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A minimax inequality for inscribed cones revisited

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Abstract. In 1993, E. Lutwak established a minimax inequality for inscribed cones of origin symmetric convex bodies. In this work, we re-prove Lutwak's result using a maxmin inequality for circumscribed cylinders. Furthermore, we explore connections between the maximum volume of inscribed double cones of a centered convex body and the minimum volume of circumscribed cylinders of its polar body.

1 Introduction

Let K be an origin symmetric convex body, and let u be a unit vector in $\mathbb{R}^d$. Throughout the paper, it will be assumed that $d \geq 3$. Let $C(K,u)$ be the unbounded cylinder circumscribed about K (i.e., the union of all lines parallel to u and intersecting K) generated by u. We define $K(u)$ to be a compact circumscribed cylinder of K obtained from $C(K,u)$ which is bounded by the two parallel supporting hyperplanes of K at the intersection of l_u (i.e., 1-subspace parallel to u) with the boundary of K. We can select these two parallel supporting hyperplanes of K perpendicular to l_u if hyperplanes are not uniquely defined. As Petty [12] pointed out, $K(u)$ has the minimum volume among all compact cylinders obtained from $C(K,u)$ and contain K. Petty [12] proved the following maxmin inequality for $K(u)$:

$$\max_{u \in S^{d-1}} \frac{\lambda_{d-1}(K|u^\perp)\lambda_1(K \cap l_u)}{\lambda(K)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d},$$

with equality if and only if K is an ellipsoid (see also [2, 6, 10, 14] for all convex bodies).

A characterization of those K for which the quantity $\lambda_{d-1}(K|u^\perp)\lambda_1(K \cap l_u)\lambda^{-1}(K)$ a constant for all $u \in S^{d-1}$ is still an open question of convex geometry (see [13]). However, it has been conjectured that ellipsoids are only convex bodies with such a property.

For a given unit vector u, one can also construct the inscribed cone of maximal volume with base $K \cap u^\perp$ and apex in K. The apex of such a cone is a point of K on a supporting hyperplane parallel to $u^\perp$. The volume of this cone is $(2d)^{-1}\lambda_{d-1}(K \cap u^\perp)\lambda_1(K|l_u)$.

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We mention that Busemann and Petty [4] posed 10 problems about centrally symmetric convex bodies. So far, only Problem 1 from the list (called *the Busemann–Petty problem*) has been solved completely (see [7] and the references therein). Problem 5 of [4] asks the following question: are the ellipsoids only convex bodies characterized by the property that the quantity $(2d)^{-1}\lambda_{d-1}(K\cap u^{\perp})\lambda_1(K|l_u)$ is a constant for all $u\in S^{d-1}$? The authors [1] proved that if K is sufficiently close to the Euclidean ball in the *Banach–Mazur metric*, then K is an ellipsoid.

Using the *dual mixed volume inequality* (see [8] for dual mixed volumes) along with other inequalities, Lutwak [9] proved the following minimax inequality for double cones inscribed in a centered convex body K of $\mathbb{R}^d$:

$$\min_{u\in S^{d-1}}\lambda_{d-1}(K\cap u^{\perp})\lambda_1(K|l_u)\leq\frac{2\varepsilon_d\varepsilon_{d-1}}{\lambda(K^{\circ})},$$

with equality if and only if K is an ellipsoid.

One can observe that the quantities $\lambda_{d-1}(K\cap u^{\perp})\lambda_1(K|l_u)\lambda^{-1}(K)$ and $\lambda_{d-1}(K|u^{\perp})\lambda_1(K\cap l_u)\lambda^{-1}(K)$ are invariant under a dilatation. Therefore, one could also set $\lambda(K)=\varepsilon_d$. In [11], it was proved that if B is a centered convex body in $\mathbb{R}^d$ with $\lambda(B)=\varepsilon_d$, then

$$\min_{u\in S^{d-1}}\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B^{\circ}|l_u)\leq 2\varepsilon_{d-1},$$

with equality if and only if B is an ellipsoid.

The purpose of this manuscript is to show that Lutwak's minimax inequality for inscribed cones of a centered convex body B can be established using the Petty's maxmin inequality for circumscribed cylinders of its polar body of B°. Furthermore, we show connections between the maximum volume of inscribed double cones of a centered convex body B and the minimum volume of circumscribed cylinders of its polar body B°. The homothety of a centered convex body B and the projection body of its polar body ΠB° will be discussed as well.

