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Quandle Cocycle Invariants for Spatial Graphs and Knotted Handlebodies

Dedicated to Professor Akio Kawauchi for his 60th birthday

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Abstract. We introduce a flow of a spatial graph and see how invariants for spatial graphs and handlebody-links are derived from those for flowed spatial graphs. We define a new quandle (co)homology by introducing a subcomplex of the rack chain complex. Then we define quandle colorings and quandle cocycle invariants for spatial graphs and handlebody-links.

1 Introduction

In this paper, we introduce flowed spatial graphs and define quandle cocycle invariants for spatial graphs and handlebody-links. Carter, Jelsovsky, Kamada, Langford, and Saito [1] defined quandle cocycle invariants for links and surface-links. It was proved that a quandle cocycle invariant detects non-invertibility for surface-links in [1] and chirality for links in [2, 12]. We remark that the fundamental quandle cannot detect them, although the fundamental quandle is stronger than the fundamental group. A quandle cocycle invariant is useful in determining the triple point number, the triple point cancelling number, the w-index, and so on (cf. [4, 5, 13]).

A spatial graph is a finite graph embedded in the 3-sphere. An invariant for links is that for spatial graphs, since equivalent spatial graphs have the equivalent constituent links. Our invariant distinguishes spatial graphs whose constituent links are equivalent. The Yamada polynomial [16] is an invariant for spatial graphs without vertices of degree greater than 3, where we remark that the Yamada polynomial of a general spatial graph is an invariant as a flat vertex graph. Our invariant is defined for all spatial graphs and distinguishes spatial graphs whose Yamada polynomials coincide.

A handlebody-link is a disjoint union of handlebodies embedded in the 3-sphere. We can use an invariant for 3-manifolds to distinguish handlebody-links. For example, the fundamental group of the exterior of a handlebody-link is an invariant. However, these invariants do not work for handlebody-links with homeomorphic exteriors, which implies that we cannot detect the chirality of a handlebody-link by using these invariants. In [3], the first author defined a weight sum invariant for handlebody-links by using Mochizuki's 3-cocycle and showed that the invariant can detect the chirality. The quandle cocycle invariant defined in this paper is a generalization of this invariant. We show that the quandle cocycle invariant is non-trivial

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for the handlebody-link represented by Kinoshita's θ-curve, where we remark that the previous weight sum invariant is trivial for the handlebody-link.

There are two steps needed to define a quandle cocycle invariant. First we have to define a quandle coloring. It is not easy to define quandle colorings for spatial graphs and handlebody-links, since a suitable coloring condition for a vertex is unknown. In [3], the first author introduced an enhanced constituent link and defined a kei coloring, where a kei is a particular type of quandle. In this paper, we introduce a flow, which is a generalization of an enhanced constituent link. In the second step, we define a suitable quandle (co)homology and a quandle cocycle invariant. We define a new subcomplex of the rack chain complex. The quotient complex gives a (co)homology for spatial graphs and handlebody-links. We also show that our quandle cocycle invariant does not depend on the choice of the representative element of a cohomology class.

This paper consists of nine sections. In Section 2, we introduce a flow of a spatial graph and see how invariants for spatial graphs and handlebody-links are derived from those for flowed spatial graphs. In Section 3, we recall the definitions of a quandle X and an X-set and define the type of a quandle. In Section 4, we define a quandle coloring for flowed spatial graphs. In Section 5, we give some examples for the quandle coloring. In Section 6, we introduce a quandle (co)homology for spatial graphs and handlebody-links. In Section 7, we define a quandle cocycle invariant for spatial graphs and handlebody-links. In Section 8, we evaluate a quandle cocycle invariant for Kinoshita's θ-curve. In Section 9, we prove the theorems that were introduced in Section 7.

2 A Flow of a Spatial Graph

We introduce a flow of a spatial graph and see how invariants for spatial graphs and handlebody-links are derived from those for flowed spatial graphs.

Let G be a finite graph without vertices of degree 0. A *spatial graph* $L = f(G)$ is a graph G embedded in the 3-sphere S^3. Two spatial graphs are *equivalent* if one can be transformed into the other by an isotopy of S^3.

Let $\mathcal{E}(L)$ be the set of edges of L. Let $\mathcal{O}_e$ be the set of two orientations of an edge $e \in \mathcal{E}(L)$. Let A be an abelian group. A map $\varphi_e : \mathcal{O}_e \to A$ is an *A-flow* of an edge e if $\varphi_e(-o) = -\varphi_e(o)$, where $-o$ is the inverse of $o \in \mathcal{O}_e$. An A-flow φ_e is represented by a pair $(o, s) \in \mathcal{O}_e \times A$ up to the equivalence relation $(o, s) \sim (-o, -s)$; see Figure 2.1, where an element of A is represented with an underline. We fix an orientation o_e for each edge e of L. A collection $\varphi = \{\varphi_e\}_{e\in\mathcal{E}(L)}$ is an *A-flow* of L if we have

$$\sum_{e\in\mathcal{E}_{\mathrm{in}}(v)} \varphi_e(o_e) = \sum_{e\in\mathcal{E}_{\mathrm{out}}(v)} \varphi_e(o_e)$$

at any vertex v, where

$$\mathcal{E}_{\mathrm{in}}(v) := \{e \mid e \text{ is an edge incident to } v \text{ such that } o_e \text{ points to } v\},$$

$$\mathcal{E}_{\mathrm{out}}(v) := \{e \mid e \text{ is an edge incident to } v \text{ such that } -o_e \text{ points to } v\}.$$

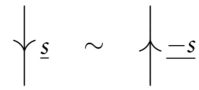

Figure 2.1

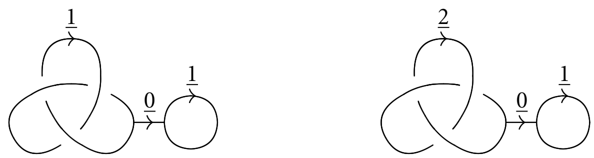

Figure 2.2

We remark that the definition of an A-flow of L does not depend on the choice of the orientations o_e. We denote by $\mathrm{Flow}(L; A)$ the set of A-flows of L.

An *A-flowed spatial graph* (L, φ) is a pair of a spatial graph L and $\varphi \in \mathrm{Flow}(L; A)$. Two A-flowed spatial graphs are *equivalent* if one can be transformed into the other by an ambient isotopy preserving an A-flow. We note that the two $\mathbb{Z}$-flowed spatial graphs depicted in Figure 2.2 are not equivalent.

By taking suitable subsets of $\mathrm{Flow}(L; A)$, we obtain many spatial graph invariants from an A-flowed spatial graph invariant. In the following proposition, we give some specific constructions of spatial graph invariants.

Proposition 2.1 *Let Ψ be an invariant for A-flowed spatial graphs.*

- *Let L be a spatial graph. Let B be a subset of A such that $B = -B$. We set*

$$\mathrm{Flow}(L; B) := \{\varphi \in \mathrm{Flow}(L; A) \mid \varphi_e(\mathcal{O}_e) \subset B \textit{ for any edge } e \textit{ of } L\}.$$

  Then the multiset $\{\Psi(L, \varphi) \mid \varphi \in \mathrm{Flow}(L; B)\}$ is an invariant of L.
- *Let (L, O) be an oriented spatial graph, where O is an assignment of an orientation $O(e) \in \mathcal{O}_e$ to each edge e of L. Let B be a subset of A. We set*

$$\mathrm{Flow}(L, O; B) := \{\varphi \in \mathrm{Flow}(L; A) \mid \varphi_e(O(e)) \in B \textit{ for any edge } e \textit{ of } L\}.$$

  Then the multiset $\{\Psi(L, \varphi) \mid \varphi \in \mathrm{Flow}(L, O; B)\}$ is an invariant of (L, O).

Then, for an A-flowed spatial graph invariant Ψ, we define the spatial graph invariant Ψ^{Σ} by

$$\Psi^{\Sigma}(L) = \{\Psi(L, \varphi) \mid \varphi \in \mathrm{Flow}(L; A)\},$$

where we note that Σ is just a symbol. Proposition 2.1 follows immediately from the fact that $\mathrm{Flow}(L; B)$ and $\mathrm{Flow}(L, O; B)$ do not depend on the embedding f for $L = f(G)$.

