# THE USE OF SPITZER'S IDENTITY IN THE INVESTIGATION OF THE BUSY PERIOD AND OTHER QUANTITIES IN THE QUEUE $GI/G/1$

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## 1. Introduction

As an illustration of the use of his identity [10], Spitzer [11] obtained the Pollaczek-Khintchine formula for the waiting time distribution of the queue $M/G/1$. The present paper develops this approach, using a generalised form of Spitzer's identity applied to a three-dimensional random walk. This yields a number of results for the general queue $GI/G/1$, including Smith's solution for the stationary waiting time, which is established under less restrictive conditions that hitherto (§ 5). A solution is obtained for the busy period distribution in $GI/G/1$ (§ 7) which can be evaluated when either of the distributions concerned has a rational characteristic function. This solution contains some recent results of Conolly on the queue $GI/E_n/1$, as well as well-known results for $M/G/1$.

## 2. Spitzer's identity

We shall quote Spitzer's identity in a rather general form which is a restatement of the results of Wendel ([13], [14] particularly § 4).

LEMMA 1. Let $\mathfrak{M}$ be the Banach algebra of totally finite complex Borel measures on a finite-dimensional Euclidean space, with convolution as multiplication, total variation as norm, and identity $\mathfrak{e}$. Let $\mathfrak{f}$ be a probability measure, and let $P$ be a linear transformation of $\mathfrak{M}$ into itself, such that

$$P^2 = P,\ ||P|| \leqq 1,\ P\mathfrak{e} = \mathfrak{e},$$

and such that $P\mathfrak{M}$ and $(I - P)\mathfrak{M}$ are sub-algebras. If we define a sequence $\mathfrak{g}_n$ in $\mathfrak{M}$ by the equations

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{e} \\ \mathfrak{g}_{n+1} &= P(\mathfrak{f}\mathfrak{g}_n) \end{aligned}$$



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then, for $|x| < 1$,

(1) $$\sum_{n=0}^{\infty} x^n \mathfrak{g}_n = \exp\left\{\sum_{n=1}^{\infty} \frac{x^n}{n} P\mathfrak{f}^n\right\}.$$

Further, if $\mathfrak{h}_n = \mathfrak{g}_{n-1} - \mathfrak{g}_n$, then

(2) $$\sum_{n=1}^{\infty} x^n \mathfrak{h}_n = \mathfrak{e} - \exp\left\{-\sum_{n=1}^{\infty} \frac{x^n}{n} (I - P)\mathfrak{f}^n\right\}.$$

## 3. Definitions and basic results

We consider the queue in which successive customers $C_n$ have service times $s_n$, where the $s_n$ are independently identically distributed with characteristic function $\beta(\theta)$. The time between the arrivals of $C_n$ and $C_{n+1}$ is $t_n$, where the $t_n$ are independently identically distributed with characteristic function $\alpha(\theta)$. The sequences $\{s_n\}$ and $\{t_n\}$ are assumed independent.

We then write

$$u_n = s_n - t_n, \quad \gamma(\theta) = E(e^{i\theta u}) = \alpha(-\theta)\beta(\theta),$$

$$S_n = \sum_{r=0}^{n-1} s_r, \quad T_n = \sum_{r=0}^{n-1} t_r, \quad U_n = \sum_{r=0}^{n-1} u_r.$$

Suppose that, at $t = 0$, $C_0$ arrives and finds the server free. Then, if $w_n$ is the waiting time of $C_n$, we have Lindley's result [7].

(3) $$\begin{cases} w_0 = 0, \\ w_{n+1} = (w_n + u_n)^+. \end{cases}$$

where $z^+ = z (z \geqq 0)$, $z^+ = 0 (z < 0)$.