2 Basic notations and facts

In this section, we recall some definitions, notations, and facts from convex geometry. A *convex body* K is a compact, convex subset of $\mathbb{R}^d$ with nonempty interior. A convex body K is said to be *centered* if it is symmetric with respect to the origin, i.e., $-K=K$. We denote by $\mathcal{K}$ the set of convex bodies in $\mathbb{R}^d$, and the set of centered convex bodies will be denoted by $\mathcal{K}_o$. As usual, S^{d-1} will stand for the standard Euclidean unit sphere in $\mathbb{R}^d$. The symbol $\lambda_i(\cdot)$ will stand for the *i-dimensional Lebesgue measure* (*volume*) in $\mathbb{R}^d$, where $1\leq i\leq d$, and when $i=d$ the subscript will be omitted. For a given direction $u\in S^{d-1}$, we use $u^{\perp}$ to denote the $(d-1)$-dimensional hyperplane (passing through the origin) orthogonal to u, and by l_u the 1-subspace parallel to u. Furthermore, $\lambda_1(K|l_u)$ denotes the *width* of K at u, and $\lambda_{d-1}(K|u^{\perp})$ the $(d-1)$-*dimensional outer cross-section measure* or *brightness* of K at $u\in S^{d-1}$, where $K|u^{\perp}$ is the orthogonal projection of K onto $u^{\perp}$ (more about these notations, the reader should refer to [5]).

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For $K \in \mathcal{K}$ with the origin an interior point of K, its *polar body* K° is defined by

$$K^\circ = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1, x \in K\},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^d$.

We mention the following simple properties of $K \in \mathcal{K}_o$: $(K^\circ)^\circ = K$, $(\alpha K)^\circ = (1/\alpha)K^\circ$ for $\alpha > 0$. For $K_1 \in \mathcal{K}_0$ and $K_2 \in \mathcal{K}_0$ if $K_1 \subseteq K_2$, then $K_2^\circ \subseteq K_1^\circ$.

The standard basis will be used to identify $\mathbb{R}^d$ and its *dual space* $\mathbb{R}^{d*}$. In that case, $\lambda_i(\cdot)$ and $\lambda_i^*(\cdot)$ coincide in $\mathbb{R}^d$. The symbol ε_i will stand for the i-dimensional volume of the unit ball in $\mathbb{R}^i$.

The *support function* $h_K : S^{d-1} \to \mathbb{R}$ of a convex body K is defined as $h_K(u) = \sup\{\langle u, y\rangle : y \in K\}$. It is well known that h_K is monotone with respect to inclusion (i.e., if $K \subseteq L$, then $h_K \leq h_L$), and positive homogeneous (i.e., $h_{\alpha K}(u) = h_K(\alpha u) = \alpha h_K$ for all $\alpha > 0$). Furthermore, if $0 \in K$, then $h_K(u)$ is the distance from the origin to the supporting hyperplane of K with outer unit normal vector u (more about support functions and properties, see [16]). When the origin is an interior point of K, its *radial function* $\rho_K(u)$ is defined by $\rho_K(u) = \max\{\alpha \geq 0 : \alpha u \in K\}$. The following relation between these two functions is well known:

(1) $$\rho_{K^\circ}(u) = \frac{1}{h_K(u)}, \quad u \in S^{d-1}.$$

It is easy to observe that if K is a centered convex body, then $2\rho_K(u) = \lambda_1(K \cap l_u)$, and $2h_K(u) = \lambda_1(K|l_u)$ for any $u \in S^{d-1}$.

For $K \in \mathcal{K}$, the *projection body* ΠK of K is defined as the convex body whose supporting hyperplane in a given direction u has a distance $\lambda_{d-1}(K|u^\perp)$ from the origin, i.e., $h_{\Pi K}(u) = \lambda_{d-1}(K|u^\perp)$ for each $u \in S^{d-1}$ (see [5, Chapter 4]). Note that any projection body is a *zonoid* (i.e., a limit of vector sums of segments) centered at the origin.

