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THE ENUMERATION OF NON-ISOMORPHIC 2-CONNECTED PLANAR MAPS

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Since Tutte initiated the systematic enumeration of planar maps [**11**], most of the literature on the subject has dealt with rooted maps (i.e., with maps whose automophism group has been trivialized by distinguishing a doubly-oriented edge). In particular, Tutte proved [**11**] that the number $B'(n)$ of rooted planar 2-connected (i.e., non-separable) maps with $n \geqq 1$ edges is expressed by the formula

$$B'(n) = \frac{2.(3n-3)!}{n!(2n-1)!}.$$

Recently one of the authors developed a general technique for enumerating unrooted planar maps considered up to orientation-preserving isomorphisms (see [**6**] and [**8**]). This technique, which is based on combinatorial map theory, Burnside's lemma [**3**, p. 181] and the concept of a quotient map (see Section 1.4), was used to find, with little algebraic manipulation, simple counting formulae for the numbers of non-isomorphic planar maps of several types [**7**]. In this work we use it to solve the more difficult problem of counting the number $B^+(n)$ of non-isomorphic non-separable planar maps with n edges. It turns out rather unexpectedly that the expression obtained directly by this method may be greatly simplified and reduces to the following simple formula:

$$B^+(n) = \frac{1}{2n}\left[B'(n) + \frac{1}{2}\sum_{\substack{t|n\\ 1\leqq t<n}} \phi\left(\frac{n}{t}\right)(9t^2-9t+2)B'(t)\right] + \begin{cases} \dfrac{n+1}{4}B'\left(\dfrac{n+1}{2}\right), & n \text{ odd},\\[2ex] \dfrac{3n-4}{16}B'\left(\dfrac{n}{2}\right), & n \text{ even},\end{cases}$$

where $\phi(t)$ is the Euler totient function.

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In Section 1 we give a brief sketch of our enumerative method which is used in Section 3 to obtain an initial formula for $B^+(n)$. The derivation is based on some structural properties of non-separable planar maps, which are investigated in Section 2. Then in Section 4 we give an analytical proof of three combinatorial identities for counting rooted maps of a special kind. This enables us to obtain the formula shown above.

It should be noted that N. Wormald [**15**] developed another enumerative method which in principle makes it possible to count planar maps of various types, including non-separable ones, up to all symmetries as well as up to orientation-preserving ones. But in the orientation-preserving case his formulae are far more complicated than ours.

1. Main definitions and the enumerative scheme. For undefined concepts the reader is referred to [**3**], [**2**] and [**10**].

1.1. A (planar) *map* is a 2-cell imbedding of a connected graph (finite, undirected, loops and multiple edges allowed) in the sphere S^+ where the sign $+$ indicates that the sphere has been assigned an orientation. In topological terms, a map is a finite cellular dissection of S^+. The expression $\Gamma = \Gamma(S^+; V, E, F)$ denotes a map Γ which has the sets of vertices V, of (open) edges E and of faces (that is, of open 2-cells) F. We suppose that $E \neq \emptyset$. The vertices, edges and faces of a map are referred to as its *elements* (as in [**2**, p. 386]), and its vertices and faces as its *principal* elements. Two elements are said to be *incident* if one of them lies on the boundary of the other.

An *isomorphism* of two maps means an orientation-preserving homeomorphism of their surfaces which takes each element of one map into an element (of the same type) of the other map. In particular, any automorphism of a map induces an incidence-preserving bijection on the set of its elements. An automorphism is called *trivial* if it takes each element into itself. We do not make a distinction between automorphisms if they differ by a trivial automorphism. And the term "automorphism" will mean a non-trivial automorphism.

We will need the following lemma, which is a well-known fact in combinatorial map theory ([**11**], [**5**], [**4**]).

1.2. LEMMA. *Each map automorphism is defined by its action on the darts* (*or edge-ends*) *and this action is a regular permutation* (*that is, it consists of independent cycles of equal length*).

A dart of a map or graph can be considered as an ordered pair (e, v), where e is an edge, v is a vertex, and e is incident to v. Each edge, even a loop, has 2 darts, and the dart (e, v) is considered to belong to e and to be incident to v. In a map the dart $d = (e, v)$ will be said to be incident to the face incident to e and lying to the right of e as one traverses e away from d.

An *unlabelled map* is an isomorphism class of maps. (It is worth noting that an unlabelled map has no specially distinguished elements or darts.)

1.3. To *root* a map is to distinguish a dart as its root, and an isomorphism of maps which have roots is assumed to take a root into a root. A *rooted map* is an isomorphism class of maps which have roots. In a rooted map the vertex and the face incident to the root are called the *root-vertex* and the *root-face*, respectively.

It follows from Lemma 1.2 (the lemma in Section 1.2) that a rooted map has no (non-trivial) automorphisms. Moreover, an unlabelled map Γ generates $2n/a$ different rooted maps, where n is the number of edges and a is the order of the automorphism group of Γ.

