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ON A SHEAF OF DIVISION RINGS*

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Introduction. R. Arens and I. Kaplansky ([1]) call a ring A biregular if every two sided principal ideal of A is generated by a central idempotent and a ring A strongly regular if for any a in A there exists b in A such that $a=a^2b$. In ([1], Sections 2 and 3), a lot of interesting properties of a biregular ring and a strongly regular ring are given. Some more properties can also be found in [3], [5], [8], [9] and [13]. For example, J. Dauns and K. Hofmann ([3]) show that a biregular ring A is isomorphic with the global sections of the sheaf of simple rings A/K where K are maximal ideals of A. The converse is also proved by R. Pierce ([9], Th. 11-1). Moreover, J. Lambek ([5], Th. 1) extends the above representation of a biregular ring to a symmetric module. In a special case, when A is strongly regular, it is immediate from ([1], Th. 3.2) that the stalks of the associated sheaf of A are division rings ([13], Lem. 1.6). Furthermore, an investigation of the category of strongly regular rings is given by J. E. Roos ([9], Section 3). The purpose of the present paper is to show two more properties of A by a sheaf technique: that any finitely generated submodule of a finitely generated and projective module over A is projective; and that the existence of a strongly separable splitting ring for A in case A is a finitely generated and projective over its center R. We shall employ the sheaf representation of a ring with identity 1 due to R. Pierce ([8]) and other results given in [6], [7], [8] and [12].

Let R be a commutative ring with identity 1. The Boolean algebra of the idempotents of R is denoted by $B(R)$ and the Boolean space of the set of maximal ideals of $B(R)$ with hull-kernel topology is denoted by Spec $B(R)$. This topological space has an open base $U(e)=\{x/x$ in Spec $B(R)$ with $(1-e)$ in $x\}$ for each e in $B(R)$. It is known that Spec $B(R)$ is totally disconnected, compact and Hausdorff. Moreover, a sheaf (Pierce's sheaf) of rings R_x $(=R/xR)$ is defined over Spec $B(R)$ and R is isomorphic with the global sections of such a sheaf ([8]). Furthermore, denote $R_x \otimes_R M$ by M_x for an R-module M and $R_x \otimes_R A$ by A_x for an R-algebra A where all modules are assumed left unitary over a ring or an algebra.

1. Some basic facts of a sheaf of division rings. In this section, we shall point out some basic facts of a ring A when A_x are division rings for all x in Spec $B(A)$ where $B(A)$ is the Boolean algebra of the central idempotents of A. Then, it can be shown that any finitely generated submodule of a finitely generated and projective module

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over a R-algebra A with A_x division rings is projective. This fact will be used to show the existence of splitting rings for such an A in section 2. We start with the definition of a regular ideal.

DEFINITION. An ideal I of a ring A is regular if $I=(I\cap B(A))A$ ([8], Def. 9.2). An important fact about regular ideals is proposition 9.3 in [8]. In our notations, we have

PROPOSITION 1.1. *Assume* $S(r)=\{x/x$ *in* Spec $B(A)$ *with* $r_x\neq 0_x\}$ for an r in A. For any set U in Spec $B(A)$, define $I(U)=\{r$ in $R/S(r)\subset U\}$. For any set $I\subset A$, define $U(I)=\cup S(r)$ for all r in I. Then $U\to I(U)$ and $I\to U(I)$ determine inverse isomorphisms between the lattice of all open subsets of Spec $B(A)$ and the set of regular ideals of A.

Also, it is proved that a ring A is biregular if and only if every ideal of A is regular ([8]), Section 11. Hence, assume A is biregular, K is a maximal ideal of A if and only if $K\cap B(A)$ is a maximal ideal in $B(A)$. Thus by noting that a strongly regular ring is biregular the following proposition is immediate.

PROPOSITION 1.2. *Let A be a ring with identity. Then the following statements are equivalent*:

(1) A_x *is a division ring for each x in* Spec $B(A)$.