We state one lemma for constructions of A-flowed spatial graph invariants, since they play a critical role in Proposition 2.1. Two spatial graph diagrams represent an equivalent spatial graph if and only if they are related by a finite sequence of the R1–R5 moves depicted in Figure 2.3 ([8,16,17]). Then we have the following lemma.

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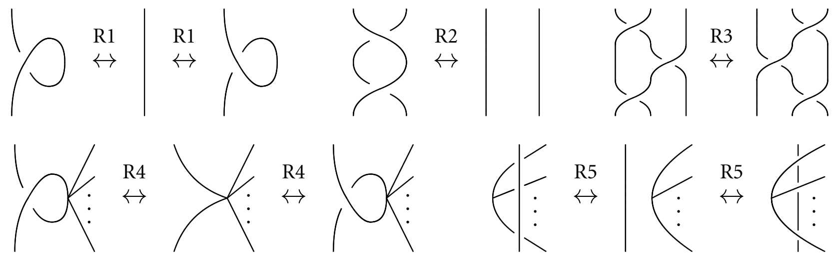

Figure 2.3

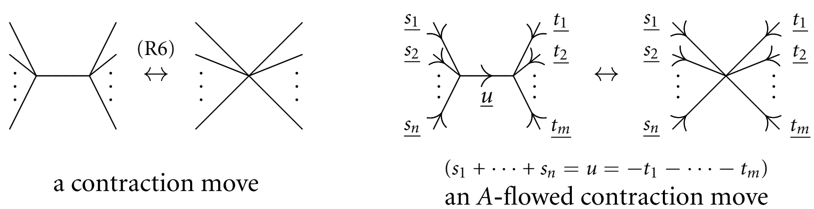

Figure 2.4

Lemma 2.2 *Two A-flowed spatial graph diagrams represent an equivalent A-flowed spatial graph if and only if they are related by a finite sequence of the A-flowed R1–R5 moves, where the A-flowed R1–R5 moves are the R1–R5 moves preserving A-flows.*

A *handlebody-knot* is a handlebody embedded in S^3. A *handlebody-link* is a disjoint union of handlebody-knots. Two handlebody-links are *equivalent* if one can be transformed into the other by an isotopy of S^3. When a handlebody-link H is a regular neighborhood of a spatial graph L, we say that H is *represented* by L. We note that two spatial graphs representing an equivalent handlebody-link are said to be neighborhood equivalent ([14]).

An *(A-flowed) contraction move* is a local change of an (A-flowed) spatial graph as described in Figure 2.4, where the replacement is applied in a disk embedded in S^3. An (A-flowed) R6 move is the diagrammatic move corresponding to the (A-flowed) contraction move. Then we have the following theorem.

Theorem 2.3 ([3]) *Two spatial graphs represent an equivalent handlebody-link if and only if they are related by a finite sequence of contraction moves and ambient isotopies.*

By Proposition 2.1 and Theorem 2.3, we have the following proposition.

Proposition 2.4 *Let Ψ be an invariant for A-flowed spatial graphs. If Ψ is invariant under A-flowed contraction moves, then the multiset $\Psi^{\Sigma}(L)$ is an invariant of a handlebody-link represented by a spatial graph L.*

We state one lemma for constructions of A-flowed spatial graph invariants that are invariant under A-flowed contraction moves. By Lemma 2.2, we have the following lemma, since we may apply an A-flowed contraction move in a small disk by an isotopy of S^3.

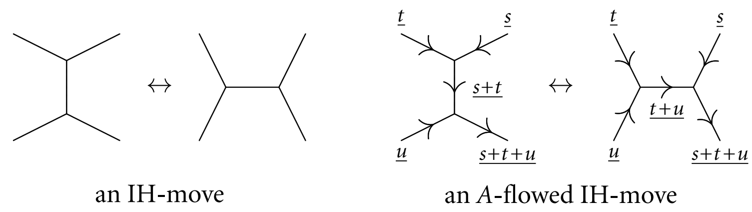

Figure 2.5

Lemma 2.5 *Let D_1 and D_2 be diagrams of A-flowed spatial graphs (L_1, φ_1) and (L_2, φ_2), respectively. The following statements are equivalent:*

- *Two A-flowed spatial graphs (L_1, φ_1) and (L_2, φ_2) are related by a finite sequence of A-flowed contraction moves and ambient isotopies preserving A-flows.*
- *Two diagrams D_1 and D_2 are related by a finite sequence of the A-flowed R1–R6 moves.*

Remark 2.6 We do not need all spatial graphs to represent all handlebody-links. Spatial trivalent graphs are sufficient to represent all handlebody-links, where a spatial trivalent graph may contain circle components. An *(A-flowed) IH-move* is a local change of an (A-flowed) spatial trivalent graph as described in Figure 2.5, where the replacement is applied in a disk embedded in S^3. Then, in Theorem 2.3, Proposition 2.4 and Lemma 2.5, we can replace spatial graphs and contraction moves with spatial trivalent graphs and IH-moves, respectively (see [3]).

3 A Quandle

We recall the definitions of a quandle X and an X-set, and define the type of a quandle.

A *quandle* ([6, 9]) is a non-empty set X with a binary operation $* : X \times X \to X$ satisfying the following axioms:

Q_1. For any $a \in X$, $a * a = a$;
Q_2. For any $a \in X$, the map $S_a : X \to X$ defined by $S_a(x) = x * a$ is a bijection;
Q_3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

We present some examples of quandles. A *trivial quandle* $(X, *)$ is a non-empty set X with the binary operation defined by $a * b = a$. The *dihedral quandle of order* p, denoted by $(R_p, *)$, is the quandle consisting of the set $\mathbb{Z}_p (:= \mathbb{Z}/p\mathbb{Z})$ with the binary operation defined by $a*b = 2b-a$. The *tetrahedral quandle*, denoted by $(S_4, *)$, is the quandle consisting of the set $\mathbb{Z}_2[t, t^{-1}]/(t^2 + t + 1)$ with the binary operation defined by $a * b = ta + (1 - t)b$. In general, an *Alexander quandle* $(M, *)$ is a Λ-module M with the binary operation defined by $a*b = ta+(1-t)b$, where $\Lambda := \mathbb{Z}[t, t^{-1}]$. Then the tetrahedral quandle is an Alexander quandle. We also remark that the dihedral quandle $(R_p, *)$ is isomorphic to the Alexander quandle $(\mathbb{Z}_p[t, t^{-1}]/(t + 1), *)$ as

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quandles. An *n-fold conjugation quandle* $(G, *)$ is a group G with the binary operation defined by $a * b = b^{-n}ab^n$.

The *associated group* of a quandle X, denoted by $\mathrm{As}(X)$, is defined by

$$\mathrm{As}(X) = \left\langle x \in X \mid x * y = y^{-1}xy \ (x, y \in X) \right\rangle.$$

An *X-set* is a set Y equipped with an action of the associated group $\mathrm{As}(X)$ from the right. We denote by $y \tilde{*} g$ the image of an element $y \in Y$ by the action $g \in \mathrm{As}(X)$. Then we have the following:

$\widetilde{\mathrm{Q}}_2$. For any $a \in X$, the map $\widetilde{S}_a : Y \to Y$ defined by $\widetilde{S}_a(y) = y \tilde{*} a$ is a bijection;
$\widetilde{\mathrm{Q}}_3$. For any $y \in Y, a, b \in X$, $(y \tilde{*} a) \tilde{*} b = (y \tilde{*} b) \tilde{*} (a * b)$.

We show examples for X-sets. We set $Y := X$ and $y \tilde{*} a := y * a$. Then $(Y, \tilde{*})$ is an X-set. We set $Y := \{y\}$ and $y \tilde{*} a := y$. Then $(Y, \tilde{*})$ is an X-set.

For $i \in \mathbb{Z}$, we define $a *^i b := S_b^i(a)$, $y \tilde{*}^i a := \widetilde{S}_a^i(y)$. The *type* of a quandle X is defined by

$$\operatorname{type} X := \min\{i \in \mathbb{Z}_{>0} \mid a *^i b = a \text{ for any } a, b \in X\}.$$

We set $\operatorname{type} X := \infty$ if we do not have such a positive integer i. If X is finite, then $\operatorname{type} X < \infty$. A trivial quandle is of type 1. The dihedral quandle $(R_p, *)$ is of type 2. A quandle of type 2 is called kei ([15]). The tetrahedral quandle $(S_4, *)$ is of type 3. We note that a quandle of type n is an n-quandle ([7]). In this paper, we set $\mathbb{Z}_\infty := \mathbb{Z}$. Then $a *^i b$ is well defined for $i \in \mathbb{Z}_{\operatorname{type} X}$. We define

$$\operatorname{type} X_Y := \min\{i \in \mathbb{Z}_{>0} \mid a *^i b = a,\ y \tilde{*}^i a = y \text{ for any } a, b \in X, y \in Y\}.$$

We set $\operatorname{type} X_Y := \infty$ if we do not have such a positive integer i. Then $a *^i b$ and $y \tilde{*}^i a$ are well-defined for $i \in \mathbb{Z}_{\operatorname{type} X_Y}$.

