Bull. Aust. Math. Soc. **79** (2009), 487–497
doi:10.1017/S0004972709000094

THE ESSENTIAL NORM OF A WEIGHTED COMPOSITION OPERATOR FROM THE BLOCH SPACE TO H^∞

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(Received 9 September 2008)

Abstract

We express the operator norm of a weighted composition operator which acts from the Bloch space $\mathcal{B}$ to H^∞ as the supremum of a quantity involving the weight function, the inducing self-map, and the hyperbolic distance. We also express the essential norm of a weighted composition operator from $\mathcal{B}$ to H^∞ as the asymptotic upper bound of the same quantity. Moreover we study the estimate of the essential norm of a weighted composition operator from H^∞ to itself.

2000 *Mathematics subject classification*: primary 47B33.

Keywords and phrases: weighted composition operator, the space of bounded analytic functions, Bloch space.

1. Introduction

Let $H(\mathbb{D})$ be the set of all analytic functions on the open unit disk $\mathbb{D}$ and $S(\mathbb{D})$ the set of all analytic self-maps of $\mathbb{D}$. Every analytic self-map $\varphi \in S(\mathbb{D})$ induces a composition operator $C_\varphi : f \mapsto f \circ \varphi$ and every analytic function $u \in H(\mathbb{D})$ induces a multiplication operator $M_u : f \mapsto u \cdot f$. Both C_φ and M_u are linear transformations from $H(\mathbb{D})$ to itself. The weighted composition operator uC_φ is the product of M_u and C_φ, that is, $uC_\varphi f = M_u C_\varphi f = u \cdot f \circ \varphi$.

Let $H^\infty = H^\infty(\mathbb{D})$ be the set of all bounded analytic functions on $\mathbb{D}$. H^∞ is a Banach algebra with the supremum norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

The Bloch space $\mathcal{B}$ is the set of all $f \in H(\mathbb{D})$ satisfying

$$|||f||| = \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

This work was supported by the Korean Research Foundation Grant funded by Korean Government (KRF-2008-314-C00012).

Then $|||\cdot|||$ defines a Möbius invariant complete semi-norm and $\mathcal{B}$ is a Banach space under the norm $\|f\|_{\mathcal{B}} = |f(0)| + |||f|||$. Note that $|||f||| \le \|f\|_\infty$ for any $f \in H^\infty$, hence $H^\infty \subset \mathcal{B}$. Let the little Bloch space $\mathcal{B}_o$ denote the subspace of $\mathcal{B}$ consisting of those functions f such that

$$\lim_{|z|\to 1}(1-|z|^2)f'(z) = 0.$$

The little Bloch space $\mathcal{B}_o$ is a closed subspace of $\mathcal{B}$. In particular, $\mathcal{B}_o$ is the closure in $\mathcal{B}$ of the polynomials.

Let w be a point in $\mathbb{D}$ and α_w be the Möbius transformation of $\mathbb{D}$ defined by

$$\alpha_w(z) = \frac{w-z}{1-\overline{w}z}.$$

For w, z in $\mathbb{D}$, the pseudo-hyperbolic distance $\rho(w, z)$ between z and w is given by

$$\rho(w, z) = |\alpha_w(z)| = \left|\frac{w-z}{1-\overline{w}z}\right|,$$

and the hyperbolic metric $\beta(w, z)$ is given by

$$\beta(w, z) = \frac{1}{2}\log\frac{1+\rho(w, z)}{1-\rho(w, z)}.$$

For $f \in \mathcal{B}$, we have that

$$|f(w) - f(0)| \le |||f|||\int_0^1 \frac{|w|\,dt}{1-|w|^2t^2}.$$

Hence we have the growth condition for the Bloch functions:

$$|f(w)| \le |f(0)| + |||f|||\beta(w, 0). \tag{1.1}$$

We have also the exact expression of the induced distance on the Bloch space:

$$\sup_{\|f\|_{\mathcal{B}}\le 1} |f(z) - f(w)| = \beta(z, w). \tag{1.2}$$

See [9] for more information on the Bloch space.

