose $\varepsilon > 0$ so that

$$K \subseteqq \{\hat{U}g > \varepsilon\}\,.$$

Pick an $\alpha > 0$ so that $\alpha\varepsilon^{-1}\langle s, g\rangle < \beta$. Since $h - h_n \downarrow 0$, a Dini argument implies the existence of a τ-open set O and an $n_0 \geqq 1$, so that

$$K \subseteqq O \subseteqq \{\hat{U}g > \varepsilon\} \quad \text{and} \quad O \subseteqq \{h - h_{n_0} < \alpha\}\,.$$

We claim that $s \neq \tilde{P}_0 s$. Assume the opposite. Choose V finely open according to Lemma 2.1 and the given set O. Since V is finely open and B is semipolar, we can find, since $s = \tilde{P}_O s = P_v s$ (see page 88 in [1]), $(\varnothing_k)_k \subseteqq \mathscr{B}(E)_{b+}$ satisfying $U\varnothing_k \uparrow s$ and $\operatorname{supp}(\varnothing_k) \subseteqq V \cap B^c \subseteqq O$ for $k \geqq 1$. Now

$$\begin{aligned} L(s, h) - L(s, h_{n_0}) &= \lim_{h\to\infty} \langle \varnothing_k, h - h_{n_0}\rangle \leqq \alpha \limsup_k \langle \varnothing_k, 1\rangle \\ &\leqq \alpha\varepsilon^{-1} \limsup_k \langle \varnothing_k, \hat{U}g\rangle \leqq \alpha\varepsilon^{-1}\langle s, g\rangle < \beta \end{aligned}$$

i.e. we have derived a contradiction.

Remark. A similar argument implies that if s in C satisfies $s = \tilde{P}_O s$ for a τ-relative compact τ-open set O, then $s \in \boldsymbol{I}$.

DEFINITION. s in $\mathscr{E}$ is said to be minimal if, whenever u, v are in $\mathscr{E}$ and $s = u + v$, both u and v are proportional to s.

NOTATION. The set of minimal elements of C, $\boldsymbol{I}$ and $\boldsymbol{H}$ will be denoted C^e, $\boldsymbol{I}^e$ and $\boldsymbol{H}^e$.

It is immediately seen that we have the following set identity

$$C^e = \boldsymbol{I}^e \cup \boldsymbol{H}^e\,. \tag{2.1}$$

LEMMA 2.3. *For s in $\boldsymbol{H}^e$ and every τ-compact set K we have $s = \tilde{P}_{K^c} s$.*

https://doi.org/10.1017/S0027763000001422 Published online by Cambridge University Press

Proof. Let s in $\boldsymbol{H}^e$ and K a τ-compact set be given. According to Lemma 2.2 there exists a τ-open set O containing K so that $s \neq \tilde{P}_O s$. Choose V finely open according to Lemma 2.1 and the given O,

$$s = P_V s + a$$

where

$$a = \begin{cases} 0 & \text{on the fine closure of } V \\ s - P_V s & \text{elsewhere}. \end{cases}$$

Using a theorem of G. Mokobodzki ([12]), we have $s = v + \underline{R}a$, where $v \in \mathscr{E}$ and $v \leqq P_V s$ and $\underline{R}a$ is the excessive regularisation of

$$Ra = \inf\{u \,|\, u \geqq a \text{ and } u \text{ supermedian}\}.$$

Since s is minimal and $a \neq 0$, we have $s = \beta \underline{R}a$ for some β in $(0, 1]$. Let O_1 denote a τ-open set so that $K \subseteqq O_1 \subseteqq \overline{O}_1^\tau \subseteqq O$, where $-\tau$ denotes the τ-closure. The existence of O_1 is ensured by the regularity of τ. We now now claim that $s = \tilde{P}_{O_2} s$, where $O_2 = (\overline{O}_1^\tau)^c$. Denote by V_2 the finely open set chosen according to Lemma 2.1 and O_2. $s = P_{V_2}$ on V_2, and since we have $V_2 \cap B^c = O_2 \cap B^c \supseteqq O^c \cap B^c = V^c \cap B^c \supseteqq (\overline{V}^f)^c \cap B^c$, $s = P_{V_2} s$ on $(\overline{V}^f)^c \setminus B$. But B is semipolar and $(\overline{V}^f)^c$ finely open, so this implies $s = P_{V_2} s$ on $(\overline{V}^f)^c$ and thus $\beta^{-1} P_{V_2} \geqq \underline{R}(s - P_V s)$. The conclusion $s = P_{V_2} s$ is now immediate, and since $K^c \supseteqq O_2$, we also have $s = \tilde{P}_{K^c} s$.

THEOREM 2.1. *For s in $\boldsymbol{H}$ and every τ-compact set K we have $s = \tilde{P}_{K^c} s$.*

Proof. Let s in $\boldsymbol{H}$ be given. A famous theorem of G. Mokobodzki ([13], see also [14]) ensures that s can be represented as an integral of elements belonging to C^e. But this implies, together with Theorem 1.1 and formula (2.1) that s can be represented as an integral of excessive functions belonging to $\boldsymbol{H}^e$. The conclusion of the theorem is now a consequence of Lemma 2.3 and Fubini's theorem.

An example which fits into the framework of this paper but not into that of [6]:

X is a Brownian motion of $(-1, 1)$ which is reflected at 0 when approaching from the right. Here no dual process exists, but the assumptions of this paper is fulfilled with τ equal to the sum topology on $(-1, 0) \cup [0, 1]$.

References

[1] Blumenthal, R. M., Getoor, R. K., Markov processes and potential theory, New York, Academic Press, 1968.
[2] Chung, K. L., Rao, M., A new setting for potential theory, Ann. Inst. Fourier, **30** (1980), 167–198.
[3] Dellacherie, C., Capacités et processus stochastiques, Berlin-Heidelberg-New York, Springer Verlag, 1972.
[4] Dellacherie, C., Meyer, P. A., Probabilités et potentiel: Théorie discrète du potentiel, Herman, 1983.
[5] Getoor, R. K., Ray processes and right processes, Lecture Notes in Math., **440**, Berlin-Heidelberg-New York, Springer Verlag, 1975.
[6] Getoor, R. K., Glover, J., Riesz decompositions in Markov process theory, Trans. Amer. Math. Soc., **285** (1984), 107–132.
[7] ——, Markov processes with identical excessive measures, Math. Z., **184** (1983), 187–300.
[8] Hunt, G. A., Markov processes and potentials I, Illinois J. Math., **1** (1957), 44–93.
[9] Liao, M., Riesz representation and duality of Markov processes, Séminaire de Probabilités XIX, 366–396, Lecture Notes in Math., **1123**.
[10] Meyer, P. A., Probability and potentials, Blaisdell, 1966.
[11] ——, Processus de Markov: la frontière de Martin, Lecture Notes in Math., **77**, Berlin-Heidelberg-New York, Springer Verlag, 1968.
[12] Mokobodzki, G., Densité relative de deux potentiels comparables, Lecture Notes in Math., **124**, Sem. de Prob. IV (1970), 170–194.
[13] ——, Dualité formelle et représentation des fonctions excessives, Actes du Congrès Int. des Mathematiciens, t. 2, 1970, 531–535.
[14] Walsh, J. B., The cofine topology revisited, Proc. Symp. in Pure Math., **31** (1977), 131–152. AMS, Providence.

Department of Mathematics
Aarhus University
Ny Munkegade, 8000 Aarhus C
Denmark