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GENERAL RINGS OF FUNCTIONS

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Introduction

Several authors have studied various types of rings of continuous functions on Tychonoff spaces and have used them to study various types of compactifications (See for example Hager (1969), Isbell (1958), Mrowka (1973), Steiner and Steiner (1970)). However many important results and properties pertaining to the Stone-Čech compactification and the Hewitt realcompactification can be extended to a more general setting by considering appropriate lattices of sets, generalizing that of the lattice of zero sets in a Tychonoff space. This program was first considered by Wallman (1938) and Alexandroff (1940) and has more recently appeared in Alo and Shapiro (1970), Banachewski (1962), Brooks (1967), Frolik (1972), Sultan (to appear) and others.

The purpose of this paper is to give a unified and more general treatment of such rings in a lattice setting. We will work with rings of functions defined on just a set. In addition to generalizing several classical theorems, we apply this method to generalize and unify several of the results appearing in Hager (1969), Isbell (1958), Mrowka (1973), Steiner and Steiner (1970). The advantage of this general lattice method is that we not only deal with several viewpoints at once, but in many cases get simpler proofs of the theorems.

We will use the notion of L-continuity which was presented and studied in detail in Alexandroff (1940).

1. Definitions, Notations and Preliminaries

By a *lattice* of subsets of a set X we will mean a collection of subsets of X closed under finite unions and finite intersections. If in addition the lattice is closed under countable intersections then the lattice is called a *delta lattice*. If a lattice is closed under complements, then it is called a *complemented lattice*. By a *delta paving* we will mean a pair (X, L) where X is a set and L is a delta

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lattice of subsets of X such that $\varnothing, X \in L$. Since our work will be only with delta pavings, many of our definitions will be cast in terms of delta pavings. A delta paving (X, L) is called a *separating delta paving* if whenever $x, y \in X$ and $x \neq y$, there exists an $A \in L$ such that $x \in A$ and $y \notin A$. It is called a *disjunctive delta paving* if whenever $x, y \in X$ and $x \in A$ and $y \notin A$, there exists a $B \in L$ such that $y \in B$ and $B \cap A = \varnothing$. It is called a *normal delta paving* if whenever $A, B \in L$ and $A \cap B = \varnothing$, there exists $C, D \in L$ such that $A \subset X - C$ and $B \subset X - D$ where $X-C$ and $X-D$ are disjoint. If a delta paving is normal, separating, and disjunctive, then it is called a *strongly normal delta paving.* If (X, L) is a delta paving and if f is a real valued function defined on the set X, then we say that f is *L-continuous with respect to the pair* (X, L) if $f^{-1}(C) \in L$ for every closed set C in R. Obviously f is L-continuous with respect to the pair (X, L) if and only if $f^{-1}(-\infty, a]$ and $f^{-1}[b, \infty) \in L$ for every $a, b \in R$. We will denote by $F(X, L)$ the collection of all bounded L-continuous functions with respect to the pair (X, L) and by $F^*(X, L)$ the collection of all the L-continuous functions with respect to the pair (X, L). If f is a real valued function defined on a set X, then $Z(f)$ will denote the *zero set* of f; that is $Z(f) = \{x \in X : f(x) = 0\}$. We will refer to $X - Z(f)$ as the *cozero set of f in X*. A delta paving (X, L) is called a *completely normal delta paving* if and only if $L = \{Z(f); f \in F^*(X,L)\} = \{Z(f) : f \in F(X, L)\}$. If a completely normal delta paving is also a separating and disjunctive delta paving, then we call it a *strongly completely normal delta paving.* We remark that every completely normal delta paving is a normal delta paving and the converse is true if and only if for each $A \in L$ there exists $A_i \in L$, $i = 1, 2, 3, \cdots$ such that $A = \cap_1^\infty (X - A_i)$ where (X, L) is the delta paving (see Alexandroff (1944). A family F of real valued functions defined on a set X is said to be *closed under inversion* if whenever $f \in F$ and $Z(f) = \varnothing$, f is invertible in F. F is said to be *closed under bounded inversion* if whenever $f, g \in F$ and $g(x) \neq 0$ for all $x \in X$, f/g is in F, whenever f/g is bounded. By an *inverse closed algebra* on a set X we mean a family of real valued functions defined on a set X which forms a uniformly closed vector lattice of functions containing constants and closed under multiplication and inversion. If an inverse closed algebra on a set X separates points of X, then we say that it is a *strongly inverse closed algebra.* If A is a family of real valued functions defined on a set X, then by $Z[A]$ we will mean $\{Z(f) : f \in A\}$. If X is a topological space, then we will denote by $C(X)$ the collection of all bounded real valued functions defined on X. In such a case when we refer to the zero sets of X we will mean the collection of all zero sets of X, that is $Z[C(X)]$. If (X, L) is a delta paving, then by an *L-filter* we will mean a nonempty collection of nonempty subsets of L closed under finite intersections and supersets which are in L. By an *L-ultrafilter* we will mean an L-filter which is maximal with respect to the finite intersection property. Each L-filter extends to an L-ultrafilter by Zorn's Lemma, and if we take $W(L)$, the collection of L-ultrafilters, and topologize them with a topology having as a base for the closed sets, sets of the form $V(A) = \{F \in W(L): A \in F\}$ where $A \in L$, $W(L)$ becomes a compact T_1 space. $W(L)$ is T_2 if and only if (X, L) is a normal delta paving (see Brooks (1967), Wallman (1938)). If (X, L) is a separating disjunctive delta paving, then X can be embedded in $W(L)$ as a dense subspace when it carries the relative topology. The embedding map takes each $x \in X$ into the unique L-ultrafilter of supersets of x in L. With the relative topology, $\{A: A \in L\}$ forms a base for the closed sets of X, and is thus a semi-normal base in the sense of Frink (1964). We will always identify X with its image in $W(L)$ when possible. Finally if T is a compactification of a space X, we will denote by $R(T)$ the collection of restrictions of continuous functions on T to X, and by $A(T)$, the smallest strongly inverse closed algebra containing $R(T)$.