Let X and Y be two Banach spaces and T be a linear operator from X to Y. Denote the operator norm of T by $\|T\|_{X\to Y}$. If $Y = X$, we write $\|T\|_X = \|T\|_{X\to X}$. For a bounded linear operator T from X to Y, the essential norm $\|T\|_{e,X\to Y}$ is defined to be the distance from T to the closed ideal of compact operators, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T+K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

Notice that T is compact from X to Y if and only if $\|T\|_{e,X\to Y} = 0$. We also write $\|T\|_{e,X} = \|T\|_{e,X\to X}$.

Ohno characterized the boundedness and the compactness of uC_φ from $\mathcal{B}$ to H^∞.

THEOREM A (Ohno *et al.* [4, 6]). *Let u be in $H(\mathbb{D})$ and φ be in $S(\mathbb{D})$.*

(i) *The weighted composition operator $uC_\varphi : \mathcal{B} \to H^\infty$ is bounded if and only if $u \in H^\infty$ and*

$$\sup_{z\in\mathbb{D}} |u(z)| \log \frac{e}{1-|\varphi(z)|} < \infty.$$

(ii) *Suppose that $uC_\varphi : \mathcal{B} \to H^\infty$ is bounded. Then $uC_\varphi : \mathcal{B} \to H^\infty$ is compact if and only if $u \in H^\infty$ and*

$$\limsup_{|\varphi(z)|\to 1} |u(z)| \log \frac{e}{1-|\varphi(z)|} = 0.$$

In [5], Kwon also studied the composition operators from $\mathcal{B}$ to H^∞ and computed the operator norm of C_φ from $\mathcal{B}^0$ to H^∞ with the condition $\varphi(0) = 0$, where $\mathcal{B}^0$ is the subspace of $\mathcal{B}$ which consists of all Bloch functions f with $f(0) = 0$.

THEOREM B (Kwon [5]). *For any $\varphi \in S(\mathbb{D})$ such that $\varphi(0) = 0$, we have*

$$\|C_\varphi\|_{\mathcal{B}^0\to H^\infty} = \sup_{z\in\mathbb{D}} \beta(\varphi(z), 0).$$

In this paper, we estimate the operator norm and the essential norm of uC_φ acting between $\mathcal{B}$ and H^∞. We give the explicit formula of the operator norm of uC_φ from $\mathcal{B}$ to H^∞ in Section 2, and of the essential norm in Section 3, which are the generalization of Theorems A and B.

Theorem A indicates that the compactness of uC_φ from $\mathcal{B}$ to H^∞ implies the compactness from H^∞ to itself. Especially, for the case of composition operators, the compactness of C_φ from $\mathcal{B}$ to H^∞ is equivalent to the compactness from H^∞ to itself (see Corollary 3.4). The exact formula of the essential norm of uC_φ from H^∞ to itself has not been obtained. Takagi, Takahashi and Ueki gave a partial solution of $\|uC_\varphi\|_{e,H^\infty}$.

THEOREM C (Takagi *et al.* [7]). *Let u be in $H(\mathbb{D})$ and φ be in $S(\mathbb{D})$. For $r > 0$, let $D_r = \{z \in \mathbb{D} : |u(z)| \geq r\}$ and*

$$\gamma = \inf\{r > 0 : \overline{\varphi(D_r)} \subset \mathbb{D}\}. \tag{1.3}$$

Then

$$\gamma \leq \|uC_\varphi\|_{e,H^\infty} \leq 2\gamma.$$

So, in Section 4, we deal with the estimate of $\|uC_\varphi\|_{e,H^\infty}$ under a certain assumption.