4 A Quandle Coloring for Flowed Spatial Graphs

We define a quandle coloring for flowed spatial graphs. The number of quandle colorings is an invariant for flowed spatial graphs. We also define a coloring by using a quandle X and an X-set, which is used to define a quandle cocycle invariant in Section 7.

Let X be a quandle. Let D be a diagram of a $\mathbb{Z}_{\operatorname{type} X}$-flowed spatial graph (L, φ). We denote by $\mathcal{A}(D)$ the set of arcs of D, where an arc is a piece of a curve such that its endpoint is an undercrossing or a vertex.

We choose an orientation $O(e) \in \mathcal{O}_e$ for each edge $e \in \mathcal{E}(L)$. Then (L, O, φ) is a $\mathbb{Z}_{\operatorname{type} X}$-flowed oriented spatial graph. For an arc α that originates from an edge e, we put $O(\alpha) := O(e)$, $\varphi_\alpha := \varphi_e$. To represent an orientation $O(e)$ in D, we may use the co-orientation obtained by rotating the orientation $O(e)$ $\pi/2$ counterclockwise. We denote it by the same symbol $O(\alpha)$. We denote by χ_0 the over-arc at a crossing χ of D. We denote by χ_1, χ_2 the under-arcs at χ such that the co-orientation $O(\chi_0)$ points to χ_2.

An *X-coloring* of D is a map $C \colon \mathcal{A}(D) \to X$ satisfying the following conditions (Figure 4.1):

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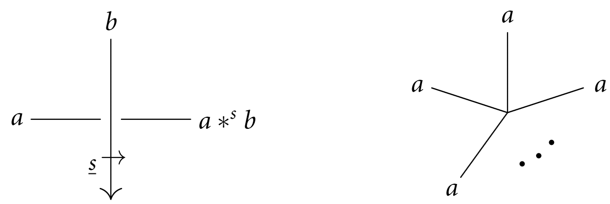

Figure 4.1

C_1. For a crossing χ, we have $C(\chi_1) *^{\varphi_{\chi_0}(O(\chi_0))} C(\chi_0) = C(\chi_2)$.
C_2. For a vertex ω, we have $C(\omega_1) = \cdots = C(\omega_d)$, where $\omega_1, \ldots, \omega_d$ are the arcs incident to ω.

An X-coloring C does not depend on the choice of the orientations $O(e)$, since the equality in C_1 is equivalent to the equality

$$C(\chi_2) *^{\varphi_{\chi_0}(-O(\chi_0))} C(\chi_0) = C(\chi_1).$$

We denote by $Col_X(D)$ the set of X-colorings of D. For two diagrams D and E that locally differ, we denote by $\mathcal{A}(D, E)$ the set of arcs that D and E share.

Theorem 4.1 *Let X be a quandle. Let D be a diagram of a $\mathbb{Z}_{\text{type}\,X}$-flowed spatial graph (L, φ). Let E be a diagram obtained by applying one of the $\mathbb{Z}_{\text{type}\,X}$-flowed R1–R6 moves to D once. For $C \in Col_X(D)$, there is a unique X-coloring $C_{D,E} \in Col_X(E)$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$.*

By Lemma 2.5, $\#Col_X(D)$ is an invariant of (L, φ), which is invariant under $\mathbb{Z}_{\text{type}\,X}$-flowed contraction moves, where $\#S$ is the number of elements in a set S. Then we put $\#Col_X(L, \varphi) := \#Col_X(D)$. When $\#Col_X(L, \varphi) = \infty$, the number of nontrivial X-colorings of D may work, where an X-coloring C of D is *trivial* if $C : \mathcal{A}(D) \to X$ is a constant map. We call $C(\xi)$ the *color* of ξ.

Proof of Theorem 4.1 The color of an edge in $\mathcal{A}(E) - \mathcal{A}(D, E)$ is uniquely determined by the colors of edges in $\mathcal{A}(D, E)$, since we have $a *^s a = a$ for the $\mathbb{Z}_{\text{type}\,X}$-flowed R1, R4 moves, and

$$(\cdots((a *^{i_1} b) *^{i_2} b) \cdots) *^{i_l} b = a \quad (i_1 + i_2 + \cdots + i_l = 0 \text{ in } \mathbb{Z}_{\text{type}\,X})$$

for the $\mathbb{Z}_{\text{type}\,X}$-flowed R2, R5 moves, and

$$(a *^s b) *^t c = (a *^t c) *^s (b *^t c)$$

for the $\mathbb{Z}_{\text{type}\,X}$-flowed R3 move, and C_2 for the $\mathbb{Z}_{\text{type}\,X}$-flowed R6 moves. ■

We denote by $\mathcal{R}(D)$ the set of connected regions of the complement of the underlying immersed graph of D. An *X_Y-coloring* of D is a map

$$C\colon \mathcal{A}(D) \cup \mathcal{R}(D) \to X \cup Y$$

such that $C|_{\mathcal{A}(D)}\colon \mathcal{A}(D) \to X$ is an X-coloring of D and that $C|_{\mathcal{R}(D)}\colon \mathcal{R}(D) \to Y$ satisfies the following condition:

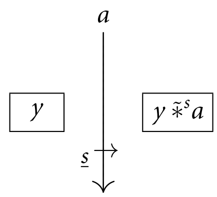

Figure 4.2

C$_3$. For regions α_1, α_2 sharing an arc α such that the co-orientation $O(\alpha)$ points to α_2, we have

$$C(\alpha_1)\,\tilde{*}^{\varphi_\alpha(O(\alpha))}C(\alpha) = C(\alpha_2)$$

(see Figure 4.2). An X_Y-coloring C does not depend on the choice of the orientations $O(e)$, since the equality in C$_3$ is equivalent to the equality

$$C(\alpha_2)\,\tilde{*}^{\varphi_\alpha(-O(\alpha))}C(\alpha) = C(\alpha_1).$$

We denote by $Col_X(D)_Y$ the set of X_Y-colorings of D. For two diagrams D and E that locally differ, we denote by $\mathcal{R}(D,E)$ the set of regions that D and E share. By $\widetilde{\mathrm{Q}}_3$, colors of regions are uniquely determined by those of arcs and one region. Therefore, by Theorem 4.1 we have the following theorem.

Theorem 4.2 *Let X be a quandle, and let Y be an X-set. Let D be a diagram of a $\mathbb{Z}_{\mathrm{type}\,X_Y}$-flowed spatial graph (L,φ). Let E be a diagram obtained by applying one of the $\mathbb{Z}_{\mathrm{type}\,X_Y}$-flowed R1–R6 moves to D once. For $C \in Col_X(D)_Y$, there is a unique X_Y-coloring $C_{D,E} \in Col_X(E)_Y$ such that*

$$C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)} \quad \text{and} \quad C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}.$$

This theorem implies that $\#Col_X(D)_Y$ is an invariant of (L,φ). Unfortunately, this invariant is not important, since we have the equality $\#Col_X(D)_Y = \#Y\#Col_X(D)$. Theorem 4.2 is used to define a quandle cocycle invariant for flowed spatial graphs in Section 7.

5 Examples for a Quandle Coloring

We give some examples for a quandle coloring. We represent the multiplicity of an element of a multiset by a subscript with an underline. For example, $\{a_{\underline{1}}, b_{\underline{2}}, c_{\underline{3}}\}$ represents the multiset $\{a, b, b, c, c, c\}$.

Let K^0 and K^1 be the spatial handcuff graphs as shown in Figure 5.1, where we ignore flows and colors. We cannot use link invariants to distinguish K^0 from K^1, since the constituent links of these spatial graphs coincide. The following example shows that K^0 and K^1 are not equivalent.