(2) *A is a strongly regular ring.*

(3) *The open set* (Spec $B(A)-\{x\}$) *corresponds to a maximal left ideal of A for each x in* Spec $B(A)$ *under Proposition* 1.1.

(4) *The open set* (Spec $B(A)-\{x\}$) *corresponds to a maximal right ideal of A for each x.*

(5) *For each $r\neq 0$ in A, there exists an element r' in A such that $rr'=e$ in $B(A)$ with $re=r$.*

It is well known that a finitely generated division algebra over its center is a central separable algebra ([4], Chapter V, Prop. 1.2). This fact can be generalized, that is, (the author later found that the finite projectivity of A can be replaced by finite generation as a ring)

THEOREM 1.3. *Let A be an R-algebra. If A_x is a central separable R_x-algebra for each x in* Spec $B(R)$ *and if A is a finitely generated and projective R-module, then A is a central separable R-algebra.*

Proof. Let K be a maximal ideal of R lying over x in Spec $B(R)$. Then $A/(KA)\cong A_x/(KA)/(xA)$ is a homomorphic image of the separable R_x-algebra A_x, and hence it is also separable over $R_x/(KA)/(xA)\cap R_x$. On the other hand, it is not hard to see that A is a faithfully projective R-module, so $R\cdot 1$ is an R-direct summand of A; and so $(KA)/(xA)\cap R_x=K/(xA)$. This implies that $A_x/(KA)/(xA)$ is separable over $R_x/K/(xA)$. Thus $A/(KA)$ is separable over R/K for each maximal ideal K of R.

Therefore A is separable over R ([4], Th. 7.1.) Moreover, since R_x is the center of A_x for all x by hypothesis, R is the center of A ([6], (1.5) and (1.6)). Consequently, A is a central separable R-algebra.

Assume A is a R-algebra with A_x division R_x-algebras. Let M be an A-module. Since M_x is a module over a division algebra A_x, it is free. We then call the number of summands of M_x the rank of M at x denoted by $\text{rank}_M(x)$. Using the proof of Theorem 3.1 in [11], we have a similar fact to Theorem 15.3 in [8]. That is, let M be a finitely generated A-module. Then M is projective if and only if rank_M is a continuous function from Spec $B(R)$ to the set of non-negative integers with discrete topology. Next we show a property of a projective A-module which is very useful in section two.

THEOREM 1.4. *Assume A is a R-algebra with A_x division R_x-algebras. Let M be a finitely generated and projective A-module. Then any finitely generated submodule P of M is projective.*

Proof. Since M is finitely generated and projective, rank_M is continuous by the above remark. Let $a_{1x}, a_{2x}, \ldots, a_{rx}, \ldots, a_{mx}$ be a basis of M_x with $a_{1x}, \ldots, a_{rx}$ in P_x. Lift $a_{1x}, \ldots, a_{rx}$ to $a_1, \ldots, a_r$ in P and $a_{r+1x}, \ldots, a_{mx}$ to $a_{r+1}, \ldots, a_m$ in M. Then the continuity of rank_M implies that there exists a basic open set $U(e)$ of x such that $ea_1, \ldots, ea_r, ea_{r+1}, \ldots, ea_m$ is a basis of eM over eA. Hence $ea_1, \ldots, ea_r$ are linearly independent. On the other hand, since P is finitely generated, there exists a basic open set $U(e')$ of x such that $e'a_1, \ldots, e'a_r$ generate $e'P$. Thus we have a basic open set $U(e'')$ contained in $(U(e)\cap U(e'))$ of x such that $e''a_1, \ldots, e''a_r$ is a basis for $e''P$. Therefore $e''P$ is a projective $e''A$-module. Consequently, by application of the partition property of Spec $B(R)$ ([10], Introduction, (2)) the proof is complete.