Suppose that the server first becomes idle at $t = Z$ after having served $C_0, C_1, \cdots, C_{N-1}$, so that $N$ is the number of customers in, and $Z$ the length of, the busy period. Then it is clear that

(4) $$\begin{cases} N = \min\{n; U_n < 0\}, \\ Z = S_N. \end{cases}$$

The object of our analysis of the busy period is to evaluate

(5) $$\Pi(x, \lambda) = E(x^N e^{-\lambda Z}).$$

## 4. The joint distribution of $S_n, T_n, w_n$

Consider the vector-valued process

$$\boldsymbol{R}_n = (S_n, T_n, w_n)$$

in the positive octant of $\{X\}$, $X = (X_1, X_2, X_3)$. From equation (3), we see that $R_n$ executes a random walk with increment

$$r_n = (s_n, t_n, u_n)$$

but with an impenetrable barrier at $X_3 = 0$. Now let $\mathfrak{g}_n$ be the probability measure of $R_n$, and $\mathfrak{f}$ that of $r_n$. Then, using the notation of § 2, we have

$$\mathfrak{g}_0 = \mathfrak{e}, \quad \mathfrak{g}_{n+1} = P(\mathfrak{f}\mathfrak{g}_n).$$

where $P$ is the linear operator on the measure algebra $\mathfrak{M}$ corresponding to the function

$$(X_1, X_2, X_3) \to (X_1, X_2, X_3^+).$$

It is not difficult to see that the conditions of Lemma 1 hold, and hence

$$\sum_{n=0}^{\infty} x^n \mathfrak{g}_n = \exp\left\{\sum_{n=1}^{\infty} \frac{x^n}{n} P\mathfrak{f}^n\right\}.$$

Since the components of $R_n$ are positive, we may apply a Laplace-Stieltjes transform to the measures $\mathfrak{g}_n$, $P\mathfrak{f}^n$. This is a continuous homomorphism of $\mathfrak{M}$ into the complex numbers, so that (cf. Wendel [14] § 4) we obtain the following theorem.

THEOREM 1. *Whenever* $|x| < 1$ *and* $\Re\kappa, \lambda, \mu \geqq 0$,

$$\sum_{n=0}^{\infty} x^n E(e^{-\kappa S_n - \lambda T_n - \mu w_n}) = \exp\left\{\sum_{n=1}^{\infty} \frac{x^n}{n} E(e^{-\kappa S_n - \lambda T_n - \mu U_n^+})\right\}. \tag{6}$$

We may, in principle, evaluate the right hand side of this expression, and we may therefore find the joint distribution of $S_n$, $T_n$, $w_n$. In particular, we may consider the joint distribution of the waiting time of a customer and his time of arrival. The use of this result will be illustrated in the next two sections.

## 5. The stationary waiting time distribution

Putting $\kappa = \lambda = 0$ in (6) we obtain, for $|x| < 1$, $\Re\mu \geqq 0$,

$$\sum_0^{\infty} x^n E(e^{-\mu w_n}) = \exp \sum_1^{\infty} \frac{x^n}{n} E(e^{-\mu U_n^+}).$$

We may now proceed as in Spitzer's paper ([11], Theorem 3), and on the assumption that $[\gamma(\xi) - 1]/\xi$ is integrable in a neighbourhood of $\xi = 0$, to the result

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$$(1-x)\sum_0^\infty x^n E(e^{-\mu w_n}) = \exp\left\{\frac{\mu}{2\pi}\int_0^x dt\int_{-\infty}^{\infty}\frac{\gamma(\xi)-1}{(1-t)(1-t\gamma(\xi))}\frac{d\xi}{\xi(\xi-i\mu)}\right\}$$

Now assume that $E(s)$, $E(t)$ exist and that $E(u) < 0$. Then $(\gamma-1)/\xi$ is bounded at $\xi = 0$, and also, by Lindley's theorem, the distribution of $w_n$ tends to a limit as $n \to \infty$. Hence

$$(1-x)\sum_0^\infty x^n E(e^{-\mu w_n}) \to \lim_{n\to\infty} E(e^{-\mu w_n})$$

as $x \to 1-$. Let the limiting cumulant generating function of $w_n$ be $K(\mu) = \lim_{n\to\infty} \log E(e^{-\mu w_n})$.
Then $K(\mu) = \lim_{x\to 1-} K_x(\mu)$, where

$$K_x(\mu) = \frac{\mu}{2\pi}\int_0^x \frac{dt}{1-t}\int_{-\infty}^{\infty}\frac{\gamma(\xi)-1}{1-t\gamma(\xi)}\frac{d\xi}{\xi(\xi-i\mu)}. \tag{7}$$

In order to simplify this expression, we suppose that, for some $c_1 > 0$, $\beta(\xi)$ is analytic in $\Im\xi > -c_1$. Then $\gamma(\xi)$ is analytic in $-c_1 < \Im\xi < 0$ and continuous in $-c_1 < \Im\xi \leqq 0$. Since $i\gamma'(0) = E(u) < 0$, we can choose $c_2 < c_1$ such that $|\gamma(\xi)| < 1$ in $-c_2 < \Im\xi < 0$. Then we can transform the integral with respect to $\xi$ in (7) into one along a contour $\Gamma$ lying in $-c_2 < \Im\xi < 0$ (using the stronger form of Cauchy's theorem, cf. [4] § 11.054).