Example 5.1 For $s,t \in \mathbb{Z}_2$, $a,b \in R_3$, we denote by $C^0_{s,t}(a,b)$ (resp. $C^1_{s,t}(a)$) the R_3-coloring of the $\mathbb{Z}_2$-flowed spatial graph diagram $D^0_{s,t}$ (resp. $D^1_{s,t}$) corresponding to

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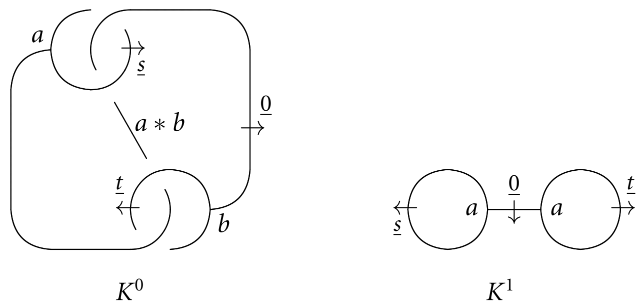

Figure 5.1

K^0 (resp. K^1) depicted in Figure 5.1. We note that type $R_3 = 2$. We have the equalities

$$Col_{R_3}(D^0_{1,1}) = \{C^0_{1,1}(a,b) \mid a,b \in R_3\}, \qquad \#Col_{R_3}(D^0_{1,1}) = 9,$$
$$Col_{R_3}(D^0_{s,t}) = \{C^0_{s,t}(a,a) \mid a \in R_3\}, \qquad \#Col_{R_3}(D^0_{s,t}) = 3$$

for $(s,t) \in \mathbb{Z}_2^2 - \{(1,1)\}$, which imply $\#Col^{\Sigma}_{R_3}(K^0) = \{9, 3_{\underline{3}}\}$. We have the equalities

$$Col_{R_3}(D^1_{s,t}) = \{C^1_{s,t}(a) \mid a \in R_3\}, \qquad \#Col_{R_3}(D^1_{s,t}) = 3$$

for $(s,t) \in \mathbb{Z}_2^2$, which imply $\#Col^{\Sigma}_{R_3}(K^1) = \{3_{\underline{4}}\}$. Thus K^0 and K^1 are not equivalent. Furthermore, K^0 and K^1 represent nonequivalent handlebody-links.

Let K^2 and K^3 be the spatial θ-curves as shown in Figure 5.2, where we ignore flows and colors. The Yamada polynomials $R(K^2)$ and $R(K^3)$ coincide:

$$R(K^2) = R(K^3) = (A^4 - A^2 + A + 1 - 2A^{-1} + A^{-2} + A^{-3} - A^{-4})^2 R(\theta),$$
$$R(\theta) = -A^2 - A - 2 - A^{-1} - A^{-2}.$$

We refer the reader to [16] for the definition and evaluation of the Yamada polynomial. The following example shows that K^2 and K^3 are not equivalent.

Example 5.2 For $s,t \in \mathbb{Z}_2$, $a,b \in R_3$, we denote by $C^2_{s,t}(a,b)$ (resp. $C^3_{s,t}(a)$) the R_3-coloring of the $\mathbb{Z}_2$-flowed spatial graph diagram $D^2_{s,t}$ (resp. $D^3_{s,t}$) corresponding to K^2 (resp. K^3) depicted in Figure 5.2. We note that type $R_3 = 2$. We have the equalities

$$Col_{R_3}(D^2_{1,1}) = \{C^2_{1,1}(a,b) \mid a,b \in R_3\}, \qquad \#Col_{R_3}(D^2_{1,1}) = 9,$$
$$Col_{R_3}(D^2_{s,t}) = \{C^2_{s,t}(a,a) \mid a \in R_3\}, \qquad \#Col_{R_3}(D^2_{s,t}) = 3$$

for $(s,t) \in \mathbb{Z}_2^2 - \{(1,1)\}$, which imply $\#Col^{\Sigma}_{R_3}(K^2) = \{9, 3_{\underline{3}}\}$. We have the equalities

$$Col_{R_3}(D^3_{s,t}) = \{C^3_{s,t}(a) \mid a \in R_3\}, \qquad \#Col_{R_3}(D^3_{s,t}) = 3$$

for $(s,t) \in \mathbb{Z}_2^2$, which imply $\#Col^{\Sigma}_{R_3}(K^3) = \{3_{\underline{4}}\}$. Thus K^2 and K^3 are not equivalent. Furthermore, K^2 and K^3 represent nonequivalent handlebody-links.

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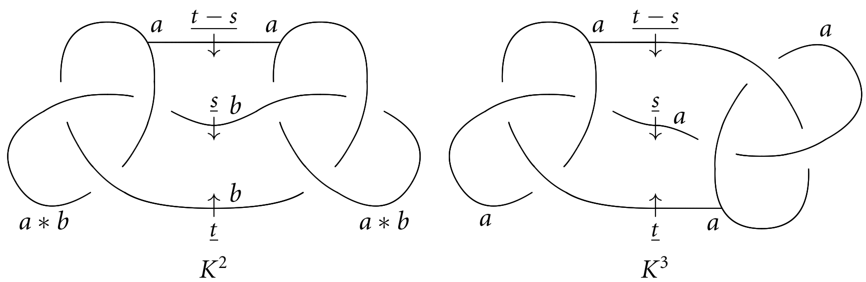

Figure 5.2

6 Quandle Homologies

Carter, Jelsovsky, Kamada, Langford, and Saito defined the quandle homology group $H_*^Q(X;A)$ and the quandle cohomology group $H_Q^*(X;A)$, and introduced quandle cocycle invariants. We note that a quandle 2-cocycle ϕ satisfies

$$\phi(a,a) = 0, \tag{6.1}$$

$$\phi(a,c) + \phi(a*c, b*c) = \phi(a,b) + \phi(a*b, c) \tag{6.2}$$

for any $a, b, c \in X$, and that a quandle 3-cocycle θ satisfies

$$\theta(a,a,b) = \theta(a,b,b) = 0, \tag{6.3}$$

$$\theta(a,c,d) + \theta(a*c, b*c, d) + \theta(a,b,c) = \theta(a*b, c, d) + \theta(a,b,d) + \theta(a*d, b*d, c*d) \tag{6.4}$$

for any $a, b, c, d \in X$. For the details we refer the reader to [1]. In this section, we introduce a new (co)homology theory to define a quandle cocycle invariant for $\mathbb{Z}_{\text{type}\, X_Y}$-flowed spatial graphs.

Let X be a quandle, and let Y be an X-set. Let $C_n^R(X)_Y$ be the free abelian group generated by $(n+1)$-tuples $(y, x_1, \dots, x_n)$, where $y \in Y$ and $x_1, \dots, x_n \in X$ if $n \geq 0$, and let $C_n^R(X)_Y = 0$ otherwise. Put

$$(y, x_1, \dots, x_n)_{i,j} := (y \,\tilde{*}^j x_i, x_1 *^j x_i, \dots, x_{i-1} *^j x_i, x_{i+1}, \dots, x_n),$$

$$(y, x_1, \dots, x_n)_{i,j}^+ := (y \,\tilde{*}^j x_i, x_1 *^j x_i, \dots, x_{i-1} *^j x_i, x_i, \dots, x_n).$$

We define a homomorphism $\partial_n : C_n^R(X)_Y \to C_{n-1}^R(X)_Y$ by

$$\partial_n(y, x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^i \big\{ (y, x_1, \dots, x_n)_{i,0} - (y, x_1, \dots, x_n)_{i,1} \big\}$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $C_*^R(X)_Y = \{C_n^R(X)_Y, \partial_n\}$ is a chain complex, since $\partial_{n-1} \circ \partial_n = 0$.

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Let $D_n^Q(X)_Y$ be the subgroup of $C_n^R(X)_Y$ generated by the elements of

$$\{(y, x_1, \dots, x_n) \in Y \times X^n \mid x_i = x_{i+1} \text{ for some } i\}$$

if $n > 1$, and let $D_n^Q(X)_Y = 0$ otherwise. Put $C_n^Q(X)_Y = C_n^R(X)_Y / D_n^Q(X)_Y$. Since $\partial_n(D_n^Q(X)_Y) \subset D_{n-1}^Q(X)_Y$, $C_*^Q(X)_Y = \{C_n^Q(X)_Y, \partial_n\}$ is a chain complex, where we denote the induced homomorphism by the same symbol ∂_n.