Proposition 1. *If (X, L) is a normal delta paving, and if $A, B \in L$ where $A \cap B = \varnothing$, then there exists an $f \in F(X, L)$ where $0 \leqq f \leqq 1$, such that $f(A) = 0$ and $f(B) = 1$.*

Proof. See Alexandroff (1940; page 317).

Proposition 2. (a) *If (X, L) is an arbitrary delta paving, then $F^*(X, L)$ is an inverse closed algebra on X. If A is an inverse closed algebra on a set X, then A consists precisely of the $Z[A]$-continuous with respect to the pair $(X, Z(A])$. Thus there is a 1-1 correspondence between algebras on a set X and completely normal delta pavings (X, L).*

(b) *The correspondence in* (a) *carries strongly completely normal delta pavings onto strongly inverse closed algebras.*

Proof. (a) See Alexandroff (1940; page 317).

(b) Suppose A is a strongly inverse closed algebra on a set X and that $x, y \in X$ where $x \neq y$. Then there exists an f in A such that $f(x) = 0$ and $f(y) = 1$. Thus $x \in Z(f)$ and $y \notin Z(f)$, and it follows that $(X, Z[A])$ is separating. The fact that $y \in Z(f-1)$ and that $Z(f-1) \cap Z(f) = \varnothing$, shows that $(X, Z[A])$ is also disjunctive.

On the other hand, if (X, L) is a strongly completely normal delta paving and $x, y \in X$ with $x \neq y$, then there exist $f, g \in F^*(X, L)$ such that $x \in Z(f)$, $y \in Z(g)$ and with $Z(f) \cap Z(g) = \varnothing$. The function $h = f^2/f^2 + g^2$ is in $F^*(X, L)$ and separates x and y.

2. The Main Theorem

Theorem. *If (X, L) is a strongly normal delta paving, then $F(X, L)$ consists precisely of the restrictions of all the continuous functions on $W(L)$ to X.*

Proof. Let $f \in F(X, L)$ and let K be any compact set containing the range

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of f. If $G \in W(L)$, then $f(G)$ is a filter base in K, and therefore has an adherence point in K. It is easy to see that if a is an adherence point of $f(G)$ and if Z is any zero set neighborhood of a in R, then $f^{-1}(Z) \in G$. It follows from this and the fact that distinct points of R are sparable by disjoint zero set neighborhoods in R, that $f(G)$ has only one adherence point in K and therefore converges. Let $\hat{f}(G) = \lim f(G)$ for any $G \in W(L)$. Then $\hat{f}$ is the unique continuous extension of f to $W(L)$. To see that $\hat{f}$ is continuous, we note that if C is any closed set in R, $C = \cap Z_a$ where the Z_a run through the zero set neighborhoods of C in R. It follows very simply from this that $(\hat{f})^{-1}(C) = \cap \overline{f^{-1}(Z_a)}$, which is closed in $W(L)$. Thus $\hat{f}$ is continuous. Suppose now that $G_1 \neq G_2$ where $G_1, G_2 \in W(L)$. Then there exist $A \in G_1$ and $B \in G_2$ such that $A \cap B = \emptyset$. By Proposition 1 there is an $h \in F(X, L)$ such that $h(A) = 0$ and $h(B) = 1$. Thus $\hat{h}(G_1) = 0$ and $\hat{h}(G_2) = 1$ and it follows that $\{\hat{f} : f \in F(X, L)\}$ separates points of $W(L)$. Since this collection is uniformly closed and contains constants, it follows from the Stone-Weierstrass Theorem, that $\{\hat{f} : f \in F(X, L)\} = C(W(L))$.