1.4. As is well known [**1**] any non-trivial orientation-preserving homeomorphism of the sphere may be uniquely (up to conjugacy) represented as a geometrical rotation. And each non-trivial rotation of the sphere is uniquely defined by its oriented axis and its rotation angle, which is positive, commensurable with 2π and less than 2π. Therefore any automorphism of a map preserves exactly two elements, which we call *axial*.

Let $\Gamma = \Gamma(S^+; V, E, F)$ be a map and ρ be an automorphism of Γ. We define (as in [**6**], [**7**], [**8**] and [**5**]) the *quotient map*

$$\Delta = \Gamma/\rho = = \Delta(S^+/\rho; V', E', F')$$

for Γ with respect to ρ as the orbit space [**10**, p. 244] of S^+/ρ and its induced map. It is clear that S^+/ρ is the oriented sphere and $E' = E/\rho$ and $F' = F/\rho$ are quotient sets. But for vertices (and also darts) a similar assertion is valid only when neither axial element is an edge. Otherwise (in which case ρ is of order 2), V' is obtained from V/ρ by adding 1 or 2 vertices of degree 1 which are images of the intersection points of the axis with 1 or 2 (open) edges of Γ. Such an additional vertex, as well as the edge and dart incident with it, are called *singular*. Thus, a quotient map is an ordinary map with s vertices of degree 1 distinguished as singular, where $0 \leqq s \leqq 2$.

With respect to Γ and ρ singular vertices of Δ and also the images of principal axial elements of Γ are called *axial elements* of the quotient map Δ. A quotient map will be called *punctured* if we want to emphasize its axial elements. It is worth noting that in a punctured quotient map the axial elements are always principal (they may be singular vertices).

We adopt the rule that no singular vertex can serve as a root of a rooted map.

Geometrically, when Γ is drawn so that ρ is a rotation of the sphere [**9**] the map Γ/ρ is obtained in the following way. We cut out a sector bounded by two great semi-circles whose diameter is the axis and which form an angle of $2\pi/l$ (where $l = l_\rho$ is the order of ρ) and then paste these semi-circles together at corresponding points.

Clearly when Γ is transformed to Γ/ρ the degrees of the principal axial elements are divided by l_ρ and the degrees of the non-axial elements are unchanged.

1.5. The natural transformation

$$\prod : \Gamma \rightarrow \Gamma/\rho$$

has a uniquely defined inverse called a *lifting*. More precisely, for every $l > 1$ every punctured map Δ (where $l = 2$ if Δ has singular vertices) may be uniquely lifted into a map denoted by $\lambda_l(\Delta) = \Gamma$ such that Γ has a corresponding rotation automorphism ρ of order l for which $\Gamma/\rho = \Delta$ [**8**].

Given $l > 1$ and a puncture map Δ with two (principal) axial elements either of which may be a singular vertex (in which case $l = 2$), the lifting $\lambda_l(\Delta) = \Gamma$ may be geometrically constructed as follows.

1) Fix two poles (i.e., opposite points) on the geometric sphere and draw Δ so that each of them coincides with an axial vertex or lies inside an axial edge or face (not the same element for both poles).

2) Cut Δ along an arbitrarily chosen semi-circle whose ends are the poles (for the sake of simplicity we might draw Δ so that the cut intersects any non-axial face in a segment and any edge in a point) and then push back the sides of the cut, shrinking the cut sphere uniformly into a sector (or lune) of angle $2\pi/l$.

3) Take l copies of this "sector-map", some elements of which are broken by the cut, put them on a sphere (of the same diameter) one after another in cyclic order and paste the neighbouring sides together (the right side of the first sector with the left side of the second and so on), identifying the corresponding points. As may be easily seen, this results in a map on the sphere formed by all the unbroken elements of our l "sector-maps" and by the broken elements (including vertices lying on the cut) pasted together.

4) The map thus constructed is the required Γ if Δ has no singular

vertices. Otherwise, Γ is obtained from it by replacing each axial vertex arising from a singular vertex of Δ and both edges incident to it by a single edge (this is possible because in this case $l = 2$, singular vertices are of degree 1 and no edge can be incident to 2 singular vertices). One can now naturally construct a rotation automorphism ρ of Γ such that $\Gamma/\rho = \Delta$. All such automorphisms are the primitive elements of a cyclic group of order l.

If X is a set of elements of Δ, $\lambda_l(X)$ will denote the set of elements of Γ lifted from them.

1.6. Now we give a sketch of the enumerative scheme which we use (see [**6**] and [**8**]). Let the symbol $\mathfrak{M} = \mathfrak{M}(n)$ denote an arbitrary set of n-edged maps. We suppose that $\mathfrak{M}$ consists of unlabelled maps, but when necessary the set of all rooted maps arising from them will be denoted by the same symbol, and this will be separately specified. Suppose a *classification* W of axes of (non-trivial) map automorphisms to be given which is *complete* for $\mathfrak{M}$: that is, for any $\Gamma \in \mathfrak{M}$ and any automorphism ρ of Γ there exists a unique class $\omega \in W$, which is designated as $\omega = \tau(\Gamma, \rho)$, to which the axis of ρ belongs. This class $\tau = \tau(\Gamma, \rho)$ is called the *type* of ρ. Thus W is a certain partition of the (finite) set of axes of all the automorphisms of all the maps in $\mathfrak{M}$ (or, more generally, of all the n-edged maps in some superset of $\mathfrak{M}$). We assume that W is a refinement of the partition of this set of axes into 3 classes: I, T and H, where an axis is in I, T or H according as it passes through 0, 1 or 2 map edges, respectively, and thus we write

$$W = \{I_\lambda, T_\mu, H_\nu\}_{\lambda \in \Lambda, \mu \in M, \nu \in N},$$

where Λ, M and N are suitably chosen finite sets.