But

$$\frac{\gamma(\xi)-1}{1-t\gamma(\xi)} = -1 + \frac{\gamma(\xi)(1-t)}{1-t\gamma(\xi)}$$

and

$$\int_\Gamma \frac{d\xi}{\xi(\xi-i\mu)} = 0.$$

Hence

$$K_x(\mu) = \frac{\mu}{2\pi}\int_0^x dt\int_\Gamma \frac{\gamma(\xi)}{1-t\gamma(\xi)}\frac{d\xi}{\xi(\xi-i\mu)}$$
$$= \frac{\mu}{2\pi}\int_\Gamma \frac{d\xi}{\xi(\xi-i\mu)}\int_0^x \frac{\gamma(\xi)dt}{1-t\gamma(\xi)}$$

by Fubini's theorem,

$$= -\frac{\mu}{2\pi}\int_\Gamma \log[1-x\gamma(\xi)]\frac{d\xi}{\xi(\xi-i\mu)}.$$

Hence, by the dominated convergence theorem,

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$$K(\mu) = -\frac{\mu}{2\pi}\int_\Gamma \log[1-\gamma(\xi)]\frac{d\xi}{\xi(\xi - i\mu)}.$$

THEOREM 2. *Let the interarrival time distribution have finite mean, and let the service time characteristic function be analytic in $\Im\xi > -c_1(c_1 > 0)$. Then, so long as $Es < Et$, there exists $c_2 > 0$ such that $\gamma(\xi)$ is analytic and $|\gamma(\xi)| < 1$ in $-c_1 < -c_2 < \Im\xi < 0$. and then, if $\Gamma$ is a contour extending from $-\infty$ to $+\infty$ in $-c_2 < \Im\xi < 0$, the stationary waiting time has cumulant generating function*

$$K(\mu) = \log E(e^{-\mu w}) = -\frac{\mu}{2\pi}\int_\Gamma \log[1-\gamma(\xi)]\frac{d\xi}{\xi(\xi-i\mu)} \tag{8}$$

*for* $\Re\mu \geqq 0$.

This result is very similar to that obtained by Smith ([8], Theorem 2) by the Wiener-Hopf technique. In our notation, Smith proves that

$$K(\mu) = -\frac{\mu}{2\pi}\int_\Gamma \log\left[\frac{\xi - i\lambda}{\xi}\{1-\gamma(\xi)\}\right]\frac{d\xi}{\xi(\xi-i\mu)}$$

for some $\lambda > 0$. Since $\int_\Gamma \log((\xi - i\tau)/\xi)\, d\xi/(\xi(\xi - i\mu)) = 0$ the factor $(\xi - i\lambda)/\xi$ is redundant, so that his solution agrees with (8). However, Smith obtained his result on the further assumption that the interarrival time characteristic function, as well as that of the service time, was analytic in $\Im\xi > -c_1$. This assumption we have shown to be unnecessary.

From his solution, Smith showed that, if $\beta(\theta)$ is rational (of degree $n$), then so is $E(e^{-\mu w})$. We now see that this result holds without any restriction on the interarrival times, except that they must have finite mean (probably an unnecessary condition). In the particular case when the service time distribution is negative exponential, this has been noted by Kendall ([6], Theorem IV).

