A GENERALIZATION OF CAUCHY'S DOUBLE ALTERNANT

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1. Introduction. The subject of alternants and alternating functions was widely studied during the last century (cf. Muir [6]). One of the best-known alternants is actually a double alternant (rows and columns) defined by Cauchy [2] in 1841. Cauchy's result may be stated as follows: If $D = [d_{pq}]$, $p, q = 1, \ldots, n$, where $d_{pq} = (x_p + y_q)^{-1}$, then

$$(1) \qquad \det D = \frac{\prod_{1 \le p < q \le n} (x_q - x_p)(y_q - y_p)}{\prod_{1 \le p,\, q \le n} (x_p + y_q)} .$$

This result is used in several recent papers (cf. Hahn [3] and Marcus and Thompson [5]). In this paper we give a generalization (no longer an alternant) of Cauchy's matrix. In [1] Carlson gives bounds on the rank and inertia of Hermitian H which satisfy $R(AH) \geq 0$, of specified rank r. For the case when A is diagonalizable, Cauchy's result may be used to prove that the bounds are best-possible. When A is not diagonalizable, perturbation arguments do not seem to work, and a special case of our result, briefly indicated in § 6 below, was employed in place of Cauchy's result in [1].

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2. <u>Definitions.</u> Let $e_1, e_2, \ldots, e_k, f_1, f_2, \ldots, f_\ell$ be positive integers such that $\sum_{p=1}^{k} e_p = \sum_{q=1}^{\ell} f_q = n$. Let $x_1, \ldots, x_k, y_1, \ldots, y_\ell$ be given complex numbers for which

(2) $x_p + y_q \neq 0$ for all p, q.

We define an $n \times n$ matrix $D = [D_{pq}]$, $p = 1, \ldots, k$; $q = 1, \ldots, \ell$, by defining each D_{pq} as an $e_p \times f_q$ matrix $[d_{ij}(x_p, y_q)]$, $i = 1, 2, \ldots, e_p$; $j = 1, 2, \ldots, f_q$. Here the functions d_{ij} are given by

$$(3) \qquad d_{ij}(x, y) = (-1)^{i+j} \binom{i+j-2}{j-1} (x+y)^{1-i-j}.$$

We illustrate the form of D for $e_1 = 3$, $e_2 = 1$, $f_1 = 1$, $f_2 = 3$.

$$D = \begin{bmatrix} (x_1+y_1)^{-1} & (x_1+y_2)^{-1} & -(x_1+y_2)^{-2} & (x_1+y_2)^{-3} \\ -(x_1+y_1)^{-2} & -(x_1+y_2)^{-2} & 2(x_1+y_2)^{-3} & -3(x_1+y_2)^{-4} \\ (x_1+y_1)^{-3} & (x_1+y_2)^{-3} & -3(x_1+y_2)^{-4} & 6(x_1+y_2)^{-5} \\ (x_2+y_1)^{-1} & (x_2+y_2)^{-1} & -(x_2+y_2)^{-2} & (x_2+y_2)^{-3} \end{bmatrix}$$

We note that if $e_1 = \ldots = e_k = f_1 = \ldots = f_\ell = 1$, we have $D = [d_{11}(x_p, y_q)] = [(x_p+y_q)^{-1}]$, which is Cauchy's double alternant.

3. THEOREM. For D defined above, we have

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$$\text{(4)} \qquad \det D = \frac{\prod_{1 \le p < q \le k} (x_q - x_p)^{e_p e_q} \prod_{1 \le p < q \le \ell} (y_q - y_p)^{f_p f_q}}{\prod_{\substack{1 \le p \le k \\ 1 \le q \le \ell}} (x_p + y_q)^{e_p f_q}} .$$

4. Note. We shall use (without proof; cf. [4], p. 205-206) the formula which follows: For any n-differentiable function f, let $f[x, \ldots, x, z]$ be the n-th divided difference of f with respect to $x, \ldots, x$ (n times), z. Then

$$\text{(5)} \qquad f(z) = \sum_{m=1}^{n-1} (1/m!) f^{(m)}(x)(z-x)^m + f[x, \ldots, x, z](z-x)^n$$

and

$$\text{(6)} \qquad \lim_{z \to x} f[x, \ldots, x, z] = (1/n!) f^{(n)}(x) .$$

5. Proof of Theorem. We shall prove the theorem inductively. For $e_1 = \ldots = e_k = f_1 = \ldots = f_\ell = 1$, and any $x_1, \ldots, x_k, y_1, \ldots, y_\ell$ satisfying (2), the matrix D reduces to Cauchy's double alternant, and (4) reduces to (1) (for a simple proof of (1) see [5], p. 7). Our inductive inference is:

(7) the conclusion (4) holds for $e_1, \ldots, e_k, f_1, \ldots, f_\ell$ $(e_1 > 1)$ and $x_1, \ldots, x_k, y_1, \ldots, y_\ell$ satisfying (2), if it holds for $e_1 - 1, 1, e_2, \ldots, e_k, f_1, \ldots, f_\ell$ and $x_1, z, x_2, \ldots, x_k, y_1, \ldots, y_\ell$ satisfying (2).