Sometimes it is more convenient to define W in terms of quotient maps.

Now let $\mathfrak{M}_{\omega,l} = \mathfrak{M}_{\omega,l}(t)$ denote the set of all maps which are (unpunctured) quotient maps for maps from $\mathfrak{M}$ with respect to automorphisms of type $\omega \in W$ and of order $l > 1$. Clearly for $\omega = T_\mu$ or H_ν only $l = 2$ makes sense, so that in both cases we shall drop the subscript l and assume that $\mathfrak{M}_{T_\mu} = \mathfrak{M}_{T_\mu,2}$ and $\mathfrak{M}_{H_\nu} = \mathfrak{M}_{H_\nu,2}$.

By the definition of quotient maps, all maps in $\mathfrak{M}_{\omega,l}$ have t edges, where t satisfies the following relationships:

$$(1)\qquad \begin{aligned} lt &= n &&\text{if } \omega = I_\lambda,\\ 2t &= n + 1 &&\text{if } \omega = T_\mu,\\ 2t &= n + 2 &&\text{if } \omega = H_\nu. \end{aligned}$$

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Now we impose the following combinatorial restriction.

1.7. *Invariance condition.* For each ω and l and for each (unpunctured) map Δ from $\mathfrak{M}_{\omega,l}(t)$ the number of ways to choose an axis such that the lifting on Δ of order l with respect to this axis gives rise to a map from $\mathfrak{M}$ with the corresponding automorphisms of type $\tau = \omega$ does not depend on Δ and depends only on ω and t (or l). This number will be denoted by $\Psi(\omega, t) = \Psi_{\mathfrak{M}}(\omega, t)$.

Of course, $\Psi(H_\nu, t) = 0$ or 1: the axis is uniquely defined by the singular vertices.

Now we may formulate the general result.

1.8. THEOREM [**6**], [**8**]. *Under the invariance condition the number $M^+(n)$ of non-isomorphic (unlabelled) maps in $\mathfrak{M} = \mathfrak{M}(n)$ is expressed by the formula*

$$(2)\quad nM^+(n) = M'(n) + \sum_{l|n,l>1} \phi(l) \sum_{I_\lambda \in W} \Psi\left(I_\lambda, \frac{n}{l}\right) M'_{I_\lambda,l}\left(\frac{n}{l}\right)$$

$$+ \begin{cases} \displaystyle\sum_{T_\mu \in W} \Psi\left(T_\mu, \frac{n+1}{2}\right) M'_{T_\mu}\left(\frac{n+1}{2}\right), & n \text{ odd,} \\ \displaystyle\sum_{H_\nu \in W} \Psi\left(H_\nu, \frac{n+2}{2}\right) M'_{H_\nu}\left(\frac{n+2}{2}\right), & n \text{ even,} \end{cases}$$

where $W = \{I_\lambda, T_\mu, H_\nu\}$ is a classification of map automorphism axes which is complete for $\mathfrak{M}$, ϕ is the Euler function, and $M'(n)$ and $M'_{\omega,l}(n/l)$ denote the numbers of rooted maps in $\mathfrak{M}(n)$ and $\mathfrak{M}_{\omega,l}(n/l)$ respectively (for $\omega = T_\mu$ or H_ν the root is non-singular).

2. Non-separable maps and their quotient maps.

2.1. A map is called *separable* if its edge-set can be partitioned into two non-null subsets such that there is exactly one vertex incident with an edge in each subset; otherwise a map is called *non-separable* or 2-*connected* [**11**]. It is clear that a non-separable map has no loops or isthmuses (an isthmus is an edge both of whose darts belong to the same face) unless it is one-edged. In the latter case there are two different non-separable maps: the link-map and the loop-map.

It is well known that any graph admits a unique decomposition into edge-disjoint 2-connected subgraphs called its *blocks*. This notion is

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naturally extended to maps: the blocks of a map are its submaps corresponding to the blocks (including loops) of its underlying graph (1-skeleton). A vertex or a face of a map belongs to a block if it is incident to an edge of this block. It is referred to as *internal* for some block if it belongs only to this block; otherwise this principal element is said to be *separating* or *external* for all the blocks containing it.

Two blocks of any map may share at most one (external) vertex and, by duality, at most one face. If they share both an external vertex v and an external face f (in which case the two blocks are called *adjacent*) then v and f must be incident. Thus the concept of the block-cutpoint tree of a connected graph [**3**, p. 37] can also be extended to maps, where a cutpoint is now called a *separator* and is an incident face-vertex pair shared by more than one block and is adjacent in the tree to all the blocks which share it. The end-vertices of the tree all correspond to blocks of the map, called its *end-blocks*. Clearly this tree must have a centre and not a bi-centre.