## 6. Moments

If we formally expand the result of Theorem 1 in a power series in $\kappa$, $\lambda$, $\mu$ and equate coefficients we obtain equations for the moments of $\boldsymbol{R}_n$. In order to justify this procedure we consider the first order terms. The higher terms, though algebraically more complicated, may be treated in a similar way,

With the notation of Lemma 1 we have

$$\sum_0^\infty x^n \mathfrak{g}_n = \exp\sum_1^\infty \frac{x^n}{n} P\mathfrak{f}^n$$

For any measure $\mathfrak{g}$ on the Euclidean space $R^3$, define

$$[\mathfrak{g}] = \mathfrak{g}(R^3), \text{ and } \quad \mathfrak{A}\mathfrak{g} = \int_{R_3} d\mathfrak{g}(z)$$

Then, if $\mathfrak{g}$ is a probability measure, $[\mathfrak{g}] = 1$, and $\mathfrak{A}\mathfrak{g}$ is the mean of the distribution. For any measures $\mathfrak{f}$ and $\mathfrak{g}$,

$$\mathfrak{A}(\mathfrak{f}+\mathfrak{g}) = \mathfrak{A}\mathfrak{f} + \mathfrak{A}\mathfrak{g} \quad \text{and} \quad \mathfrak{A}(\mathfrak{f}\mathfrak{g}) = [\mathfrak{g}]\mathfrak{A}\mathfrak{f} + [\mathfrak{f}]\mathfrak{A}\mathfrak{g}$$

so that

$$\mathfrak{A}\exp\mathfrak{g} = \mathfrak{A}\sum\frac{\mathfrak{g}^n}{n!} = \sum\frac{\mathfrak{A}\mathfrak{g}^n}{n!} = \sum\frac{n[\mathfrak{g}]^{n-1}\mathfrak{A}\mathfrak{g}}{n!} = \exp[\mathfrak{g}]\mathfrak{A}\mathfrak{g}$$

Hence

$$\sum_0^\infty x^n\mathfrak{A}\mathfrak{g}_n = \exp\sum_1^\infty\frac{x^n}{n}[P\mathfrak{f}^n]. \quad \sum_1^\infty\frac{x^n}{n}\mathfrak{A}P\mathfrak{f}^n = (1-x)^{-1}\sum_1^\infty\frac{x^n}{n}\mathfrak{A}P\mathfrak{f}^n$$

so long as $\mathfrak{A}\mathfrak{f}$ is finite. Equating coefficients, $\mathfrak{A}\mathfrak{g}_n = \sum_{r=1}^n \mathfrak{A}P\mathfrak{f}^r/r$. This is exactly the result which would be obtained by formally taking the first terms in the Laplace transforms. Hence a little algebraic manipulation yields

THEOREM 3.

If $Es$, $Et < \infty$, then

$$E(w_n) = \sum_{r=1}^n r^{-1}E(U_r^+).$$

If $Es^2$, $Et^2 < \infty$, then

$$\operatorname{var}(w_n) = \sum_{r=1}^n r^{-1}E(U_r^+)^2 - \sum_{r+s>n}^{r,s\leq n}\sum (rs)^{-1}E(U_r^+)E(U_s^+)$$

and

$$\operatorname{cov}(w_n, T_n) = \sum_{r=1}^n r^{-1}\operatorname{cov}(T_r, U_r^+)$$

If, moreover, $Eu < 0$, and $w$ is the stationary waiting time, then

$$E(w) = \sum_1^\infty n^{-1}E(U_n^+)$$

$$\operatorname{var}(w) = \sum_1^\infty n^{-1}E(U_n^+)^2.$$

## 7. The busy period - formal solution

The main object of this paper is to evaluate the joint generating function

$$\Pi(x, \lambda) = E(x^N e^{-\lambda Z})$$

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of the number $N$ of customers in, and the length $Z$ of, the busy period. (We assume that $Eu < 0$, so that $N$ and $Z$ are finite with probability one.) This problem has been considered in the case $M/G/1$ by Kendall [6] and Takacs [12], and by Conolly in the cases $GI/M/1$ [1], $GI/E_n/1$ [2] and $E_n/G/1$ [3]. Their results are all contained in the theorem established below.

Since writing this paper my attention has been drawn to some work of P. D. Finch [15] in which a very similar method to the present one is used to obtain a result (Theorem 1) equivalent to our equation (9). This is applied to give explicit expressions for

$$E(e^{-\lambda Z}; N = n)$$

which are valid for both stable and unstable queues.