3. Consequences

In the rest of the paper if $f \in F(X, L)$ where (X, L) is some strongly normal delta paving, we will denote by $\hat{f}$ its unique continuous extension to $W(L)$. It should be mentioned concerning the corollaries to be presented that Corollaries 3 and 4 generalize Theorem 2.9 and Corollary 2.10 in Steiner and Steiner (1970). Corollary 2(b) is Theorem 2.3 in Steiner and Steiner (1970). Corollaries 6,7 extend Theorem 7.1 and Proposition 7.2 in Hager (1969) and Corollary 9 is the main theorem in Mrowka (1973). In all cases one should compare the proofs given here to the ones in the above quoted papers.

Corollary 1. *If (X, L) and (Y, M) are two strongly normal delta pavings, then $F(X, L)$ and $F(Y, M)$ are isomorphic if and only if $W(L)$ is homeomorphic to $W(M)$.*

Proof. If $F(X, L)$ and $F(Y, M)$ are isomorphic, then by the theorem, so are $C(W(L))$ and $C(W(M))$. The result now follows from the Banach-Stone Theorem (Gillman and Jerrison (1960; page 57)). The converse is trivial.

Corollary 2. (a) *If (X, L) is a strongly normal delta paving, then the trace of the zero sets of $W(L)$ on X is contained in L.*

(b) *If (X, L) is a strongly completely normal delta paving, then the trace of the zero sets of $W(L)$ on X coincides with L.*

Proof. (a) The trace of a zero set in $W(L)$ on X is the zero set of an L-continuous function with respect to the pair (X, L) by the theorem. This is clearly in L by the definition of L-continuity with respect to the pair (X, L).

(b) If (X, L) is a strongly completely normal delta paving and if $A \in L$, then $A = Z(f)$ for some $f \in F(X, L)$. The result follows from the theorem and part (a).

COROLLARY 3. *If* (X, L) *is a strongly normal delta paving, and if* $X \subset T \subset W(L)$, *then if* $L' = \{Z \cap T : Z \in Z[C(WL))]$, *then* $W(L')$ *is homeomorphic to* $W(L)$.

PROOF. If $f \in F(X, L)$, then its continuous extension to T is L'-continuous with respect to the pair (T, L') since $(\hat{f})^{-1}(C) \in Z[C(W(L))]$. On the other hand, if f' is L'-continuous with respect to the pair (T, L'), then its restriction to X is K-continuous with respect to the pair (X, K) where $K = \{Z \cap X : Z \in Z[C(W(L))]\}$. But by Corollary 2(a), $K \subset L$. Thus the restriction of f' to X is L-continuous with respect to the pair (X, L). It follows that $F(X, L)$ and $F(T, L')$ are isomorphic and hence that $W(L)$ is homeomorphic to $W(L')$.

COROLLARY 4. *If* (X, L) *is a strongly normal delta paving and if* S *is any cozero set in* $W(L)$ *containing* X, *then* $\beta S = W(L)$ *where* βS *is the Stone-Čech compactification of* S.

PROOF. Although the statement of the theorem is more general than that of Corollary 2.10 in Steiner and Steiner (1970), the proof is exactly the same.

COROLLARY 5. *If* $T = W(L)$ *where* (X, L) *is a strongly normal delta paving, then* $A(T) = F^*(X, L)$.

PROOF. By the theorem $R(T) = F(X, L)$ hence $A(T) = F^*(X, L)$.

COROLLARY 6. *If* (X, L) *is a strongly completely normal delta paving,* $Z[A(W(L)]$ *is exactly the family of countable intersections of elements of* L.