Let $D_n^I(X)_Y$ be the subgroup of $C_n^R(X)_Y$ generated by the elements of

$$\Big\{ \sum_{j=0}^{\operatorname{type} X_Y - 1} (y, x_1, \dots, x_n)^+_{i,j} \;\Big|\; (y, x_1, \dots, x_n) \in Y \times X^n,\ i = 1, \dots, n \Big\}$$

if $n > 0$ and $\operatorname{type} X_Y < \infty$, and let $D_n^I(X)_Y = 0$ otherwise. Then we have the following lemma.

Lemma 6.1 *We have $\partial_n(D_n^I(X)_Y) \subset D_{n-1}^I(X)_Y$.*

Proof We may suppose that $n > 0$ and $\operatorname{type} X_Y < \infty$. Let

$$\sigma = \sum_{j=0}^{\operatorname{type} X_Y - 1} (y, x_1, \dots, x_n)^+_{i,j} \in D_n^I(X)_Y,$$

where $i \in \{1, \dots, n\}$. We have $\sigma_{i,0} = \sigma_{i,1}$ by the equalities

$$a *^{\operatorname{type} X_Y} b = a, \quad y \mathbin{\tilde{*}}^{\operatorname{type} X_Y} a = y$$

for any $a, b \in X$, $y \in Y$. By $(a *^s b) *^t c = (a *^t c) *^s (b *^t c)$, we have

$$((y, x_1, \dots, x_n)^+_{i,j})_{k,1} = \begin{cases} ((y, x_1, \dots, x_n)_{k,1})^+_{i,j} & \text{if } k > i, \\ ((y, x_1, \dots, x_n)_{k,1})^+_{i-1,j} & \text{if } k < i. \end{cases}$$

Then $\sigma_{k,1} \in D_{n-1}^I(X)_Y$ if $k \neq i$, where

$$\sigma_{k,l} = \sum_{j=0}^{\operatorname{type} X_Y - 1} \big((y, x_1, \dots, x_n)^+_{i,j}\big)_{k,l}.$$

Since $\sigma_{k,0} \in D_{n-1}^I(X)_Y$ for $k \neq i$, we have

$$\begin{aligned} \partial_n(\sigma) &= \sum_{k=1}^{i-1} (-1)^k \sigma_{k,0} + (-1)^i \sigma_{i,0} + \sum_{k=i+1}^{n} (-1)^k \sigma_{k,0} \\ &\quad - \sum_{k=1}^{i-1} (-1)^k \sigma_{k,1} - (-1)^i \sigma_{i,1} - \sum_{k=i+1}^{n} (-1)^k \sigma_{k,1} \in D_{n-1}^I(X)_Y. \end{aligned}$$

■

We put $C^I_n(X)_Y = C^R_n(X)_Y/(D^Q_n(X)_Y + D^I_n(X)_Y)$. Then $C^I_*(X)_Y = \{C^I_n(X)_Y, \partial_n\}$ is a chain complex. For an abelian group A, we define the chain and cochain complexes

$$C^W_*(X;A)_Y = C^W_*(X)_Y \otimes A, \qquad \partial = \partial \otimes \mathrm{id};$$

$$C^*_W(X;A)_Y = \mathrm{Hom}(C^W_*(X)_Y, A), \qquad \delta = \mathrm{Hom}(\partial, \mathrm{id}),$$

where W is R, Q, or I. We denote by $H^W_n(X;A)_Y$ and $H^n_W(X;A)_Y$ the n-th homology group and the n-th cohomology group of $C^W_*(X;A)_Y$ and $C^*_W(X;A)_Y$, respectively. We note that, if type $X_Y = \infty$, then $C^I_*(X;A)_Y = C^Q_*(X;A)_Y$ and $C^*_I(X;A)_Y = C^*_Q(X;A)_Y$.

A map $f \in C^2_R(X;A)_Y$ induces a 2-cocycle of $C^*_Q(X;A)_Y$ if and only if f satisfies the conditions

$$f(y,a,a) = 0, \tag{6.5}$$

$$f(y,b,c) + f(y \,\bar{*}\, b, a * b, c) + f(y,a,b) = f(y \,\bar{*}\, a, b, c) + f(y,a,c) + f(y \,\bar{*}\, c, a * c, b * c), \tag{6.6}$$

for any $y \in Y$ and $a,b,c \in X$. We suppose that type $X_Y < \infty$. A map $f \in C^2_R(X;A)_Y$ induces a 2-cocycle of $C^*_I(X;A)_Y$ if and only if f satisfies the conditions (6.5), (6.6) and

$$\sum_{i=0}^{\mathrm{type}\, X_Y - 1} f(y \,\bar{*}^i a, a, b) = \sum_{i=0}^{\mathrm{type}\, X_Y - 1} f(y \,\bar{*}^i b, a *^i b, b) = 0, \tag{6.7}$$

for any $y \in Y$ and $a,b \in X$. Then, by the equalities (6.1)–(6.4), we have the following proposition, which is useful in finding 2-cocycles of $C^*_I(X;A)_Y$.

Proposition 6.2 *Let X be a quandle such that* type $X < \infty$. *For a quandle 2-cocycle ϕ, we define $1 \otimes \phi \in C^2_R(X;A)_{\{y\}}$ by $(1 \otimes \phi)(y,a,b) = \phi(a,b)$ for $a,b \in X$. Then $1 \otimes \phi$ is a 2-cocycle of $C^*_Q(X;A)_{\{y\}}$. Furthermore, if ϕ satisfies*

$$\mathrm{type}\, X \phi(a,b) = \sum_{i=0}^{\mathrm{type}\, X - 1} \phi(a *^i b, b) = 0$$

for any $a,b \in X$, then $1 \otimes \phi$ is a 2-cocycle of $C^*_I(X;A)_{\{y\}}$.

A quandle 3-cocycle θ is a 2-cocycle of $C^*_Q(X;A)_X$. Furthermore, if θ satisfies

$$\sum_{i=0}^{\mathrm{type}\, X - 1} \theta(a *^i b, b, c) = \sum_{i=0}^{\mathrm{type}\, X - 1} \theta(a *^i c, b *^i c, c) = 0$$

for any $a,b,c \in X$, then θ is a 2-cocycle of $C^*_I(X;A)_X$.

Example 6.3 (dihedral quandle R_p) Let p be an odd prime. The quandle cohomology group $H^3_Q(R_p;\mathbb{Z}_p) \cong \mathbb{Z}_p$ is generated by the cohomology class $[\theta_p]$ defined by

$$\theta_p(x,y,z) = (x-y)\frac{y^p + (2z-y)^p - 2z^p}{p},$$

where we remark that the right-hand side of the equality represents a polynomial with coefficients in $\mathbb{Z}_p$. We call θ_p *Mochizuki's* 3*-cocycle* [10]. We note that type $R_p = 2$. Since we have the equalities

$$\theta_p(x,y,z) + \theta_p(x*y,y,z) = \big((x-y)+(y-x)\big)\,\frac{y^p + (2z-y)^p - 2z^p}{p} = 0,$$

$$\theta_p(x,y,z) + \theta_p(x*z,y*z,z) = \big((x-y)+(y-x)\big)\,\frac{y^p + (2z-y)^p - 2z^p}{p} = 0,$$

θ_p is a 2-cocycle of $C^*_I(R_p;\mathbb{Z}_p)_{R_p}$.

T. Satoh and the authors discussed the cohomology group $H^3_I(R_p;\mathbb{Z}_p)_{R_p}$ in Osaka, and showed that $H^3_I(R_3;\mathbb{Z}_3)_{R_3} \cong \mathbb{Z}_3$ by direct calculation.

Example 6.4 (tetrahedral quandle S_4) Put $A := \mathbb{Z}_2[t,t^{-1}]/(t^2+t+1)$. The quandle cohomology group $H^3_Q(S_4;A) \cong A^3$ is generated by the cohomology classes $[f_1],[f_2],[f_3]$ defined by

$$f_1(x,y,z) = (x-y)(y-z)^2,$$
$$f_2(x,y,z) = t(x-y)(y-z)z,$$
$$f_3(x,y,z) = t^2(x-y)^2(y-z)^2z^2$$

(see [11]). We note that type $S_4 = 3$. Since we have the equalities

$$\begin{aligned}
&f_2(x,y,z) + f_2(x*y,y,z) + f_2(x*^2 y,y,z)\\
&\quad = t\big((x-y)+(tx-ty)+(t^2x-t^2y)\big)\,(y-z)z\\
&\quad = 0,\\
&f_2(x,y,z) + f_2(x*z,y*z,z) + f_2(x*^2 z, y*^2 z, z)\\
&\quad = t(x-y)(y-z)z + t(tx-ty)(ty-tz)z + t(t^2x-t^2y)(t^2y-t^2z)z\\
&\quad = 0,
\end{aligned}$$

f_2 is a 2-cocycle of $C^*_I(S_4;A)_{S_4}$. Similarly, f_3 is a 2-cocycle of $C^*_I(S_4;A)_{S_4}$.