2. The operator norm of uC_φ from $\mathcal{B}$ to H^∞

In this section, we consider the operator norm of uC_φ from $\mathcal{B}$ to H^∞. For $z \in \mathbb{D}$, we shall use the short-hand notation

$$\widetilde{\beta}(z) = \max\{1, \beta(z, 0)\}.$$

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THEOREM 2.1. *Let u be in $H(\mathbb{D})$ and φ be in $S(\mathbb{D})$. Then we have*

$$\|uC_\varphi\|_{\mathcal{B}\to H^\infty} = \|uC_\varphi\|_{\mathcal{B}_o\to H^\infty} = \sup_{z\in\mathbb{D}} |u(z)|\widetilde{\beta}(\varphi(z)).$$

PROOF. Since $\mathcal{B}_o \subset \mathcal{B}$, it is easy to see that $\|uC_\varphi\|_{\mathcal{B}_o\to H^\infty} \le \|uC_\varphi\|_{\mathcal{B}\to H^\infty}$. For $f \in \mathcal{B}$, the growth condition (1.1) implies

$$\begin{aligned}\|uC_\varphi f\|_\infty &= \sup_{z\in\mathbb{D}} |u(z)\ f(\varphi(z))| \\ &\le \sup_{z\in\mathbb{D}}\{|u(z)|(|f(0)| + \|\|f\|\|\ \beta(\varphi(z)))\} \\ &\le \|f\|_{\mathcal{B}}\ \sup_{z\in\mathbb{D}} |u(z)|\widetilde{\beta}(\varphi(z)).\end{aligned}$$

Hence we obtain

$$\|uC_\varphi\|_{\mathcal{B}\to H^\infty} \le \sup_{z\in\mathbb{D}} |u(z)|\widetilde{\beta}(\varphi(z)).$$

Next we show that

$$\|uC_\varphi\|_{\mathcal{B}_o\to H^\infty} \ge \sup_{z\in\mathbb{D}} |u(z)|\widetilde{\beta}(\varphi(z)).$$

To prove this, we use two test functions. The first one is the constant function 1. Since 1 is a unit vector of $\mathcal{B}_o$, we have

$$\|uC_\varphi\|_{\mathcal{B}_o\to H^\infty} \ge \|uC_\varphi\ 1\|_\infty = \|u\|_\infty. \tag{2.1}$$

The second one is defined as following. Let $r \in (0, 1)$ and $\lambda \in \mathbb{D}$. We put $\eta_\lambda = \lambda/|\lambda|$ and

$$f_{r,\lambda}(z) = \frac{1}{2}\log\frac{1 + r\ \overline{\eta_\lambda} z}{1 - r\ \overline{\eta_\lambda} z}.$$

Then $f_{r,\lambda}(0) = 0$ and

$$(1 - |z|^2)|f'_{r,\lambda}(z)| = \frac{r(1 - |z|^2)}{|1 - r^2\ \overline{\eta_\lambda}^2 z^2|}.$$

Hence $f_{r,\lambda} \in \mathcal{B}_o$ and $\|f_{r,\lambda}\|_{\mathcal{B}} = r$. For arbitrary $w \in \mathbb{D}$, put $\lambda = \varphi(w)$. Then

$$\begin{aligned}\|uC_\varphi\|_{\mathcal{B}_o\to H^\infty} &\ge \frac{\|uC_\varphi\ f_{r,\varphi(w)}\|_\infty}{\|f_{r,\varphi(w)}\|_{\mathcal{B}}} \\ &= \frac{1}{r}\ \sup_{z\in\mathbb{D}}\left|u(z)\ \frac{1}{2}\log\frac{1 + r\ \overline{\eta_{\varphi(w)}}\varphi(z)}{1 - r\ \overline{\eta_{\varphi(w)}}\varphi(z)}\right| \\ &\ge \frac{1}{r}\ |u(w)|\ \beta(r\varphi(w), 0).\end{aligned}$$

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Taking the limit as $r \to 1$ and the supremum over all $w \in \mathbb{D}$, we get

$$\|uC_\varphi\|_{\mathcal{B}_o \to H^\infty} \geq \sup_{z \in \mathbb{D}} |u(z)| \, \beta(\varphi(z), 0). \tag{2.2}$$