We consider the process $\boldsymbol{R}_n^*$ obtained from $\boldsymbol{R}_n$ by replacing the impenetrable barrier at $X_3 = 0$ by an absorbing one. Then, if we denote the probability measure of $\boldsymbol{R}_n^*$ by $\mathfrak{g}_n^*$, and the effect of the boundary by an operator $P^*$, it is easy to see that the conditions of Lemma 1 are again satisfied. Hence, if

$$\mathfrak{h}_n = \mathfrak{g}_{n-1}^* - \mathfrak{g}_n^*$$

we have, from (2)

$$\sum_1^\infty x^n \mathfrak{h}_n = \mathfrak{e} - \exp\left\{-\sum_1^\infty \frac{x^n}{n}(I - P^*)\mathfrak{f}^n\right\}.$$

Now, if $A_n$ is the event corresponding to absorption at $n$ (i.e. $N = n$),

$$\mathfrak{h}_n = P\Big(\sum_0^{n-1} \boldsymbol{r}_j; A_n\Big)$$

where $P(\phi; A) = P(\phi|A)P(A)$.

Hence, for any $\phi(\boldsymbol{z})$,

$$\sum_{n=1}^\infty x^n \int \phi(\boldsymbol{z}) d\mathfrak{h}_n(\boldsymbol{z}) = E\left(x^N \phi\left(\sum_0^{N-1} \boldsymbol{r}_j\right)\right)$$

Thus

$$E(x^N e^{-\kappa T_N - \lambda S_N - \mu U_N}) = 1 - \exp\left\{-\sum_1^\infty \frac{x^n}{n} E(e^{-\kappa T_n - \lambda S_n - \mu U_n}; U_n < 0)\right\}$$

In particular, we have, for $|x| < 1$, $\Re\lambda \geqq 0$,

$$\Pi(x, \lambda) = 1 - \exp\left\{-\sum_{n=1}^\infty \frac{x^n}{n} E(e^{-\lambda S_n}; U_n < 0)\right\}. \tag{9}$$

We may now apply Hewitt's inversion theorem (cf. [11] Theorem 3) to give

$$E(e^{-\lambda S_n}; S_n < Y) = \frac{1}{2\pi} \lim_{A\to\infty} \int_{-A}^{A} \beta^n(\xi) \int_0^Y e^{-\lambda s} e^{-i\xi s} ds\, d\xi$$

for any $Y > 0$,

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \beta^n(\xi)\, \frac{1 - e^{-i(\xi - i\lambda)Y}}{i(\xi - i\lambda)}\, d\xi$$

Hence

$$E(e^{-\lambda S_n}; S_n < T_n) = \frac{1}{2\pi i} E \int_{-\infty}^{\infty} \beta^n(\xi) \frac{1 - e^{-i(\xi - i\lambda)T_n}}{\xi - i\lambda}\, d\xi$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \beta^n(\xi)\{1 - \alpha^n(\xi - i\lambda)\} \frac{d\xi}{\xi - i\lambda}$$

Substituting this in the formula for $\Pi(x, \lambda)$ we obtain

THEOREM 4. *For* $|x| < 1$, $\Re\lambda \geqq 0$, *the generating function* $\Pi(x, \lambda)$ *for the busy period is given by*

$$(10) \quad \Pi(x, \lambda) = 1 - \exp\left\{\frac{1}{2\pi i} \int_{-\infty}^{\infty} \log\left[\frac{1 - x\beta(\xi)}{1 - x\alpha(-\xi + i\lambda)\beta(\xi)}\right] \frac{d\xi}{\xi - i\lambda}\right\}.$$

This is the complete formal solution for the busy period, which is analogous to Smith's formal solution ([8], Theorem 2) for the waiting time. We shall show that, if either $\alpha(\theta)$ or $\beta(\theta)$ is a rational function, the integral can be evaluated. As usual we shall denote by $K_n$ the class of distributions over $[0, \infty)$ which have a rational characteristic function of degree $n$.

## 8. The busy period in $GI/K_n/1$

Write $\phi(\xi) = \phi(\xi, x, \lambda) = (1 - x\beta(\xi))/(1 - x\alpha(-\xi + i\lambda)\beta(\xi))$ so that

$$(11) \qquad \log\{1 - \Pi(x, \lambda)\} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log \phi\,(\xi, x, \lambda)\, \frac{d\xi}{\xi - i\lambda}$$