7 A Quandle Cocycle Invariant for Flowed Spatial Graphs

A quandle cocycle invariant is a weight sum invariant. We define the Boltzmann weight at a crossing, and then we define a quandle cocycle invariant for flowed spatial graphs.

$\sum_{i=0}^{s-1}\sum_{j=0}^{t-1} f((y\,\bar{*}^i a)\,\bar{*}^j b, a *^j b, b)$ $\qquad$ $-\sum_{i=0}^{s-1}\sum_{j=0}^{t-1} f((y\,\bar{*}^i a)\,\bar{*}^j b, a *^j b, b)$

Figure 7.1

Let X be a quandle, and let Y be an X-set. Let f be a 2-cocycle of $C_I^*(X;A)_Y$. Let D be a diagram of a $\mathbb{Z}_{\operatorname{type} X_Y}$-flowed spatial graph (L,φ). We choose an orientation $O(e) \in \mathcal{O}_e$ for each edge $e \in \mathcal{E}(L)$ (such that $\varphi_e(O(e)) \geq 0$ if $\operatorname{type} X_Y = \infty$). Then (L, O, φ) is a $\mathbb{Z}_{\operatorname{type} X_Y}$-flowed oriented spatial graph. We denote by $\epsilon(\chi) \in \{1,-1\}$ the sign of a crossing χ of D. We denote by $\chi_{i,1}, \chi_{i,2}$ the regions sharing a crossing χ and the under-arc χ_i such that the co-orientation $O(\chi_i)$ points to $\chi_{i,2}$. We put

$$\bar{f}(y,a,s,b,t) := \sum_{i=0}^{s-1}\sum_{j=0}^{t-1} f((y\,\bar{*}^i a)\,\bar{*}^j b, a *^j b, b),$$

where we remark that $\bar{f}(y,a,s,b,t) = 0$ if $s = 0$ or $t = 0$. For an X_Y-coloring $C \in Col_X(D)_Y$, the Boltzmann weight $B_f(\chi;C)$ at a crossing χ is defined by

$$B_f(\chi;C) = \epsilon(\chi)\bar{f}\Big(C(\chi_{1,1}), C(\chi_1), \varphi_{\chi_1}\big(O(\chi_1)\big), C(\chi_0), \varphi_{\chi_0}\big(O(\chi_0)\big)\Big), \tag{7.1}$$

where we regard $\varphi_{\chi_1}(O(\chi_1))$ and $\varphi_{\chi_0}(O(\chi_0))$ as integers in $\{0,1,\ldots,\operatorname{type} X_Y - 1\}$ (see Figure 7.1).

Lemma 7.1 *The Boltzmann weight $B_f(\chi;C)$ does not depend on the choice of the orientations $O(e)$.*

Proof If $\varphi_{\chi_1}(O(\chi_1)) = 0$ or $\varphi_{\chi_0}(O(\chi_0)) = 0$, then the Boltzmann weight $B_f(\chi;C) = 0$ does not depend on the choice of the orientations, since we have $\varphi_{\chi_1}(-O(\chi_1)) = 0$ or $\varphi_{\chi_0}(-O(\chi_0)) = 0$. Then we may suppose that $\varphi_{\chi_1}(O(\chi_1)) \neq 0$, $\varphi_{\chi_0}(O(\chi_0)) \neq 0$ and $\operatorname{type} X_Y < \infty$. For the orientations $O(\chi_0)$, $-O(\chi_1)$, $-O(\chi_2)$, the Boltzmann weight $B_f(\chi;C)$ is given by

$$-\epsilon(\chi)\bar{f}\Big(C(\chi_{1,2}), C(\chi_1), \varphi_{\chi_1}\big(-O(\chi_1)\big), C(\chi_0), \varphi_{\chi_0}\big(O(\chi_0)\big)\Big). \tag{7.2}$$

For the orientations $-O(\chi_0)$, $O(\chi_1)$, $O(\chi_2)$, the Boltzmann weight $B_f(\chi;C)$ is given by

$$-\epsilon(\chi)\bar{f}\Big(C(\chi_{2,1}), C(\chi_2), \varphi_{\chi_2}\big(O(\chi_2)\big), C(\chi_0), \varphi_{\chi_0}\big(-O(\chi_0)\big)\Big). \tag{7.3}$$

For the orientations $-O(\chi_0)$, $-O(\chi_1)$, $-O(\chi_2)$, the Boltzmann weight $B_f(\chi;C)$ is given by

$$\epsilon(\chi)\bar{f}(C(\chi_{2,2}),C(\chi_2),\varphi_{\chi_2}(-O(\chi_2)),C(\chi_0),\varphi_{\chi_0}(-O(\chi_0))). \tag{7.4}$$

The values (7.1)–(7.4) coincide by the cocycle condition (6.7) and the following equalities:

$$\begin{aligned}
&C(\chi_{1,2}) = y\,\tilde{*}^s a,\ C(\chi_{2,1}) = y\,\tilde{*}^t b,\ C(\chi_{2,2}) = (y\,\tilde{*}^s a)\,\tilde{*}^t b,\\
&C(\chi_2) = a *^t b,\\
&\varphi_{\chi_1}(-O(\chi_1)) = \operatorname{type} X_Y - s,\ \varphi_{\chi_2}(O(\chi_2)) = s,\\
&\varphi_{\chi_2}(-O(\chi_2)) = \operatorname{type} X_Y - s,\ \varphi_{\chi_0}(-O(\chi_0)) = \operatorname{type} X_Y - t,
\end{aligned}$$

where $y = C(\chi_{1,1})$, $a = C(\chi_1)$, $s = \varphi_{\chi_1}(O(\chi_1))$, $b = C(\chi_0)$, and $t = \varphi_{\chi_0}(O(\chi_0))$. For example, the values (7.1) and (7.2) coincide, since we have

$$\begin{aligned}
&-\sum_{i=0}^{\operatorname{type} X_Y - s-1}\sum_{j=0}^{t-1} f\Big(\big((y\tilde{*}^s a)\,\tilde{*}^i a\big)\,\tilde{*}^j b, a *^j b, b\Big)\\
&= -\sum_{i=s}^{\operatorname{type} X_Y -1}\sum_{j=0}^{t-1} f\big((y\tilde{*}^i a)\tilde{*}^j b, a *^j b, b\big)\\
&= \sum_{j=0}^{t-1}\Big(-\sum_{i=s}^{\operatorname{type} X_Y -1} f((y\tilde{*}^j b)\tilde{*}^i(a *^j b), a *^j b, b)\Big)\\
&= \sum_{j=0}^{t-1}\sum_{i=0}^{s-1} f\big((y\tilde{*}^j b)\tilde{*}^i(a *^j b), a *^j b, b\big)\\
&= \sum_{i=0}^{s-1}\sum_{j=0}^{t-1} f\big((y\tilde{*}^i a)\tilde{*}^j b, a *^j b, b\big). \qquad \blacksquare
\end{aligned}$$

We set

$$B_f(C) := \sum_{\chi} B_f(\chi;C),$$

where χ runs over all crossings of D. Then we define the multiset

$$\Phi_f(D) := \{B_f(C) \mid C \in Col_X(D)_Y\}.$$

Theorem 7.2 *Let X be a quandle, and let Y be an X-set. Let f be a 2-cocycle of $C_I^*(X;A)_Y$. Let D be a diagram of a $\mathbb{Z}_{\operatorname{type} X_Y}$-flowed spatial graph (L,φ). The multiset $\Phi_f(D)$ is an invariant of (L,φ), which is invariant under $\mathbb{Z}_{\operatorname{type} X_Y}$-flowed contraction moves.*

Then we put $\Phi_f(L,\varphi) := \Phi_f(D)$.

Theorem 7.3 *The invariant $\Phi_f(L,\varphi)$ does not depend on the choice of a representative element of $[f] \in H_I^2(X;A)_Y$.*

8 An Example for a Quandle Cocycle Invariant

We give an example for a quandle cocycle invariant. Let K be Kinoshita's θ-curve as shown in Figure 8.1, where we ignore flows and colors. Kinoshita's θ-curve has the following significant property. When we remove any one edge from Kinoshita's θ-curve, then the remainder is trivial. The following example shows that K is nontrivial. We note that the invariant introduced in [3] does not work for this spatial graph.