PROOF. By the previous corollary, we have that $Z[A(W(L))] = Z[F^*(X, L)]$. However by our assumption $Z[F^*(X, L)] = L$, and since L is closed under countable intersections, the proof is complete.

COROLLARY 7. *If* L *is a complemented lattice, and* (X, L) *is a separating disjunctive delta paving, then for each cozero sets in* $W(L)$ *containing* X, $\beta S = W(L)$ *where* βS *is the Stone-Čech compactification of* S.

PROOF. As usual $Z[F(X, L)] \subset L$. On the other hand, if $A \in L$ and if K_A is the characteristic function of A, then since L is complemented, K_A is L-continuous with respect to the pair (X, L). Since $A = Z(1 - K_A)$, we see that $L = Z[F(X, L)]$. It follows that (X, L) is a strongly completely normal delta paving, and the result follows from Corollary 4.

4. Further Consequences

The following corollaries follow directly from the method of proof of the main theorem. In all cases when T is a T_2 compactification of a space X, we will denote by L the trace of the zero sets of T on X. That is $L = Z[R(T)]$.

COROLLARY 8. *Suppose that T is a T_2 compactification of a space X. Suppose that whenever $Z(f)$ and $Z(g)$ are disjoint elements of L there is an $h \in R(T)$, such that $h(Z(f)) = 0$ and $h(Z(g)) = 1$. Then T is homeomorphic to $W(L)$.*

PROOF. Since $R(T)$ is contained in $F(X, L)$, each f in $R(T)$ extends to a continuous $\hat{f}$ defined on $W(L)$. Exactly as in the proof of the main theorem $\{\hat{f} : f \in R(T)\} = C(W(L))$. It follows that $C(T)$ and $C(W(L))$ are isomorphic and thus the corollary follows.

COROLLARY 9. *If T is a T_2 compactification of a space X, and if $R(T)$ is closed under bounded inversion, then T is homeomorphic to $W(L)$.*

PROOF. Suppose that $Z(f)$ and $Z(g)$ are disjoint elements of L. The function $h = f^2/f^2 + g^2$ is in $R(T)$ by our hypothesis and $h(Z(f)) = 0$ while $h(Z(g) = 1$. The result now follows from the previous corollary.

COROLLARY 10. *If T is a T_2 compactification of a space X and if for every cozero set S in T containing X, $\beta S = T$ where βS is the Stone-Čech compactification of S, then T is homeomorphic to $W(L)$.*

PROOF. If $Z(f)$ and $Z(g)$ are disjoint elements of L, then the function h constructed as in the proof of the previous corollary is a bounded continuous function on the cozero set $T - Z(f^2 + g^2)$, and is therefore, by our assumption, continuously extendable to T. Thus h is in $R(T)$, $h(Z(f)) = 0$ and $h(Z(g)) = 1$ and the result follows from Corollary 8.

5. Remarks

According to our main theorem, if (X, L) is a strongly normal delta paving, then $W(L)$ "behaves" like the ordinary Stone-Čech compactification in that every bounded L-continuous function with respect to the pair (X, L) has a unique continuous extension to $W(L)$, and these extensions constitute all of the (bounded) continuous functions on $W(L)$. Corollary 1 strengthens this assertion by generalizing the well known theorem which states that for completely regular Hausdorff spaces X and Y, $C(X)$ is isomorphic to $C(Y)$ iff βX is homeomorphic to βX where βX and βY denote the Stone-Čech compactifications of X and Y respectively. It therefore seems reasonable to call a Wallman compactification arising from a strongly normal delta paving a "generalized Stone-Čech compactification''. Mrowka's β-like compactifications (Mrowka (1973)), and Wallman compactifications arising from nest generated intersection rings (Steiner and Steiner (1970)) are examples of generalized Stone-Čech compactifications. Not only do these compactifications admit the usual algebraic and uniform copletion interpretations, but also admit measure theoretic interpretations. One can consult Sultan (unpublished) for a complete discussion of this. We also remark in passing that any strongly inverse closed algebra provides us with a generalizd Stone-Čech compactification, namely, the Wallman compact space associated with the zero sets of the algebra. We also remark that Corollary 5 can in fact be strengthened to the following: If T is any T_2 compactification of space X, then $A(T) = F^*(X, L)$ where L is the trace of the zero sets of T on X. The proof of this amounts to the realization that $Z[A(T)]$ and $Z[F^*(X, L)]$ coincide, and then using Proposition 2. This observation explains the connection between the work done and those of Hager (1969), Isbell (1958), Steiner and Steiner (1970).

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