2.2. *Definition*. A *series map* or *s-map* is a separable map whose block-separator tree is a chain. The end-blocks of this chain are called *extremal* and the other blocks are called *internal*.

We need the following three properties of s-maps.

(1) If B is a block of a map Γ and if it contains a separating vertex v then B is adjacent to at least one block containing v with which it shares an external face f incident to v. The same with f and v reversed is valid. Therefore in a series map Γ each extremal block contains a unique separating vertex and a unique separating face and each non-extremal block contains at most two such vertices and faces. For the same reason at least one of the two separating elements of an extremal block is shared by only two blocks.

(2) Property (1) defines the general construction of s-maps. Namely we take any non-separable map B and distinguish a vertex v and a face incident to v. Then we take any map Δ which is either an s-map or a non-separable map and again distinguish a vertex and a face incident to it, which in the case of an s-map belong to one of the extremal blocks and not both are separating. Then we paste B and Δ together, identifying the distinguished faces with each other and similarly the distinguished vertices (topologically this is done by cutting small circular holes inside the distinguished faces which touch the boundaries only in the distinguished vertices and then pasting both maps together by the boundaries of the holes).

(3) This third party property is as follows. If v is a vertex of degree 1 belonging to an s-map, then v belongs to a block B which is a link-map and is one of the extremal blocks (in fact B is a link-map and contains no internal faces and only one external face), for if B shared this face and its other vertex with two blocks B_1 and B_2, then B, B_1 and B_2 would be pairwise adjacent, which would contradict the definition of an s-map. The same (with the roles of faces and vertices exchanged) is valid for a 1-gonal face; that is, a face bounded by a loop.

2.3. PROPOSITION. *Let Γ be any map other than a link-map or a loop-map and let Δ be the quotient map of Γ with respect to any (non-trivial) automorphism ρ of Γ. Then Γ is non-separable if and only if Δ is of one of the following two forms*:

1) *Δ is a (punctured) non-separable map with an arbitrary pair of (principal) axial elements other than a vertex v and a face incident to v*;
2) *Δ is a (punctured) s-map one of whose axial elements is an internal element of one of its extremal blocks and the other axial element is an internal element of the other extremal block.*

The "only if" part of this proposition will follow from Lemmas 2.4 and 2.5 and the "if" part from Lemma 2.6.

2.4. LEMMA. *If Γ is non-separable then every end-block of Δ (if Δ is separable) has an internal axial element.*

Proof. Suppose that an end-block of Δ has no internal axial element. Then Δ can be drawn so that a cut whose ends are the axial elements does not intersect with any internal element of that end-block. Then when the geometrical lifting is performed, copies of that end-block are end-blocks of Γ.

2.5. LEMMA. *If Γ is non-separable then the axial elements of Δ are not incident with each other.*

Proof. If the axial elements are a vertex and a face incident to each other, the geometrical lifting of Δ can be performed using a cut which intersects no non-axial elements. Therefore the axial vertex if Δ is lifted to a separating vertex of Γ if it is non-singular, and corresponds to an isthmus of Γ otherwise.

2.6. The "only if" part of Proposition 2.3 follows from Lemmas 2.4 and 2.5 since the only separable maps in which every end-block has an internal axial element are s-maps. The converse is contained in the following:

LEMMA. *If Γ is separable and the axial elements are not incident, then Δ is separable and has an end-block with no internal axial element.*

Proof. We first show that there exists a unique block B of Γ invariant under ρ. Consider the automorphism ρ' of the block-separator tree T of Γ induced by ρ. Since Γ is separable, T has at least one "separator-vertex" (i.e., one representing a separator of Γ); and since the axial elements of Γ under ρ are not incident, ρ' can fix no separator-vertex of T: otherwise the corresponding incident face and vertex of Γ would be axial. Therefore ρ' fixes exactly one element of T: its centre, which must correspond not to a separator but to a block of Γ. Thus B is the unique block of Γ fixed by ρ.

Aside from B, none of the blocks of Γ can have an internal axial element; so we can cut out a sector both sides of which do not intersect any block except B. Such a sector will include part of B and one entire block from each orbit of blocks under ρ' not containing B. So Δ will have at least one block B' obtained from B and one block obtained from each orbit not containing B. Thus Δ is separable and must therefore have at least two end-blocks. Of all the blocks of Δ, only B' can have an internal axial element; so at least one of the endblocks of Δ has no internal axial element. This completes the proof of the lemma and of Proposition 2.3.

3. An initial formula for the number of non-separable maps.

3.1. Now we apply the general enumerative scheme of Theorem 1.8 to non-separable maps using Proposition 2.3. Let $\mathfrak{B} = \mathfrak{B}(n)$ denote the set of non-separable maps with n edges. An appropriate classification $W = W_B$ of the axes of map automorphisms is

$$W_B = \{I_0, I_{a,b}, T_a, H\}_{a \geqq b \geqq 1},$$

which we will show to be complete for $\mathfrak{B}$ and to satisfy invariance condition 1.7. Here I_0 corresponds to those automorphisms with respect to which the quotient maps are non-separable, and $I_{a,b}$, T_a and H to those with respect to which the quotient maps are s-maps whose extremal blocks have, respectively, a and b non-singular edges, a non-singular edges and one singular edge, and one singular edge apiece. By Proposition 2.3 this classification exhausts all possible cases; i.e., W_B is complete for $\mathfrak{B}$.