Example 8.1 Put $X := S_4$, $Y := S_4$. For $r, s \in \mathbb{Z}_3$, $y, a, b \in S_4$, we denote by $C_{r,s}(y, a, b)$ the S_4-coloring of the $\mathbb{Z}_3$-flowed spatial graph diagram $D_{r,s}$ depicted in Figure 8.1. We note that $\operatorname{type} X_Y = \operatorname{type} S_4 = 3$. We have

$$\begin{aligned} Col_X(D_{1,1})_Y &= \{C_{1,1}(y, a, b) \mid y, a, b \in S_4\}, \\ Col_X(D_{2,2})_Y &= \{C_{2,2}(y, a, b) \mid y, a, b \in S_4\}, \\ Col_X(D_{r,s})_Y &= \{C_{r,s}(y, a, a) \mid y, a \in S_4\} \end{aligned}$$

for $(r, s) \in \mathbb{Z}_3^2 - \{(1, 1), (2, 2)\}$.

Let f_2 be the 2-cocycle of $C_I^*(S_4; A)_{S_4}$ defined in Example 6.4. By the equality

$$B_{f_2}(C_{1,1}(y, a, b)) = t(a - b)^3 = \begin{cases} 0 & \text{if } a = b, \\ t & \text{otherwise,} \end{cases}$$

we have $\Phi_{f_2}(D_{1,1}) = \{0_{\underline{16}}, t_{\underline{48}}\}$, where we refer the reader to Section 5 for the notation of the multiset $\{0_{\underline{16}}, t_{\underline{48}}\}$. By the equality

$$B_{f_2}(C_{2,2}(y, a, b)) = t(a - b)^3 = \begin{cases} 0 & \text{if } a = b, \\ t & \text{otherwise,} \end{cases}$$

we have $\Phi_{f_2}(D_{2,2}) = \{0_{\underline{16}}, t_{\underline{48}}\}$. By the equality $B_{f_2}(C_{r,s}(y, a, a)) = 0$, we have $\Phi_{f_2}(D_{r,s}) = \{0_{\underline{16}}\}$ for $(r, s) \in \mathbb{Z}_3^2 - \{(1, 1), (2, 2)\}$. Then we have

$$\Phi_{f_2}^{\Sigma}(K) = \{\Phi_{f_2}(D_{r,s}) \mid r, s \in \mathbb{Z}_3\} = \{\{0_{\underline{16}}, t_{\underline{48}}\}_{\underline{2}}, \{0_{\underline{16}}\}_{\underline{7}}\} \neq \{\{0_{\underline{16}}\}_{\underline{9}}\},$$

where we remark that $\Phi_{f_2}^{\Sigma}$ of the trivial spatial θ-curve is $\{\{0_{\underline{16}}\}_{\underline{9}}\}$. Thus K is nontrivial. Furthermore, K represents a nontrivial handlebody-link.

9 Proofs of Theorems 7.2 and 7.3

We state one lemma and prove Theorems 7.2 and 7.3 for $\operatorname{type} X_Y < \infty$. The proofs for $\operatorname{type} X_Y = \infty$ are easier than those for $\operatorname{type} X_Y < \infty$.

We suppose that $\operatorname{type} X_Y < \infty$. Let (L, O, φ) be a $\mathbb{Z}_{\operatorname{type} X_Y}$-flowed oriented spatial graph, and let D be a diagram of (L, O, φ). We denote by $\overline{D}$ the diagram obtained by replacing an edge $e \in \mathcal{E}(L)$ with $\varphi_e(O(e))$ parallel edges if $\varphi_e(O(e)) \neq 0$ and two antiparallel edges otherwise as shown in Figure 9.1. Let $(\overline{L}, \overline{O})$ be the oriented

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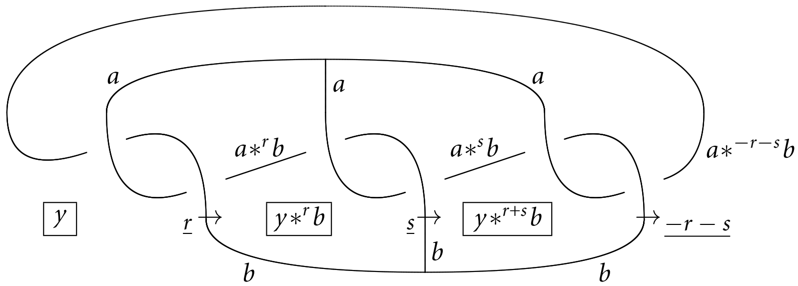

Figure 8.1

spatial graph represented by $\overline{D}$. We define a $\mathbb{Z}_{\text{type}\, X_Y}$-flow $\overline{\varphi}$ of $\overline{L}$ by $\overline{\varphi}_e(\overline{O}(e)) = 1$ for $e \in \mathcal{E}(\overline{L})$. We denote the $\mathbb{Z}_{\text{type}\, X_Y}$-flowed oriented spatial graph diagram obtained by adding $\overline{\varphi}$ to the diagram $\overline{D}$ by the same symbol $\overline{D}$. A $\mathbb{Z}_{\text{type}\, X_Y}$-flowed oriented spatial graph (L, O, φ) is *single* if $\varphi(O(e)) = 1$ for any edge $e \in \mathcal{E}(L)$. Then $(\overline{L}, \overline{O}, \overline{\varphi})$ is single.

Lemma 9.1 *Let (L, O, φ) be a $\mathbb{Z}_{\text{type}\, X_Y}$-flowed oriented spatial graph, and let D be a diagram of (L, O, φ). Then we have $\Phi_f(D) = \Phi_f(\overline{D})$.*

Proof Let χ be a crossing of D. Put $s := \varphi_{\chi_1}(O(\chi_1))$, $t := \varphi_{\chi_0}(O(\chi_0))$. We denote by $\overline{\chi}_{(i,j)}$ $(i = 0, \dots, s-1,\ j = 0, \dots, t-1)$ the crossings that originate from χ (see Figure 9.1). For $C \in Col_X(D)_Y$, there is a unique X_Y-coloring $\overline{C} \in Col_X(\overline{D})_Y$ such that parallel (antiparallel) arcs that originate from an arc α of D have the same color as α. This correspondence gives a bijection between $Col_X(D)_Y$ and $Col_X(\overline{D})_Y$. By the equality $B_f(\chi; C) = \sum_{i=0}^{s-1} \sum_{j=0}^{t-1} B_f(\overline{\chi}_{(i,j)}; \overline{C})$, we have $\Phi_f(D) = \Phi_f(\overline{D})$. ■

Proof of Theorem 7.2 By Lemma 2.5, it is sufficient to show that $\Phi_f(D)$ is invariant under the $\mathbb{Z}_{\text{type}\, X_Y}$-flowed R1–R6 moves. We have the invariance under the $\mathbb{Z}_{\text{type}\, X_Y}$-flowed R6 move immediately, since the Boltzmann weight is a weight at a crossing.

If D_1 and D_2 are related by a finite sequence of the $\mathbb{Z}_{\text{type}\, X_Y}$-flowed R1–R5 moves, then so are $\overline{D_1}$ and $\overline{D_2}$. By Lemma 9.1, it is sufficient to show that $\Phi_f(D)$ is invariant under the $\mathbb{Z}_{\text{type}\, X_Y}$-flowed R1–R5 moves preserving orientations for a diagram D of a single $\mathbb{Z}_{\text{type}\, X_Y}$-flowed oriented spatial graph.