2. **Splitting rings.** Let D be a finitely generated central division algebra over a field F. It is well known that any maximal sub-field K of D is a splitting field for D. In general, let A be a central separable R-algebra. A commutative ring extension of R, S, is called a splitting ring for A if $A\cong\text{Hom}_S(P, P)$ for a finitely generated projective and faithful S-module P ([2], P. 382). It is proved that if a maximal commutative subalgebra K of A is separable then it is a splitting ring for A ([2], Th. 5.6). On the other hand, there exist central separable algebras without strongly separable splitting rings ([4], Ex. 3, P. 148). In general, it is not known which type of central separable algebras has a strongly separable splitting ring. However, we are going to generalize the fact for a division ring to a central separable algebra under our consideration.

Here we recall that K is a maximal commutative R-algebra of A if the commutant A^K of K in A is equal to K. That is, $A^K=\{a/a \text{ in } A \text{ with } as=sa \text{ for all } s \text{ in } K\}=K$ ([4], P. 64). Also, a commutative R-algebra S is called a quasi-separable cover of R if S_x is a locally strongly separable R_x-algebra for each x in Spec $B(R)$. S is called a separable cover of R if it is a separable quasi-separable cover of R ([7], Def. 2.1).

THEOREM 2.1. *Let A be a central separable R-algebra such that A_x is a division R_x-algebra for each x in* Spec $B(R)$. *If K is a maximal commutative subalgebra of A and a finitely generated R-module, then K is a projective splitting ring for A. Moreover, if such a K is a quasi-separable cover of R then K is a strongly separable splitting ring for A.*

Proof. Since A is a central separable R-algebra, it is finitely generated and projective over R. But A_x is a central division R_x-algebra for each x in Spec $B(R)$, then the finitely generated submodule K of A is projective by Theorem 1.4. Now we claim that K_x is also a maximal commutative subalgebra of A_x for each x. In fact, let b_x be an element in the commutant of K_x in A_x with b in A. Since A_x is a free R_x-module with K_x as a finitely generated submodule, there is a basis of A_x, $\{a_{1x}, \ldots, a_{rx}, \ldots, a_{mx}$ for some integer $m\}$, such that $a_{1x}, \ldots, a_{rx}$ are in K_x. The proof of Theorem 1.4 implies that there is a basic open set $U(e)$ of x such that $\{ea_1, \ldots, ea_r\}$ is a basis for eK over eR. On the other hand, b_x is in the commutant of K_x in A_x, so $b_x a_{ix} = a_{ix} b_x$ for $i=1, \ldots, r$. But then there is a basic open set $U(e')$ of x such that $e'ba_i = e'a_i b$ for $i=1, \ldots, r$ ([12], (2.9)). Hence we have a basic open set $U(e'')$ of x contained in $U(e) \cap U(e')$ so that $e''ba_i = e''a_i b$ for $i=1, \ldots, r$ and $\{e''a_i / i=1, \ldots, r\}$ is a basis for $e''K$ over $e''R$. Thus $e''b$ is in the commutant of $e''K$ in $e''A$. Noting that $K \cong e''K \oplus (1-e'')K$, we conclude that $e''b$ is in the commutant of K in A. By hypothesis, $K = A^K$ (= the commutant of K in A), so $e''b$ is in K; and so $(e''b)_x = b_x$ is in K_x. Therefore $K_x = (A_x)^{K_x}$. That is, K_x is a maximal commutative subalgebra of A_x. But then K_x is a splitting ring for A_x for each x. This implies that K is a splitting ring for A [(6], Cor. 1.11).

Moreover, since K is finitely generated over R and a quasi-separable cover of R by hypothesis, K is a separable cover of R ([7], Prop. 2.3). But K is also projective over R from the first part of this theorem, then it is a strongly separable splitting ring for A. The proof is thus complete.

REMARK. We note that if R_x is a perfect field for each x then K given in the theorem is a quasi-separable cover of R. More properties of such an A can be derived from [11].

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