As was shown in [**11**], in a non-separable map with $t > 1$ edges each vertex is incident to each face at most once. Therefore any vertex is incident with exactly as many faces as edges. Summing over all vertices we obtain $2t$ incident vertex-face pairs (this is valid for $t = 1$ too). But there are $\binom{t+2}{2}$ pairs of different principal elements in all because by the Euler formula ($|V| - |E| + |F| = 2$) in a map with t edges there are $t + 2$ principal elements. Thus if $\Delta \in \mathfrak{B}_{I_0,l}$ then it has $t = n/l$ edges and admits $\binom{t+2}{2} - 2t$ different axes which lead to non-separable liftings. In other words, the invariance condition holds for I_0 and

$$\Psi_{\mathfrak{B}}(I_0, l) = \binom{t+2}{2} - 2t, \quad t = n/l.$$

For the case of $I_{a,b}$, by Proposition 2.3 the axial elements of a quotient map are arbitrary internal principal elements of distinct extremal blocks. Such a block with a edges contains a internal principal elements: from the total number $a + 2$ we must subtract one separating vertex and one separating face (see 2.2). Therefore

$$\Psi(I_{a,b}, l) = ab.$$

Similarly,

$$\Psi(T_a, 2) = a,$$

$$\Psi(H, 2) = 1.$$

Hence the invariance condition holds for all cases.

Let $\mathfrak{C}(a, b; t)$ denote the set of t-edged s-maps whose extremal blocks have a and b edges. Then

$$\mathfrak{B}_{I_{a,b},l} = \mathfrak{C}(a, b; \frac{n}{l}),$$

$$\mathfrak{B}_{T_a} = \mathfrak{C}_*\left(a, 1; \frac{n+1}{2}\right) \quad \text{if } n \text{ is odd, and}$$

$$\mathfrak{B}_H = \mathfrak{C}_{**}(1, 1; \frac{n+2}{2}) \quad \text{if } n \text{ is even,}$$

where $\mathfrak{C}_*$ and $\mathfrak{C}_{**}$ are the same sets of s-maps as $\mathfrak{C}$ but with one and two vertices of degree 1 distinguished as singular. Let $C'(a, b; t)$, $C'_*(a, 1; t)$ and $C'_{**}(1, 1; t)$ be the corresponding numbers of rooted maps, $B'(n)$ be the number of rooted maps in $\mathfrak{B}(n)$ and $B^+(n)$ be the number of non-isomorphic maps in $\mathfrak{B}(n)$. Note that $\mathfrak{B}_{I_0,l} = \mathfrak{B}(n/l)$. Substituting all these expressions into the general formula (2) we obtain the following result.

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3.2 THEOREM.

$$(3)\qquad 2nB^{+}(n) = B'(n) + \frac{1}{2}\sum_{\substack{t|n\\ 1\leqq t<n}} \phi\left(\frac{n}{t}\right)(t^2 - t + 2)B'(t)$$

$$+ \sum_{\substack{t|n\\ 1<t<n}} \phi\left(\frac{n}{t}\right) \sum_{\substack{1\leqq b\leqq a\\ a+b\leqq t}} abC'(a, b; t)$$

$$+ \begin{cases} \sum_{a\geqq 1} aC'_*\left(a, 1; \dfrac{n+1}{2}\right), & n \text{ odd},\\ C'_{**}\left(1, 1; \dfrac{n+2}{2}\right), & n \text{ even}. \end{cases}$$

3.3. In this section we express C'_* and C'_{**} in terms of C'.

1) We begin with the case $C'_*(a, 1; n)$ for $a > 1$. Each map in $\mathfrak{C}(a, 1; n)$ has a unique vertex or face of degree 1. Consequently by Lemma 1.2 it has no automorphisms and admits $2n$ different rootings. We may partition the set $\mathfrak{C}(a, 1; n)$ into pairs, in which unlabelled maps in a given pair differ only in that one has a link-map and the other has a loop-map as the extremal 1-edged block, and both have the same incidence structure. Each pair contributes $4n$ to $C'(a, 1; n)$. Only the first map in any pair may be turned into a map from $\mathfrak{C}_*(a, 1; n)$ by declaring the vertex of degree 1 to be singular. In general such a map obtained by declaring one or two vertices to be singular is called *derived* (for the corresponding map without singular vertices). Any map in $\mathfrak{C}_*(a, 1; n)$ admits $2n - 1$ rootings (it can be rooted in every dart except for the singular one). As our derived maps exhaust $\mathfrak{C}_*(a, 1; n)$ we have

$$(4)\qquad C'_*(a, 1; n) = \frac{2n-1}{4n} C'(a, 1; n), \quad n > a > 1.$$

2) Now let $a = 1$. There are three types of maps in $\mathfrak{C}(1, 1; n)$ which differ only in the pair of extremal blocks; two link-maps, a link-map and a loop-map, or two loop-maps. These maps are said to be *ii-*, *io-*, or *oo-maps*, respectively. Two maps of different types in $\mathfrak{C}(1, 1; n)$ are called *related* if they differ only by their extremal blocks and have the same incidence structure; i.e., one may be obtained from the other by replacing one or both its extremal blocks which was a link-map by a loop-map or vice-versa.