The invariance under the $\mathbb{Z}_{\text{type}\, X_Y}$-flowed R1, R4 moves follows from (6.5). The invariance under the $\mathbb{Z}_{\text{type}\, X_Y}$-flowed R2 move follows from the signs of the crossings that appear in the diagram for the move. The invariance under the $\mathbb{Z}_{\text{type}\, X_Y}$-flowed R3 move follows from (6.6). The invariance under the $\mathbb{Z}_{\text{type}\, X_Y}$-flowed R5 move follows from (6.7), since the number of edges incident and directed in minus the number of edges incident and directed out vanishes modulo type X_Y. ■

Proof of Theorem 7.3 If 2-cocycles f_1, f_2 of $C_I^*(X; A)_Y$ are cohomologous, then $f_1 - f_2$ is null-cohomologous. By the equality $B_{f_1}(C) - B_{f_2}(C) = B_{f_1 - f_2}(C)$, it is sufficient to show that

$$B_f(C) = 0 \tag{9.1}$$

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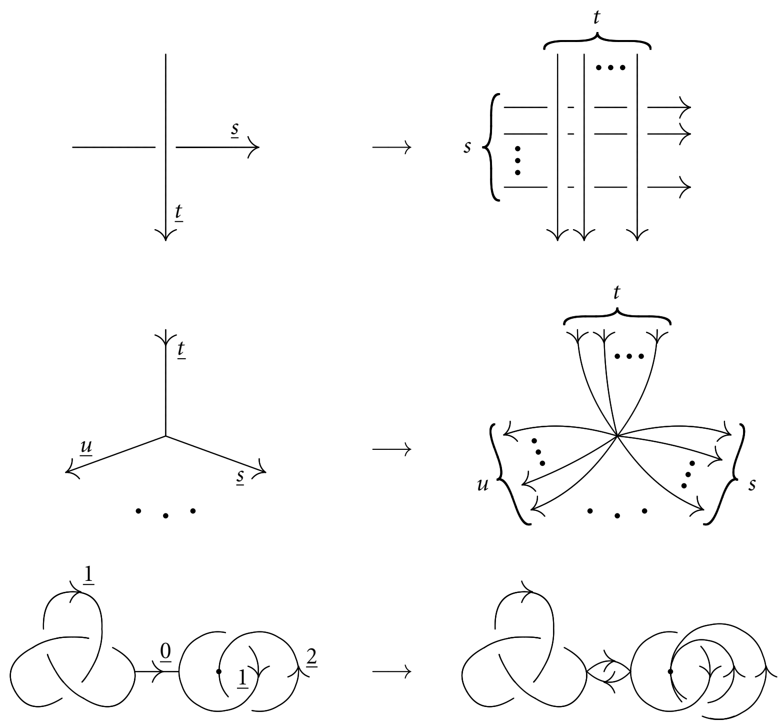

Figure 9.1

for a null-cohomologous 2-cocycle f of $C_I^*(X;A)_Y$. Let g be a 1-cocycle of $C_I^*(X;A)_Y$ such that $f = \delta^1 g$. Furthermore, by Lemma 9.1, it is sufficient to show the equality (9.1) for a diagram D of a single $\mathbb{Z}_{\mathrm{type}\, X_Y}$-flowed oriented spatial graph (L, O, φ).

We denote by $\mathcal{SA}(D)$ the set of curves obtained from D by removing (small neighborhoods of) crossings and vertices. We call a curve in $\mathcal{SA}(D)$ a *semi-arc* of D. We note that a semi-arc is obtained by dividing an over-arc at crossings. For a semi-arc α that originates from an arc $\hat{\alpha}$, we define the orientation and the color of α by those of $\hat{\alpha}$: $O(\alpha) := O(\hat{\alpha}), C(\alpha) := C(\hat{\alpha})$.

For a semi-arc α, there is a unique region R_α facing α such that the orientation $O(\alpha)$ points from the region R_α. Then we define $b(\alpha) := g(C(R_\alpha), C(\alpha))$. For a semi-arc α whose endpoint χ is a crossing or a vertex, we define

$$\epsilon(\alpha;\chi) := \begin{cases} 1 & \text{if the orientation } O(\alpha) \text{ points to } \chi, \\ -1 & \text{otherwise.} \end{cases}$$

We denote by $\chi_{(1)}, \chi_{(2)}$ the semi-arcs that originate from under-arcs at a crossing χ such that the co-orientation $O(\chi_0)$ points to $\chi_{(2)}$. We denote by $\chi_{(3)}, \chi_{(4)}$ the semi-arcs which originate from over-arcs at a crossing χ such that the co-orientation $O(\chi_1)$

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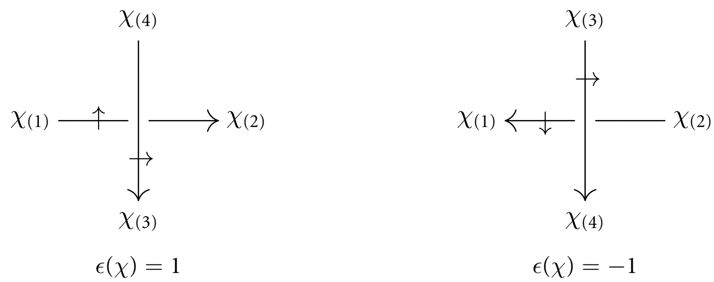

Figure 9.2

$(= O(\chi_2))$ points to $\chi_{(4)}$. For a crossing χ, we have

(9.2)

$$\begin{aligned}
B_f(\chi; C) &= \epsilon(\chi) f\big(C(\chi_{1,1}), C(\chi_1), C(\chi_0)\big) \\
&= \epsilon(\chi)(\delta^1 g)\big(C(\chi_{1,1}), C(\chi_1), C(\chi_0)\big) \\
&= \epsilon(\chi) g\big(C(\chi_{1,1}), C(\chi_1)\big) - \epsilon(\chi) g\big(C(\chi_{1,1}) \,\tilde{*}\, C(\chi_0), C(\chi_1) \,\tilde{*}\, C(\chi_0)\big) \\
&\quad - \epsilon(\chi) g\big(C(\chi_{1,1}), C(\chi_0)\big) + \epsilon(\chi) g\big(C(\chi_{1,1}) \,\tilde{*}\, C(\chi_1), C(\chi_0)\big) \\
&= \epsilon(\chi) g\big(C(\chi_{1,1}), C(\chi_{(1)})\big) - \epsilon(\chi) g\big(C(\chi_{1,1}) \,\tilde{*}\, C(\chi_{(0)}), C(\chi_{(2)})\big) \\
&\quad - \epsilon(\chi) g\big(C(\chi_{1,1}), C(\chi_{(3)})\big) + \epsilon(\chi) g\big(C(\chi_{1,1}) \,\tilde{*}\, C(\chi_{(1)}), C(\chi_{(4)})\big) \\
&= \sum_{i=1}^{4} \epsilon(\chi_{(i)}; \chi) b(\chi_{(i)}).
\end{aligned}$$

See Figure 9.2 for the last equality.

For semi-arcs $\omega_{(1)}, \dots, \omega_{(d_\omega)}$ incident to a vertex ω of degree d_ω, we show the equality

$$\sum_{i=1}^{d_\omega} \epsilon(\omega_{(i)}; \omega) b(\omega_{(i)}) = 0. \tag{9.3}$$

For integers i and j such that $R_{\omega_{(i)}} = R_{\omega_{(j)}}$, we have the equalities

$$\epsilon(\omega_{(i)}; \omega) = -\epsilon(\omega_{(j)}; \omega), \quad g\big(C(R_{\omega_{(i)}}), C(\omega_{(i)})\big) = g\big(C(R_{\omega_{(j)}}), C(\omega_{(j)})\big),$$

which imply that

$$\epsilon(\omega_{(i)}; \omega) b(\omega_{(i)}) + \epsilon(\omega_{(j)}; \omega) b(\omega_{(j)}) = 0.$$

Then we may suppose that the orientations of all semi-arcs agree with each other. Thus we have

$$\sum_{i=1}^{d_\omega} \epsilon(\omega_{(i)}; \omega) b(\omega_{(i)}) = \pm \sum_{k=0}^{n \,\mathrm{type}\, X_Y - 1} b(\omega_{(i_k)}) = 0$$

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for some positive integer n, where the last equality follows from the equality

$$\sum_{i=0}^{\text{type}\, X_Y - 1} g(y \,\bar{*}^i a, a) = 0.$$

By equalities (9.2) and (9.3), we have

$$\begin{aligned} B_f(C) &= \sum_{\chi:\ \text{crossing}} B_f(\chi; C) \\ &= \sum_{\chi:\ \text{crossing}} \sum_{i=1}^{4} \epsilon(\chi_{(i)}; \chi) b(\chi_{(i)}) + \sum_{\omega:\ \text{vertex}} \sum_{i=1}^{d_\omega} \epsilon(\omega_{(i)}; \omega) b(\omega_{(i)}) \\ &= \sum_{\alpha:\ \text{semi-arc}} \big(b(\alpha) - b(\alpha) \big) = 0. \end{aligned}$$

■

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