Suppose that $\beta(\xi)$ is a rational function of degree $n$, so that $\phi(\xi)$ is meromorphic in $\Im\xi < 0$. Clearly $\phi(\xi) \to 1$ as $|\xi| \to \infty$, and so, if $\Re\lambda > 0$, we may convert the integral into one around a contour $\Gamma_1$ consisting of the real axis together with a large semicircle in $\Im\xi < 0$ standing on it. Then

$$\log(1 - \Pi) = -\frac{1}{2\pi i} \int_{\Gamma_1} \log \phi(\xi)\, \frac{d\xi}{\xi - i\lambda}$$

$$= -\frac{1}{2\pi i} [\log \phi \log(\xi - i\lambda)]_{\Gamma_1} + \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\phi'(\xi)}{\phi(\xi)} \log(\xi - i\lambda) d\xi.$$

Suppose that the roots of $1 - x\beta(\xi) = 0$ are $\xi = \xi_1(x), \cdots, \xi_n(x)$. Then it is not difficult to show, by Rouché's theorem, that there are exactly $n$ roots of

$$1 - x\alpha(-\xi + i\lambda)\beta(\xi) = 0$$

Let these be $\xi = \xi_1(x, \lambda), \cdots, \xi_n(x, \lambda)$. Then it is clear that $\log\phi \log(\xi - i\lambda)$ is single-valued on $\Gamma_1$, and $\phi'/\phi$ has poles at $\xi = \xi_i(x)$ with residues 1, and at $\xi = \xi_i(x, \lambda)$ with residues $-1$. Hence

$$\log(1 - \Pi) = \sum_{j=1}^{n} \log(\xi_j(x) - i\lambda) - \log(\xi_j(x, \lambda) - i\lambda).$$

THEOREM 5. *If the service time characteristic function is a rational function of degree $n$, then*

$$\Pi(x, \lambda) = 1 - \prod_{j=1}^{n} \frac{\xi_j(x) - i\lambda}{\xi_j(x, \lambda) - i\lambda} \tag{12}$$

*where $\xi_j(x)$ are the roots of $1 - x\beta(\xi) = 0$, and $\xi_j(x, \lambda)$ those of $1 - x\alpha(-\xi + i\lambda)\,\beta(\xi) = 0$, in $\Im\xi < 0$.*

COROLLARY. If $\beta(\xi) = (1 - ib\xi)^{-1}$ then

$$\Pi(x, \lambda) = 1 - (1 - x + b\lambda)/b\eta(\lambda) \tag{13}$$

where $\eta(\lambda)$ satisfies

$$1 + b\lambda - b\eta = x\alpha(i\eta)$$

## 9. The busy period in $K_n/G/1$

If $\alpha(\xi)$ is rational, the function $\phi(\xi)$ is meromorphic in $\Im\xi > 0$, so that if $\Gamma_2$ is the real axis and a large semicircle standing on it in $\Im\xi > 0$,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \log\phi(\xi)\frac{d\xi}{\xi - i\lambda} = \frac{1}{2\pi i}\int_{\Gamma_2} \log\phi(\xi)\frac{d\xi}{\xi - i\lambda}$$

We can now modify $\Gamma_2$ by adding to it a small circle (described in a clockwise sense) around $\xi = i\lambda$. If the new contour is $\Gamma_3$

$$\frac{1}{2\pi i}\int_{\Gamma_2} \log\phi(\xi)\frac{d\xi}{\xi - i\lambda} = \frac{1}{2\pi i}\int_{\Gamma_3} \log\phi(\xi)\frac{d\xi}{\xi - i\lambda} + \log\phi(i\lambda)$$

But $\log\phi(i\lambda) = \log(1 - x\beta(i\lambda))/(1 - x\beta(i\lambda)) = 0$, so that

$$\begin{aligned}\log(1 - \Pi) &= \frac{1}{2\pi i}\int_{\Gamma_3} \log\phi(\xi)\frac{d\xi}{\xi - i\lambda} \\ &= \frac{1}{2\pi i}[\log\phi\log(\xi - i\lambda)]_{\Gamma_3} - \frac{1}{2\pi i}\int_{\Gamma_3}\frac{\phi'(\xi)}{\phi(\xi)}\log(\xi - i\lambda)d\xi\end{aligned}$$