If Γ is an ii-map it either has no automorphisms or has a unique automorphism which transposes the extremal "link-blocks" and is of order 2. Both cases require separate considerations.

a) Let Γ be an unlabelled ii-map without automorphisms. Then Γ has two different related io-maps Γ_1 and Γ_2 which are obtained from it by replacing one or the other extremal link-map by a loop-map. Moreover, Γ has one related oo-map Γ_0. Each of these maps has no automorphisms and admits $2n$ rootings. Hence Γ and its related maps contribute $8n$ to $C'(1, 1; n)$. Similarly there are two derived maps with one singular vertex for Γ, only one such map for each of Γ_1 and Γ_2 and no derived map for Γ_0. Each of these 4 derived maps admits $2n - 1$ rootings. So they contribute $4(2n - 1)$ to $C'_*(1, 1; n)$. Finally for Γ there exists a (unique) derived map with two singular vertices and it contributes $2n - 2$ to $C'_{**}(1, 1; n)$, and for Γ_1, Γ_2 and Γ_0 there exist no derived maps.

As a result we have the following proportions of the numbers of rooted maps which arise from an ii-map without automorphisms and from its related maps, and have 0, 1, or 2 singular vertices, respectively (all numbers have been divided by 2):

$$(5) \qquad 4n:2(2n - 1):(n - 1).$$

b) Let Γ be an ii-map with a unique automorphism of order 2. Now Γ has one related oo-map Γ_0 and only one related io-map Γ_1 since its extremal blocks are indistinguishable. Γ admits $\frac{1}{2} \cdot 2n = n$ different rootings and the same is valid for Γ_0, but Γ_1 has no automorphisms and admits $2n$ rootings. All together they contribute $n + n + 2n = 4n$ to $C'(1, 1; n)$. By the same argument there is a unique derived map with one singular vertex for Γ and it has no automorphisms. The same holds for Γ_1, while Γ_0 has no derived maps. All together both derived maps contribute $2(2n - 1)$ to $C'_*(1, 1; n)$. Again only Γ has a (unique) derived map with two singular vertices which admits $\frac{1}{2}(2n - 2) = n - 1$ rootings since declaring both vertices of degree 1 to be singular preserves the symmetry. Thus we have the same proportions (5) of the contributions.

When Γ runs through all ii-maps with n edges, it and its related maps run through $\mathfrak{C}(1, 1; n)$ without repetitions and their derived maps run through $\mathfrak{C}{*}(1, 1; n)$ and $\mathfrak{C}_{**}(1, 1; n)$. Hence the same proportions (5) hold for the numbers of rooted maps in these three sets. This proves the following relationships.

$$(6) \qquad C'_*(1, 1; n) = \frac{2n - 1}{2n} C'(1, 1; n),$$

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$$(7) \qquad C'_{**}(1, 1; n) = \frac{n-1}{4n} C'(1, 1; n).$$

Substituting expressions (4), (6) and (7) into (3) we obtain (after simple manipulations) the following result.

3.4. PROPOSITION.

$$(8) \qquad B^+(n) = \frac{1}{2n} \Bigg[B'(n) + \frac{1}{2} \sum_{\substack{t \mid n \\ 1 \leqq t < n}} \phi\left(\frac{n}{t}\right) (t^2 - t + 2) B'(t)$$

$$+ \sum_{\substack{t \mid n \\ 1 < t < n}} \phi\left(\frac{n}{t}\right) \sum_{\substack{a \geq b \geq 1 \\ a + b \leqq t}} abC'(a, b; t) \Bigg]$$

$$+ \begin{cases} \dfrac{1}{2(n+1)} C'\left(1, 1; \dfrac{n+1}{2}\right) \\ \qquad\qquad + \dfrac{1}{4(n+1)} \displaystyle\sum_{a \geqq 2} aC'\left(a, 1; \dfrac{n+1}{2}\right), & n \text{ odd}, \\ \dfrac{1}{8(n+2)} C'\left(1, 1; \dfrac{n+2}{2}\right), & n \text{ even}. \end{cases}$$

4. Enumeration of series maps. The aim of this section is to prove the following three identities.

4.1. PROPOSITION. *For* $n \geqq 2$,

$$(9) \qquad C'(1, 1; n) = 2n(3n - 5) B'(n - 1);$$

$$(10) \qquad \sum_{a \geqq 2} aC'(a, 1; n) + 2C'(1, 1; n) = 4n^2 B'(n);$$

$$(11) \qquad \sum_{a \geqq b \geqq 1} abC'(a, b; n) = 4n(n - 1) B'(n),$$

where for $n \geqq 1$

$$(12) \qquad B'(n) = \frac{2.(3n - 3)!}{n!(2n - 1)!}$$

is the number of rooted n-edged non-separable maps.