Let us see why (7) is enough to make the induction go. First, both sides of (4) are affected the same by rearrangements of rows of D; hence the fact that (7) refers to confluence of the first rows with the e_1th row is no restriction. (7) will prove that any confluence of a single row with a group of other

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rows preserves (4). Second, the columns and rows enter into (4) symmetrically; thus it suffices to prove (7).

Let D denote (in accord with the previous notation) the matrix for which we are to prove (4). Let a matrix agreeing with D in all rows except the e_1th, and having in that row the elements g_{jq} $(q = 1, \ldots, \ell; j = 1, \ldots, f_q)$, be denoted $F(g_{jq})$. Thus $D = F(d_{e_1 j}(x_1, y_q))$. The matrix for which (4) is asserted by the inductive hypothesis is $F(d_{1j}(z, y_q))$. Here d_{ij} is defined by (3). We know $x_1 + y_q \neq 0$ for all q, and we will soon let $z \to x_1$, so we are assuming $z + y_q \neq 0$.

Now d_{ij} is an infinitely differentiable function of its first argument, so let us apply (5) to $d_i(x) \equiv d_{ij}(x, y_q)$ (the dependence on j and y_q is not indicated in the next few equations):

$$(8) \qquad d_i(z) = \sum_{m=0}^{e_1 - 1} m!^{-1} d_i^{(m)}(x_1)(z - x_1)^m + d_i[x_1, \ldots, x_1, z](z - x_1)^{e_1 - 1}.$$

From (3) we compute $d_i' = i d_{i+1}$ and hence by induction

$$(9) \qquad d_1^{(m)} = m!\, d_{m+1}.$$

Substituting (9) in (8), we have

$$d_1(z) = \sum_{m=1}^{e_1 - 1} d_m(x_1)(z - x_1)^{m-1} + d_1[x_1, \ldots, x_1, z](z - x_1)^{e_1 - 1},$$

and from this we obtain

$$(10) \qquad d_1[x_1, \ldots, x_1, z] = d_1(z)(z - x_1)^{1 - e_1} - \sum_{m=1}^{e_1 - 1} d_m(x_1)(z - x_1)^{m - e_1}.$$

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On the other hand, by (6) and (9) we have

$$(11)\quad \lim_{z\to x_1} d_1[x_1,\dots,x_1,z] = ((e_1-1)!)^{-1} d_1^{(e_1-1)}(x_1) = d_{e_1}(x_1) .$$

This completes the preliminaries to relating the determinants of D and $F(d_{1j}(z,y_q))$.

Applying the inductive hypothesis and dividing the e_1th row of $F(d_{1j}(z,y_q))$ by $(z-x_1)^{e_1-1}$, we obtain the following:

$$(12)\quad \det F(d_{1j}(z,y_q)(z-x_1)^{1-e_1}) =$$

$$\frac{\prod_{2\le p<q\le k}(x_q-x_p)^{e_pe_q}\ \prod_{2\le p\le k}(x_p-x_1)^{(e_1-1)e_p}\ \prod_{2\le p\le k}(x_p-z)^{e_p}\ \prod_{1\le p<q\le \ell}(y_q-y_p)^{f_pf_q}}{\prod_{1\le q\le \ell}\left\{(x_1+y_q)^{e_1-1}(z+y_q)\prod_{2\le p\le k}(x_p+y_q)^{e_p}\right\}^{f_q}} .$$

Without affecting this value for the determinant, we can modify the e_1th row of the matrix $F(d_{1j}(z,y_q)(z-x_1)^{1-e_1})$ by subtracting from it $(z-x_1)^{m-e_1}$ times the mth row, for each $m = 1,\dots,e_1-1$. Referring to (10), one sees that we have proved $\det F(d_1[x_1,\dots,x_1,z])$ equals (12). But now let z approach x_1. The matrix, by (11), approaches $F(d_{e_1j}(x_1,y_q)) = D$, while (12) plainly approaches the desired expression (4). This completes the proof.

6. <u>Remark.</u> If $e_p = f_p$ and $y_p = \bar{x}_p$ (in this case, (2) is equivalent to $x_p + \bar{x}_q \neq 0$ for all p and q) then the matrix D is Hermitian. By numbering the blocks of D appropriately

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we can assume for suitable s and t $(0 \leq s \leq t \leq k)$ the following properties:

(13) $\{x_1, \ldots, x_t\}$ is a maximal set of distinct elements of

$x_1, \ldots, x_k$,

(14) $Re(x_p) > 0$ if $1 \leq p \leq s$, $Re(x_p) < 0$ if $s + 1 \leq p \leq t$,

and

(15) for $p \leq t$, $e_p \geq e_q$ if $x_p = x_q$ (necessarily $q \geq t$).

Then it is an easy consequence of (4) and the theorems of [1] that D has $\sum_{p=1}^{s} e_p$ positive and $\sum_{p=s+1}^{t} e_p$ negative eigenvalues.

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