Then (2.1) and (2.2) imply that

$$\begin{aligned} \|uC_\varphi\|_{\mathcal{B}_o \to H^\infty} &\geq \max\Big\{ \|u\|_\infty, \sup_{z \in \mathbb{D}} |u(z)| \, \beta(\varphi(z), 0) \Big\} \\ &= \sup_{z \in \mathbb{D}} |u(z)| \, \widetilde{\beta}(\varphi(z)). \end{aligned}$$

□

Considering the case that $u \equiv 1$, we obtain the characterization of the boundedness of $C_\varphi : \mathcal{B} \to H^\infty$.

COROLLARY 2.2. *Let φ be in $S(\mathbb{D})$. Then the following statements are equivalent:*

(i) *C_φ is bounded from $\mathcal{B}$ to H^∞;*
(ii) *C_φ is bounded from $\mathcal{B}_o$ to H^∞;*
(iii) $\|\varphi\|_\infty < 1$.

If uC_φ is bounded from $\mathcal{B}$ to H^∞ and $|\varphi(z)| \to 1$ as $z \to \zeta \in \partial\mathbb{D}$, then the radial limit of u must vanish at ζ. Thus we can conclude that if u is not the zero function and φ has the radial limits of modulus 1 on a set of positive measure, then uC_φ is never bounded. More especially, considering the case that $\varphi(z) = z$, it follows that that the multiplication operator M_u is bounded from $\mathcal{B}$ to H^∞ if and only if u is the zero function. Then we have the following corollary.

COROLLARY 2.3. *Let u be an analytic function on $\mathbb{D}$. Then the following statements are equivalent:*

(i) *M_u is bounded from $\mathcal{B}$ to H^∞;*
(ii) *M_u is bounded from $\mathcal{B}_o$ to H^∞;*
(iii) *M_u is compact from $\mathcal{B}$ to H^∞;*
(iv) *M_u is compact from $\mathcal{B}_o$ to H^∞;*
(v) $u \equiv 0$.

3. The essential norm of uC_φ from $\mathcal{B}$ to H^∞

In this section, we estimate the essential norm of uC_φ from $\mathcal{B}$ to H^∞. To do this, we prepare two lemmas.

LEMMA 3.1. *Let u be in $H(\mathbb{D})$ and φ be in $S(\mathbb{D})$. Suppose that uC_φ is bounded from $\mathcal{B}$ to H^∞. Then uC_φ is compact from $\mathcal{B}$ to H^∞ if and only if $\|uC_\varphi f_n\|_\infty \to 0$ for any bounded sequence $\{f_n\}$ in $\mathcal{B}$ that converges to 0 uniformly on every compact subset of $\mathbb{D}$.*

The lemma above is a generalization of a well-known result called the weak convergence lemma and we omit its proof (see [1, Proposition 3.11]).

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LEMMA 3.2. *For z, w in $\mathbb{D}$, let $L(z, w)$ be the positive function on $\mathbb{D} \times \mathbb{D}$ defined by*

$$L(z, w) = \left| \frac{\log(e/(1 - \bar{z}w))}{\log(e/(1 - |z|^2))} \right|.$$

Then the following conditions hold:

(i) $L(z, w)$ *is bounded on* $\mathbb{D} \times \mathbb{D}$;

(ii) $\lim_{|z| \to 1} \sup_{w \in \mathbb{D}} L(z, w) = 1$.

PROOF. (i) We can see that

$$L(z, w) \leq \frac{\log(e/(1 - |z|)) + 2\pi}{\log(e/(1 - |z|)) - \log 2} < \frac{1 + 2\pi}{1 - \log 2}.$$

Hence we have that $L(z, w)$ is bounded on $\mathbb{D} \times \mathbb{D}$.

(ii) We have

$$\lim_{|z| \to 1} \sup_{w \in \mathbb{D}} L(z, w) \leq \lim_{|z| \to 1} \frac{\log(e/(1 - |z|)) + 2\pi}{\log(e/(1 - |z|)) - \log 2} = 1.$$

On the other hand, since $L(z, z) = 1$, we get the assertion. □

Here we give the explicit formula of the essential norm of uC_φ.