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4.2. *Proof.* 1) An *initial* s-map is a rooted s-map such that the root belongs to one of the two extremal blocks and is not incident to the external face of that block. We first count initial s-maps.

An initial s-map Γ may be constructed as follows. Take any rooted non-separable map B and choose any incident face-vertex pair other than the pair "root-face, root-vertex". Take any rooted non-separable map or initial series map Γ_1 and choose the pair "root-face, root-vertex". Identify these two pairs by property (2) of Section 2.2. It is clear that for any initial map Γ the maps B and Γ_1 are uniquely defined. If B has n_1 edges it has $2n_1$ darts and, since it is a block, exactly $2n_1 - 1$ incident face-vertex pairs other than the pair "root-face, root-vertex". So if we let $f(n_1) = B'(n_1)$ and $f(x_1, x_2, \dots, x_k)$ be the number of initial s-maps with $k \geqq 2$ blocks, where the i-th block has n_i edges and the root is in the first block, then

$$(13)\quad f(n_1, n_2, \dots, n_k) = (2n_1 - 1)\, B'(n_1) f(n_2, \dots, n_k) = \dots$$
$$= (2n_1 - 1)(2n_2 - 1) \dots (2n_{k-1} - 1) B'(n_1) B'(n_2) \dots B'(n_k).$$

Therefore the number of initial s-maps with $a + b + c$ edges in which the extremal blocks have a and b edges and the root belongs to the block with a edges is

$$(14)\quad (2a - 1)B'(a)B'(b) \sum_{n_2 + \dots + n_{k-1} = c} \prod_{i=2}^{k-1} (2n_i - 1)B'(n_i).$$

If we let

$$\beta(x) = \sum_{n=1}^{\infty} B'(n)x^n,$$

then

$$\sum_{n=1}^{\infty} (2n - 1)B'(n)x^n = 2x\beta'(x) - \beta(x),$$

where $\beta'(x)$ means $d\beta(x)/dx$, and the sum in (14) is the coefficient of x^c in $[1 - (2x\beta'(x) - \beta(x))]^{-1}$. It was shown in [**11**] that

$$(15)\quad \beta(x) = 2t - 3t^2,$$

where $t = x(1 - t)^{-2}$ so that

$$(16)\quad x = t(1 - t)^2.$$

Hence

(17) $\beta'(x) = \dfrac{d\beta}{dx} = \dfrac{d\beta/dt}{dt/dx} = 2(1 - t)^{-1}$,

and so

(18) $[1 - (2x\beta'(x) - \beta(x))]^{-1} = (1 - t)^{-2}$.

By Lagrange's inversion formula [**14**],

$$(1 - t)^{-2} = 1 + \sum_{c=1}^{\infty} \frac{x^c}{c} \cdot \mathrm{coef}_{t^{c-1}}\,[2(1 - t)^{-(2c+3)}].$$

But

$$2(1 - t)^{-(2c+3)} = \sum_{i=0}^{\infty} \frac{x^c \cdot 2(3c + 1)!}{(2c + 2)!\, c\,!},$$

in which the coefficient of x^c is $\dfrac{3c + 1}{2}$ $\mathrm{B}'(c + 1)$. Hence (14) is equal to

$$(2a - 1)\mathrm{B}'(a)B'(b)\,\frac{3c + 1}{2}\,B'(c + 1).$$

The number of rooted s-maps, not necessarily initial, with $a + b + c$ edges in which the extremal blocks have a and b edges and the root belongs to the block with a edges is

(19) $2aB'(a)B'(b)\,\dfrac{3c + 1}{2}B'(c + 1)$:

the only difference with initial maps is that we not exclude the pair "root-face, root-vertex" in B.

2) Let Γ be an unrooted s-map with $a + b + c$ edges in which the extremal blocks have a and b edges.

If Γ has no non-trivial automorphisms then it admits $2(a + b + c)$ distinct rootings. Of these, the number with the root in an internal block is $2c$, and the number with the root in an extremal block having a edges is $2a$ if $a \neq b$, and $4a$ if $a = b$.

If Γ has a non-trivial automorphism, then $a = b$ and Γ admits $2a + c$ distinct rootings. Of these, the number with the root in an internal block is c and the number with the root in an extremal block with a edges is $2a$.

In either case, the ratio of the number of rooted s-maps with the root in an internal block to the number of rooted s-maps with the root in an extremal block having a edges is c/a if $a \neq b$, and $c/(2a)$ if $a = b$. Hence, from (19), the number of rooted s-maps with $a + b + c$ edges whose extremal blocks and a and b edges and whose root lies in an internal block is

$$(20)\quad B'(a)B'(b)\frac{c}{2}(3c + 1)\, B'(c + 1)\cdot\begin{cases} 2 \text{ if } a \neq b \\ 1 \text{ if } a = b. \end{cases}$$