The following well-known fact (see [5]) will be used here for centered convex bodies; if S be a subspace of $\mathbb{R}^d$ and $B \in \mathcal{K}_o$, then

(2) $$(B \cap S)^\circ = B^\circ|S.$$

For $B \in \mathcal{K}_o$, the special case of *Blaschke–Santaló inequality* states that

$$\lambda(B)\lambda(B^\circ) \leq \varepsilon_d^2,$$

with equality if and only if B is an ellipsoid (proved by Blaschke [3] in $\mathbb{R}^2$ and $\mathbb{R}^3$, then by Santaló [15] for all dimensions; more about this inequality the reader should refer to [5] and the references therein).

3 Inequalities for inscribed double cones and circumscribed cylinders

First, we give the following alternative proof of Lutwak's minimax inequality for inscribed double cones. For the sake of simplicity, we will omit the constant $(2d)^{-1}$ in the volume equation of inscribed double cones.

Theorem 1 *If $B \in \mathcal{K}_o$, then*

$$\min_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)\lambda(B^\circ) \le 2\varepsilon_{d-1}\varepsilon_d,$$

with equality if and only if B is an ellipsoid.

Proof It follows from Petty's result that there exists a unit vector $u \in S^{d-1}$ such that

$$\frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)} \ge \frac{2\varepsilon_{d-1}}{\varepsilon_d}. \tag{3}$$

From the Blaschke–Santaló inequality and the identity (2), we have

$$\lambda_{d-1}(B^\circ|u^\perp) \le \frac{\varepsilon_{d-1}^2}{\lambda_{d-1}((B^\circ|u^\perp)^\circ)} = \frac{\varepsilon_{d-1}^2}{\lambda_{d-1}(B \cap u^\perp)}, \tag{4}$$

with equality if and only if $B^\circ|u^\perp$ is an ellipsoid.

The identity (1) can be written as $2\rho_{B^\circ}(u)2h_B(u) = 4$ for every $u \in S^{d-1}$. It is equivalent to

$$\lambda_1(B^\circ \cap l_u)\lambda_1(B|l_u) = 4. \tag{5}$$

Therefore, (3)–(5) yield that there exists $u \in S^{d-1}$ such that

$$\frac{\varepsilon_{d-1}^2}{\lambda_{d-1}(B \cap u^\perp)}\frac{4}{\lambda_1(B|l_u)} \ge \lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u) \ge \frac{2\varepsilon_{d-1}}{\varepsilon_d}\lambda(B^\circ).$$

Thus, the result follows. Obviously, if B is an ellipsoid, then equality holds. Now, assume that

$$\min_{u \in S^{d-1}} \lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)\lambda(B^\circ) = 2\varepsilon_{d-1}\varepsilon_d.$$

Then, for every $u \in S^{d-1}$,

$$\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)\lambda(B^\circ) \ge 2\varepsilon_{d-1}\varepsilon_d.$$

Applying (2), (4), and the Blaschke–Santaló inequality to the last inequality yields

$$\frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)} \le \frac{2\varepsilon_{d-1}}{\varepsilon_d},$$

for all $u \in S^{d-1}$. It follows from Petty's result that B° must be an ellipsoid. Hence, B is an ellipsoid. ∎

Problem 6 of [4] asks to find a centered convex body B of $\mathbb{R}^d$ for which

$$\max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)}$$

is minimal. It was conjectured that the ellipsoid is the answer to this problem. Related to this conjecture, we prove the following result.

Theorem 2 *Let B be a centered convex body in $\mathbb{R}^d$. If*

$$\max_{u\in S^{d-1}} \frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d},$$

then

$$\min_{u\in S^{d-1}} \lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ\cap l_u)\lambda(B) \leq 2\varepsilon_{d-1}\varepsilon_d.$$

Proof It follows from the hypothesis of the theorem that there exists $u\in S^{d-1}$ such that

$$\frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \geq \frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Using the Blaschke–Santaló inequality and (2), we get

$$\lambda_{d-1}(B\cap u^\perp) \leq \frac{\varepsilon_{d-1}^2}{\lambda_{d-1}(B^\circ|u^\perp)},$$

with equality if and only if $B\cap u^\perp$ is an ellipsoid. The result can be established using the identity (5). One can also see that if B is an ellipsoid, then equality holds for both, the hypothesis and the conclusion of theorem. Now, assume that

$$\min_{u\in S^{d-1}} \lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ\cap l_u)\lambda(B) = 2\varepsilon_{d-1}\varepsilon_d.$$

Then

$$\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ\cap l_u)\lambda(B) \geq 2\varepsilon_{d-1}\varepsilon_d,$$

for all $u\in S^{d-1}$. Applying (5), the Blaschke–Santaló inequality, and (2) to the last inequality, we get

$$\frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \leq \frac{2\varepsilon_{d-1}}{\varepsilon_d},$$

for all $u\in S^{d-1}$. Thus,

$$\max_{u\in S^{d-1}} \frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} = \frac{2\varepsilon_{d-1}}{\varepsilon_d}. \quad \blacksquare$$

In [4], it was also mentioned that the maximum value of the quantity

$$\min_{u\in S^{d-1}} \frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)}$$

is still unsolved when $d\geq 3$. Similar to Theorem 2, one can easily deduce the following related result.