Let the zeros of the denominator of $\alpha(\xi)$ be $-i\eta_1, \cdots, -i\eta_n (\Re\eta_j > 0)$, and let the roots of

$$1 - x\alpha(-\xi + i\eta)\beta(\xi) = 0$$

in $\Im\xi > 0$ be $\xi_j(x, \lambda)$ (as before, Rouché's theorem shows that there are $n$ of them). Then

$$[\log \phi \log(\xi - i\lambda)]_{\Gamma_3} = 0$$

and

$$-\frac{1}{2\pi i}\int_{\Gamma_3} \frac{\phi'(\xi)}{\phi(\xi)} \log(\xi - i\lambda) d\xi = -\sum_{j=1}^{n} \log[i(\lambda + \eta_j) - i\lambda] - \log[\xi_j - i\lambda]$$
$$= \log \prod_{j=1}^{n} \frac{\xi_j - i\lambda}{i\eta_j}$$

THEOREM 6. *When the interarrival time has a rational characteristic function with poles at* $\xi = -i\eta_j$ ($j = 1, \cdots, n; \Re\eta_j > 0$),

$$\Pi(x, \lambda) = 1 - \prod_{j=1}^{n} \frac{\xi_j(x, \lambda) - i\lambda}{i\eta_j} \tag{14}$$

*where* $\xi_j$ *are the roots, in* $\Im\xi > 0$, *of*

$$1 - x\alpha(-\xi + i\lambda)\beta(\xi) = 0$$

COROLLARY. If the arrivals from a Poisson process with rate $\nu$, so that $\alpha(\xi) = (1 - i\xi/\nu)^{-1}$, then $\Pi(x, \lambda)$ satisfies

(15) $\quad \Pi = x\beta\{i\lambda + i\nu(1 - \Pi)\}$ (cf. Kendall [5], equation (59)).

## 10. The Wiener-Hopf factorisation

Spitzer's identity has a very intimate connection with the method of Hopf and Wiener [9], which was used by Smith [8] to drive the waiting time distribution. In the analysis of the busy period the relevant Wiener-Hopf factorisation is of

$$\phi(\xi, x, \lambda) = \frac{1 - x\beta(\xi)}{1 - x\alpha(-\xi + i\lambda)\beta(\xi)}$$

We shall not discuss the possibility of effecting a suitable factorisation (see [9]), but we shall show that, once such a factorisation has been made, the solution for $\Pi$ follows.

THEOREM 7. *Suppose there exist functions* $\phi_+(\xi, x, \lambda)$, $\phi_-(\xi, x, \lambda)$ *which satisfy*

(i) $\phi = \phi_+\phi_-$ *for all real* $\xi$,
(ii) $\phi_+$ *is analytic and zero-free in* $\Im\xi > 0$, *and tends to* 1 *as* $|\xi| \to \infty$,
(iii) $\phi_-$ *is analytic and zero-free in* $\Im\xi < 0$, *and tends to* 1 *as* $|\xi| \to \infty$,
(iv) $\phi_+$ *and* $\phi_-$ *are continuous in* $\Im\xi \geqq 0$ *and* $\Im\xi \leqq 0$ *respectively*.

*Then* $\Pi(x, \lambda) = 1 - \phi_+(i\lambda, x, \lambda)$.

PROOF: Under the given conditions

$$\begin{aligned}\log(1-\Pi) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \log\phi \frac{d\xi}{\xi - i\lambda} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\log\phi_+ + \log\phi_-)\frac{d\xi}{\xi - i\lambda} \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} \log\phi_- \frac{d\xi}{\xi - i\lambda} + \frac{1}{2\pi i}\int_{\Gamma_2} \log\phi_+ \frac{d\xi}{\xi - i\lambda} \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} \log\phi_- \frac{d\xi}{\xi - i\lambda} + \frac{1}{2\pi i}\int_{\Gamma_3} \log\phi_+ \frac{d\xi}{\xi - i\lambda} + \log\phi_+(i\lambda) \\ &= \log\phi_+(i\lambda).\end{aligned}$$

Hence $\qquad \Pi = 1 - \phi_+(i\lambda)$.

## 11. Conclusion

Spitzer's identity, applied to appropriate multi-dimensional random walks, is a very powerful tool in the analysis of queueing problems. It yields formal solutions, which can be evaluated when either the interarrival time or the service time has a distribution of the class $K_n$. The limitations on this method are similar to those on the method of Hopf and Wiener in restricting the form of the barrier to a single hyperplane, but the results are obtained under somewhat weaker analytic conditions than is usual in the Wiener-Hopf approach.

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