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# On invariant measures for simple branching processes (Summary)

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Problems pertaining to invariant measures of a non-critical Galton-Watson process, whether with or without immigration, may be discussed in terms of measures of a subcritical process with a possibly defective immigration distribution. There is in fact only one such measure satisfying a regular variation condition. This result provides a unifying principle for several contexts of Galton-Watson theory. A full discussion with analytical details will appear elsewhere.

## 1. The central result

Let $F(s)$ and $B(s)$ be probability generating functions (p.g.f.'s) on the non-negative integers. We assume that $F(1-) = 1$ and $0 < F(0) < 1$, which implies that $q \in (0, 1]$ where $q$ is the smallest root in the interval $[0, 1]$ of $F(x) = x$; that $F(s) \not\equiv \alpha + \beta s$ and $m \equiv F'(1-) < \infty$; and that $0 < B(0) < 1$, but $B(1-) \leq 1$.

Our central result pertains to solutions of the form

$$V(s) = \sum_{i=0}^{\infty} v_i s^i \;, \quad \left(\{v_i\} \geq \{0\},\ \{v_i\} \not\equiv \{0\}\right) \text{ convergent for } s \in [0, 1)$$

to the functional equation

$$V(s) = B(s)V\left(F(s)\right) \;, \quad s \in [0, 1) \tag{1}$$

where $m < 1$. This amounts to an investigation of stationary measures of a subcritical Galton-Watson process, with offspring p.g.f. $F(s)$ and a

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*possibly defective* immigration p.g.f. $B(s)$ . The possibility that $B(1-) < 1$ is the only novel feature which necessitates a treatment different from [5], Section 5.3; however it is *precisely this* which enables us to answer not only the open problems in this reference, but also to provide information on the uniqueness of invariant measures for the Galton-Watson process when $m \neq 1$ , either with or without immigration. The main result is as follows.

THEOREM. *There exists a unique solution to* (1) *(as usual to a constant multiplier) of the kind we seek satisfying* $V(1-x) = x^{-\delta}L(x)$ *, where* $L(x)$ *is slowly varying as* $x \to 0+$ *, and* $\delta$ *is a finite real number. In fact,* $\delta$ *is given by* $\log B(1-)/\log m$ *, and this solution by*

$$V(s) = c\{1 - G(s)\}^{-\delta}V_1(s) \ , \quad c = \text{const.} > 0 \ . \tag{2}$$

Here $G(s)$ is the p.g.f. of the asymptotic conditional distribution of the Galton-Watson process generated by $F(s)$ , [1]; and $V_1(s)$ generates the unique invariant measure for such a process with immigration, with offspring p.g.f. $F(s)$ and (proper) immigration p.g.f. $B(s)/B(1-)$ , [5], Section 5.3. Before proceeding to list some of the consequences of this result and the preliminary results involved in its proof, we note that a power-series variant of Karamata's Tauberian Theorem implies that for the solution (2) as $n \to \infty$

$$\sum_{i=0}^{n} v_i \sim n^{\delta}L(1/n)/\Gamma(\delta+1) \ .$$

## 2. Consequences for the ordinary Galton-Watson process

Kingman [2] has shown that in general quite distinct invariant measures may exist for the Galton-Watson process generated by $F(s)$ with $m \neq 1$ . Let $\Pi(s)$ , $s \in [0, q)$ generate any such measure. Our result implies that there is *only one* satisfying $V(s) \equiv \exp\Pi(qs) = (1-s)^{-\delta}L(1-s)$ ; in fact $\delta = -1/\log F'(q-)$ , etc. A second consequence of similar sort concerns uniqueness of p.g.f. solutions $Q(s)$ to the functional equation $\left\{1 - Q\left(F(s)\right)\right\} = m^{\nu}\{1 - Q(s)\}$ , $s \in [0, 1)$ , where $m < 1$ and $0 < \nu \le 1$ , raised by Rubin and Vere-Jones

[3]. Our result implies that the only solutions satisfying, for some $\delta$, $Q(s) = 1 - (1-s)^{1-\delta}L(1-s)$ are given by $Q(s) = 1 - c(1 - G(s))^{\nu}$, $0 < c \le 1$.

A side-product of our work pertains to a random variable $W$ which has the distribution of the limit law of the process when $m < 1$, discussed in [1]; or for that of the process when $m > 1$, discussed in [4]. In either case, $\int_0^y Pr(W > x)dx = L(y)$, where $L(y)$ is slowly varying as $y \to \infty$. (This implies $E(W^\rho) < \infty$, $0 < \rho < 1$.)

## 3. Consequences for the Galton-Watson process with a proper immigration distribution

In the case $m > 1$, an invariant measure always exists, but is not in general unique. There is however *only one* satisfying $V(qs) = (1-s)^{-\delta}L(1-s)$; and in fact $\delta = \log B(q-)/\log F'(q-)$ etc. (In the case $m < 1$, the unique invariant measure *always* satisfies $V(s) = L(1-s)$.)

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