THEOREM 3.3. *Let u be in $H^\infty(\mathbb{D})$ and φ be in $S(\mathbb{D})$. Suppose that uC_φ is bounded from $\mathcal{B}$ to H^∞ (then uC_φ is also bounded from $\mathcal{B}_o$ to H^∞). Then we have the following estimation:*

$$\|uC_\varphi\|_{e, \mathcal{B} \to H^\infty} = \|uC_\varphi\|_{e, \mathcal{B}_o \to H^\infty} = \limsup_{|\varphi(z)| \to 1} |u(z)| \beta(\varphi(z), 0)$$

where we define the limit supremum above as equal to zero if $\|\varphi\|_\infty < 1$.

PROOF. By the inclusion $\mathcal{B}_o \subset \mathcal{B}$, we can see that $\|uC_\varphi\|_{e, \mathcal{B}_o \to H^\infty} \leq \|uC_\varphi\|_{e, \mathcal{B} \to H^\infty}$.

Since uC_φ is bounded from $\mathcal{B}$ to H^∞, we have $u \in H^\infty$. First, suppose that $\|\varphi\|_\infty < 1$. For any bounded sequence $\{f_n\}$ in $\mathcal{B}$ such that f_n converges to zero uniformly on every compact subset of $\mathbb{D}$, we have

$$\|uC_\varphi f_n\|_\infty \leq \|u\|_\infty \sup_{z \in \varphi(\mathbb{D})} |f_n(z)| \to 0$$

as $n \to \infty$. Hence Lemma 3.1 implies that uC_φ is compact from $\mathcal{B}$ to H^∞, and we get $\|uC_\varphi\|_{e, \mathcal{B} \to H^\infty} = 0$. Next we assume that $\|\varphi\|_\infty = 1$ and put $\varphi_r(z) = r\varphi(z)$ for $r \in (0, 1)$. Then, by the argument above, we have that uC_{φ_r} is compact from $\mathcal{B}$ to H^∞. Using (1.2), we obtain

$$\begin{aligned} \|uC_\varphi\|_{e, \mathcal{B} \to H^\infty} &\leq \|uC_\varphi - uC_{\varphi_r}\|_{\mathcal{B} \to H^\infty} \\ &= \sup_{z \in \mathbb{D}} \left\{ |u(z)| \sup_{\|f\|_{\mathcal{B}} \leq 1} |f(\varphi(z)) - f(\varphi_r(z))| \right\} \\ &= \sup_{z \in \mathbb{D}} |u(z)| \, \beta(\varphi(z), \varphi_r(z)). \end{aligned}$$

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Letting $r \to 1$, we have that

$$\|uC_\varphi\|_{e,\mathcal{B}\to H^\infty} \le \lim_{r\to 1} \sup_{z\in\mathbb{D}} |u(z)|\beta(\varphi(z), \varphi_r(z)).$$

For $s \in (0, 1)$, we divide $\mathbb{D}$ into two parts: $D_1 = \{z : |\varphi(z)| \le s\}$ and $D_2 = \{z : |\varphi(z)| > s\}$. Since φ_r converges uniformly to φ on D_1, we obtain that

$$\lim_{r\to 1} \sup_{z\in D_1} |u(z)|\beta(\varphi(z), \varphi_r(z)) = 0.$$

Hence it follows that

$$\|uC_\varphi\|_{e,\mathcal{B}\to H^\infty} \le \lim_{r\to 1} \sup_{z\in D_2} |u(z)|\beta(\varphi(z), \varphi_r(z)).$$

Since

$$\rho(\varphi(z), \varphi_r(z)) = \left|\frac{(1-r)\varphi(z)}{1-r|\varphi(z)|^2}\right| \le |\varphi(z)|,$$

we get the following estimate independent of r:

$$\|uC_\varphi\|_{e,\mathcal{B}\to H^\infty} \le \sup_{z\in D_2} |u(z)|\beta(\varphi(z), 0).$$