Theorem 3 *Let $B\in\mathcal{K}_o$. If*

$$\max_{u\in S^{d-1}} \lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ\cap l_u)\lambda(B) \geq 2\varepsilon_{d-1}\varepsilon_d,$$

then

$$\min_{u\in S^{d-1}} \frac{\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)} \le \frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Furthermore, if $\min_{u\in S^{d-1}} \lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)\lambda(B)^{-1} = 2\varepsilon_{d-1}\varepsilon_d^{-1}$, *then* $\max_{u\in S^{d-1}} \lambda_{d-1}(B^{\circ}|u^{\perp})\lambda_1(B^{\circ}\cap l_u)\lambda(B) = 2\varepsilon_{d-1}\varepsilon_d$.

One of the challenging problems in convex geometry is that whether a centered convex body B and ΠB° (or B° and ΠB) are homothetic if and only if B is an ellipsoid. We mention that the *isoperimetrix* of *Holmes-Thompson measure* is defined as $\hat{I}_B = \varepsilon_d \lambda^{-1}(B^{\circ})\varepsilon_{d-1}^{-1}\Pi B^{\circ}$ (see [17]). The *relative inner radius* $r(B,\hat{I}_B)$ of B with respect to $\hat{I}_B$ is the largest $\alpha > 0$ such that $\alpha\hat{I}_B \subseteq B$, and the *relative outer radius* $R(B,\hat{I}_B)$ of B with respect to $\hat{I}_B$ is the smallest $\alpha > 0$ such that $B \subseteq \alpha\hat{I}_B$. In [11], it was shown that

$$R(B,\hat{I}_B) = \max_{u\in S^{d-1}} \frac{2\varepsilon_{d-1}}{\varepsilon_d} \frac{\lambda(B^{\circ})}{\lambda_{d-1}(B^{\circ}|u^{\perp})\lambda_1(B^{\circ}\cap l_u)} \tag{6}$$

and

$$r(B,\hat{I}_B) = \min_{u\in S^{d-1}} \frac{2\varepsilon_{d-1}}{\varepsilon_d} \frac{\lambda(B^{\circ})}{\lambda_{d-1}(B^{\circ}|u^{\perp})\lambda_1(B^{\circ}\cap l_u)}. \tag{7}$$

Theorem 4 *Let B be a centered convex body in $\mathbb{R}^d$. If*

$$\max_{u\in S^{d-1}} \lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)\lambda(B^{\circ}) \ge 2\varepsilon_{d-1}\varepsilon_d,$$

then B and ΠB° are homothetic if and only if B is an ellipsoid.

Proof Similar to Theorem 1, one can easily establish that if

$$\max_{u\in S^{d-1}} \lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)\lambda(B^{\circ}) \ge 2\varepsilon_{d-1}\varepsilon_d,$$

then

$$\min_{u\in S^{d-1}} \frac{\lambda_{d-1}(B^{\circ}|u^{\perp})\lambda_1(B^{\circ}\cap l_u)}{\lambda(B^{\circ})} \le \frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Therefore,

$$1 \le \frac{2\varepsilon_{d-1}}{\varepsilon_d} \max_{u\in S^{d-1}} \frac{\lambda(B^{\circ})}{\lambda_{d-1}(B^{\circ}|u^{\perp})\lambda_1(B^{\circ}\cap l_u)}.$$

Hence, the identity (6) yields $R(B,\hat{I}_B) \ge 1$.

Applying Petty's result to (7), we obtain that $r(B,\hat{I}_B) \le 1$, with equality if and only if B is an ellipsoid (see also [17]). One can observe that B and ΠB° are homothetic if and only if $r(B,\hat{I}_B) = R(B,\hat{I}_B)$. It could be the case only if B is an ellipsoid. ■

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