3) To find the number $C'(a, b; n)$ we add (19) to (20), but if $a \neq b$ we also add (19) with a and b interchanged, since the root may also belong to the extremal block with b edges. We get

$$B'(a)\, B'(b)\,\frac{3c + 1}{2}\, B'(c + 1)\cdot\begin{cases} 2c + 2a + 2b \text{ if } a \neq b \\ c + 2a \text{ if } a = b, \end{cases}$$

and putting $c = n - a - b$, we obtain the equation

$$(21)\quad \begin{aligned} C'(a, b; n) &= n\cdot\frac{3(n - a - b) + 1}{2}\cdot \\ B'(a)\, B'(b)\, B'(n - a - b + 1)&\cdot \begin{cases} 2 \text{ if } a \neq b \\ 1 \text{ if } a = b. \end{cases} \end{aligned}$$

Setting $a = b = 1$ in (21) we get (9) directly.
Also,

$$\left[\sum_{a\geqq 2} aC'(a, 1; n)\right] + 2C'(1, 1; n)$$

$$= \frac{2\cdot 2n}{2}\sum_{a=1}^{n-1} aB'(a)[3(n - a) - 2]\, B'(n - a)$$

$$= 6n\sum_{a=1}^{n-1} aB'(a)(n - a)\, B'(n - a) - 4n\sum_{a=1}^{n-1} aB'(a)\, B'(n - a)$$

$$= 6n\cdot \operatorname{coef}_{x^n}[x\beta'(x)]^2 - 4n\cdot\operatorname{coef}_{x^n}[x\beta'(x)\,\beta(x)].$$

Using (15), (16) and (17) and simplifying we see that this is equal to

$$n\cdot\operatorname{coef}_{x^n}[8t^2(1 - t)].$$

By Lagrange's formula, this is

$$8\cdot\operatorname{coef}_{t^{n-1}}[(2t - 3t^2)(1 - t)^{-2n}],$$

which simplifies to (10).

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Finally,

$$\sum_{\substack{a \geqq b \geqq 1 \\ a+b \leqq n}} abC'(a, b; n)$$

$$= n \cdot \sum_{\substack{a, b \geqq 1 \\ a+b \leqq n}} aB'(a)\, bB'(b) \cdot \frac{3(n - a - b) + 1}{2} \cdot B'(n - a - b + 1)$$

$$= n \cdot \operatorname{coef}_{x^{n+1}} \left[(x\beta'(x))^2 \cdot \frac{3x\beta'(x) - 2\beta(x)}{2} \right]$$

$$= n \cdot \operatorname{coef}_{x^{n+1}} [4t^3(1 - t^2)] = n \cdot \operatorname{coef}_{x^n} [4t^2] \quad \text{(by (16))}.$$

By Lagrange's formula, the coefficient of x^n in t^2 is

$$\frac{2}{n} \cdot \operatorname{coef}_{t^{n-2}} [(1 - t)^{-2n}] = \frac{2}{n} \binom{3n - 3}{n - 2} = (n - 1)\, B'(n),$$

which yields (11).

5. The result. Simplifying (8) with the aid of expressions (9), (10) and (11) we get the final result.

5.1. THEOREM. *The number $B^+(n)$ of non-isomorphic (up to orientation-preserving isomorphisms) non-separable planar maps with $n \geqq 1$ edges is expressed by the following formula*:

$$(22) \quad B^+(n) = \frac{1}{2n} \Big[B'(n) + \frac{1}{2} \sum_{\substack{t \mid n \\ 1 \leqq t < n}} \phi\left(\frac{n}{t}\right) (9t^2 - 9t + 2)\, B'(t)\Big] + \begin{cases} \dfrac{n + 1}{4} B'\left(\dfrac{n + 1}{2}\right), & n \text{ odd}, \\[2ex] \dfrac{3n - 4}{16} B'\left(\dfrac{n}{2}\right), & n \text{ even}, \end{cases}$$

where $B'(n) = 2 \cdot (3n - 3)!/n!(2n - 1)!$ is the number of rooted non-separable n-edged planar maps.

The values of $B'(n)$ and $B^+(n)$ for $1 \leqq n \leqq 20$ are given in the table in figure 1.

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The values of $B^{+}(n)$ for $n \leqq 10$ were independently checked by generating the maps by computer [**12**].

5.2. *Remark.* Identities (9), (10) and (11) seem very surprising and it would be interesting to find direct combinatorial interpretations of them. It is especially intriguing that $B^{+}(n)$ proved to be expressed in terms of the values of $B'(t)$, $t \leqq n$, alone by such a simple formula as (22), even though the general scheme leads to using the auxiliary function $C'(a, b; n)$. Is there a more direct proof of (22)?

The results obtained here and in [**6**], [**7**], [**8**] have been used to count other sets of non-isomorphic planar maps including 3-connected maps [**13**].

n	$B'(n)$	$B^{+}(n)$
1	2	2
2	1	1
3	2	2
4	6	3
5	22	6
6	91	16
7	408	42
8	1938	151
9	9614	596
10	49335	2605
11	260130	12098
12	1402440	59166
13	7702632	297684
14	42975796	1538590
15	243035536	8109078
16	1390594458	43476751
17	8038677054	236474942
18	46892282815	1302680941
19	275750636070	7256842362
20	1633292229030	40832979283

FIGURE 1

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