Here, letting $s \to 1$, we conclude that

$$\|uC_\varphi\|_{e,\mathcal{B}\to H^\infty} \le \limsup_{|\varphi(z)|\to 1} |u(z)|\beta(\varphi(z), 0).$$

It remains only to prove that

$$\|uC_\varphi\|_{e,\mathcal{B}_o\to H^\infty} \ge \limsup_{|\varphi(z)|\to 1} |u(z)|\beta(\varphi(z), 0). \tag{3.1}$$

For $\lambda \in \mathbb{D}$ such that $|\lambda| > 1/2$ and for $p \in (0, 1)$, we put

$$g_{\lambda,p}(z) = \beta(\lambda, 0)^{-p}\left(\frac{1}{2}\log\frac{(1+|\lambda|)^2}{1-\overline{\lambda}z}\right)^{p+1}.$$

We have

$$|g_{\lambda,p}(0)| = \beta(\lambda, 0)^{-p}\left(\tfrac{1}{2}\log(1+|\lambda|)^2\right)^{p+1} < (\log 4)^2$$

and

$$(1-|z|^2)\,|g'_{\lambda,p}(z)| = \frac{p+1}{2}\left\{(1-|z|^2)\left|\frac{\overline{\lambda}}{1-\overline{\lambda}z}\right|\beta(\lambda, 0)^{-p}\left|\frac{1}{2}\log\frac{(1+|\lambda|)^2}{1-\overline{\lambda}z}\right|^p\right\}.$$

For the moment fix $p \in (0, 1)$. Let $\{\lambda_n\}$ be a sequence in $\mathbb{D}$ with $|\lambda_n| \to 1$. By (i) of Lemma 3.2, we conclude $\{g_{\lambda_n,p}\}$ is a bounded sequence of functions in $\mathcal{B}_o$

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which converges to zero uniformly on every compact subset of $\mathbb{D}$. For any compact operator K, the image $\{Kg_{\lambda_n,p}\}$ is a relatively compact subset in H^∞, and its limit point is only the zero function. Hence we have that $\|Kg_{\lambda_n,p}\|_\infty \to 0$.

By some calculation, we have

$$\sup_{z\in\mathbb{D}} \frac{1-|z|^2}{|1-\bar{\lambda}z|} = \frac{2-2\sqrt{1-|\lambda|^2}}{|\lambda|^2}.$$

Since $|g_{\lambda_n,p}(0)| \to 0$, (ii) of Lemma 3.2 implies that

$$\begin{aligned}
&\lim_{n\to\infty} \|g_{\lambda_n,p}\|_{\mathcal{B}} \\
&\quad = \lim_{n\to\infty} |||g_{\lambda_n,p}||| \\
&\quad \le (p+1) \lim_{n\to\infty} \left\{ \frac{1-\sqrt{1-|\lambda_n|^2}}{|\lambda_n|} \sup_{z\in\mathbb{D}} \left| \frac{\log(e/(1-\overline{\lambda_n}z)) + \log((1+|\lambda_n|)^2/e)}{\log(e/(1-|\lambda_n|^2)) + \log((1+|\lambda_n|)^2/e)} \right|^p \right\} \\
&\quad = p+1.
\end{aligned}$$

Here, take a sequence $\{z_n\}$ in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$. Then we have

$$\begin{aligned}
\|uC_\varphi - K\|_{\mathcal{B}_o\to H^\infty} &\ge \limsup_{n\to\infty} \frac{\|uC_\varphi g_{\varphi(z_n),p}\|_\infty - \|Kg_{\varphi(z_n),p}\|_\infty}{\|g_{\varphi(z_n),p}\|_{\mathcal{B}}} \\
&\ge \frac{1}{p+1} \limsup_{n\to\infty} |u(z_n)\, g_{\varphi(z_n),p}(\varphi(z_n))| \\
&= \frac{1}{p+1} \limsup_{n\to\infty} |u(z_n)| \beta(\varphi(z_n), 0).
\end{aligned}$$

Therefore we get

$$\|uC_\varphi\|_{e,\mathcal{B}_o\to H^\infty} \ge \frac{1}{p+1} \limsup_{n\to\infty} |u(z_n)| \beta(\varphi(z_n), 0).$$

Letting $p \to 0$, we obtain

$$\|uC_\varphi\|_{e,\mathcal{B}_o\to H^\infty} \ge \limsup_{n\to\infty} |u(z_n)| \beta(\varphi(z_n), 0).$$

Taking the supremum over all sequences $\{z_n\}$ such that $|\varphi(z_n)| \to 1$, we get (3.1). Our proof is accomplished. □

Recall that uC_φ is compact from H^∞ to itself if and only if $|u(z)| \to 0$ whenever $|\varphi(z)| \to 1$. Hence it follows that C_φ is compact from H^∞ to itself if and only if $\|\varphi\|_\infty < 1$. Combining this fact with Corollary 2.3, we have the following corollary.

Corollary 3.4. *Let φ be in $S(\mathbb{D})$. Then the following are equivalent:*

(i) *C_φ is bounded from $\mathcal{B}$ to H^∞;*
(ii) *C_φ is bounded from $\mathcal{B}_o$ to H^∞;*

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(iii) C_φ *is compact from* $\mathcal{B}$ *to* H^∞;
(iv) C_φ *is compact from* $\mathcal{B}_o$ *to* H^∞;
(v) C_φ *is compact from* H^∞ *to* H^∞;
(vi) $\|\varphi\|_\infty < 1$.

Next we give an example which indicates the difference between the boundedness and compactness of uC_φ from $\mathcal{B}$ to H^∞.

EXAMPLE 3.5. Put $\varphi(z) = (1+z)/2$, $u(z) = 1-z$, $v(z) = (\log(e/(1-z)))^{-1}$, and $w(z) = (\log\log(e^e/(1-z)))^{-1}$. Then $\varphi(1) = 1$ and $|\varphi(z)| < 1$ for $z \neq 1$. Since these three weight functions tend to 0 as $z \to 1$, uC_φ, vC_φ, and wC_φ are compact from H^∞ to H^∞. By Theorems 2.1 and 3.3, it follows that uC_φ is compact, vC_φ is bounded but is not compact, and wC_φ is not bounded from $\mathcal{B}$ to H^∞.

4. The essential norm of uC_φ from H^∞ to H^∞

In this section, we consider the essential norm of uC_φ from H^∞ to H^∞. We recall that $\|uC_\varphi\|_{H^\infty} = \|u\|_\infty$. From this 'big-oh' condition, we expect the compactness of uC_φ can be described by the corresponding 'little-oh' condition. Indeed, we can interpret Theorem C into the following form.

PROPOSITION 4.1. *Let u be in $H(\mathbb{D})$ and φ be in $S(\mathbb{D})$. Then*

$$\limsup_{|\varphi(z)|\to 1} |u(z)| \leq \|uC_\varphi\|_{e,H^\infty} \leq 2 \limsup_{|\varphi(z)|\to 1} |u(z)|. \tag{4.1}$$

The proof of this proposition is straightforward and, hence, is omitted here.

Our intuition suggests that the coefficient '2' in (4.1) could be removable. In [8], Zheng gave the exact formula of the essential norm of C_φ on H^∞ which supports our intuition.

THEOREM D (Zheng [8]). *Let φ be in $S(\mathbb{D})$. Then*

$$\|C_\varphi\|_{e,H^\infty} = \begin{cases} 0 & \text{if } \|\varphi\|_\infty < 1, \\ 1 & \text{if } \|\varphi\|_\infty = 1. \end{cases}$$

We give a sufficient condition for the essential norm of uC_φ to coincide with its operator norm.

PROPOSITION 4.2. *Let u be in $H(\mathbb{D})$ and φ be an inner function. Then*

$$\|uC_\varphi\|_{e,H^\infty} = \|uC_\varphi\|_{H^\infty} = \|u\|_\infty.$$

PROOF. It is trivial that $\|uC_\varphi\|_{e,H^\infty} \leq \|uC_\varphi\|_{H^\infty}$. By the maximum modulus principle,

$$\|uC_\varphi\|_{e,\,H^\infty} \geq \limsup_{|\varphi(z)|\to 1} |u(z)| = \limsup_{|z|\to 1} |u(z)| = \|u\|_\infty = \|uC_\varphi\|_{H^\infty}. \qquad \square$$

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From this result above, we can observe the fact that if $\varphi(z)$ gets too close to the unit circle too often, then the essential norm of uC_φ would be close to its operator norm. As we know, the inner functions are of the high extremeness of the closed unit ball of H^∞. Recall the de Leeuw–Rudin characterization of the extreme points of the closed unit ball of H^∞, that is, φ is an extreme point of the closed unit ball of H^∞ if and only if

$$\int_0^{2\pi} \log(1 - |\varphi(e^{i\theta})|)\, d\theta = -\infty$$

(see [2] or [3, Ch. 9]).

Here we consider the problem of how small is the essential norm of uC_φ induced by a nonextreme self-map φ. For $\varphi \in S(\mathbb{D})$ which is not an extreme point of the closed unit ball of H^∞, put

$$\omega(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log(1 - |\varphi(e^{i\theta})|)\, d\theta\right). \tag{4.2}$$

Then ω is a bounded analytic function. Moreover ω is an outer function in H^∞ such that $|\varphi| + |\omega| \le 1$ on $\mathbb{D}$ and $|\varphi(e^{i\theta})| + |\omega(e^{i\theta})| = 1$ almost everywhere (see [3, Ch. 9]). Hence we have that $|\omega(z)| \to 0$ if $|\varphi(z)| \to 1$. Using this outer function ω, we obtain the following theorem.

THEOREM 4.3. *Let u be in H^∞ and φ be in $S(\mathbb{D})$. If φ is not an extreme point of the closed unit ball of H^∞, then*

$$\limsup_{|\varphi(z)|\to 1} |u(z)| \le \|uC_\varphi\|_{e,H^\infty} \le \limsup_{|\omega(z)|\to 0} |u(z)|$$

where ω is the outer function defined by (4.2).

PROOF. It is enough to prove the upper estimate. Since ω has no zero on $\mathbb{D}$, the nth root of ω can be defined as a function in H^∞. We put $v_n = u\,\omega^{1/n}$ for any positive integer n. Then $v_n C_\varphi$ is compact on H^∞ and

$$\|uC_\varphi\|_{e,H^\infty} \le \|uC_\varphi - v_n C_\varphi\|_{H^\infty} = \|u(1-\omega^{1/n})\|_\infty.$$

Letting $n \to \infty$, we have that

$$\|uC_\varphi\|_{e,H^\infty} \le \lim_{n\to\infty} \|u(1-\omega^{1/n})\|_\infty.$$

For any sequence $\{z_j\} \subset \mathbb{D}$ such that $\limsup |\omega(z_j)| > 0$,

$$\lim_{n\to\infty} \limsup_{j\to\infty} |u(z_j)(1-\omega(z_j)^{1/n})| = 0.$$

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Hence we have that

$$\begin{aligned}\lim_{n\to\infty}\|u(1-\omega^{1/n})\|_\infty &= \lim_{n\to\infty}\limsup_{|\omega(z)|\to 0}|u(z)(1-\omega(z)^{1/n})|\\ &= \lim_{n\to\infty}\limsup_{|\omega(z)|\to 0}|u(z)|\\ &= \limsup_{|\omega(z)|\to 0}|u(z)|.\end{aligned}$$

Our proof is accomplished. □

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https://doi.org/10.1017/S0004972709000094 Published online by Cambridge University Press