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IDENTITIES AND LEFT CANCELLATION IN DISTRIBUTIVELY GENERATED NEAR-RINGS

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Abstract

Given a semigroup S, we define $\{N(S), +, \cdot\}$ to be the 'free' distributively generated near-ring. Since all words in $N(S)$ can be expressed as the sum and difference of elements of S, we are able to define a length function on the words of $N(S)$. The following theorems then follow:

THEOREM 1. *$N(S)$ contains a multiplicative identity e if and only if $e \in S$.*

THEOREM 2. *If S is the free semigroup in the variety of all semi-groups then $N(S)$ is left cancellative.*

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1. Introduction

We want to prove several results about distributively generated near-rings 'freely' generated by semigroups. We will show for an arbitrary semigroup S that the 'freely' generated near-ring contains a multiplicative identity if and only if the original semigroup contains a multiplicative identity. Secondly, we will show that if S is the free semigroup, then the resulting near-ring is left cancellative.

Let us say more carefully what we mean by the distributively generated near-ring 'freely' generated by a semigroup S. Our *near-rings* will be left near-rings; that is, a set with two binary operations $+$ and $\cdot$, so that the system is a group with respect to $+$, a semigroup with respect to $\cdot$, and $\cdot$ is left distributive over $+$. One is *distributively generated* if it is additively generated by a set of elements that distribute from the right as well as from the left.

2. Preliminaries

We are interested here in a special class of distributively generated near-rings.

Given any semigroup S, we define the distributively generated near-ring $\{N(S), +, \cdot\}$ as follows : if $s, t \in S$ and $u, v, w \in N(S)$, then $s \cdot t = st$ (the product of s and t in S), and $(-u) \cdot t = -(u \cdot t)$, and $(w + u) \cdot s = w \cdot s + u \cdot s$, and $w \cdot (u + v) = w \cdot u + w \cdot v$, and $w \cdot (-u) = -(w \cdot u)$, and $w \cdot 0 = 0$. It has been shown in Evans and Neff (1964) and Fröhlich (1960) that $N(S)$ is a distributively generated near-ring.

All words in $N(S)$ can be expressed as the sum and difference of elements of S; that is, $w = \sum_i \pm s_i$. If $w = \sum_i \pm s_i$ and $v = \sum_j \pm t_j$, then the product $wv = \sum_j \pm (\sum_i \pm s_i t_j)$ which we call the *linear form* of the product. A word is reduced if the word is reduced as a word in the free group on S.

We define length of a word in $N(S)$ as follows :

1. length of 0 is 0;
2. length of $\pm s$, where $s \in S$, is 1;
3. length of $\pm u \pm v$ is the sum of the length of u and the length of v;
4. length of $u \cdot v$ is the product of the length of u and the length of v.

On a free semigroup S (possibly with identity added), we define the standard length, written as $|s|$,

1. if g is a generator of S, $|g| = 1$;
2. if e is the identity, $|e| = 0$;
3. $|st| = |s|\,|t|$.

Let u and v be reduced words of $N(S)$, $u = \sum_i \pm s_i$ and $v = \sum_j \pm t_j$. In the word $uv = \sum_j \pm (\sum_i \pm s_i t_j)$ we call the subword $\sum_i \pm s_i t_j$ the t_j segment. The segment is called positive or negative depending on the sign of t_j. Each t_j segment is reduced, and if the signs of t_j and t_{j-1} are different then $\pm(\sum_i \pm s_i t_j) \mp (\sum_i \pm s_i t_{j+1})$ is reduced. We will refer to the 'place' where two adjacent segments adjoin as a 'joint'.

3. Identities

If a generating set contains an identity, then so must the distributively generated near-ring; however, the converse does not always hold. For example $\{Z_6, +, \cdot\}$ is a distributively generated near-ring where a generating set is $\{0, 2, 3, 4\}$. We will show that the converse does hold in $N(S)$.

LEMMA 1. *If $N(S)$ has an identity e and $e = \sum_{i=1}^{n} \pm a_i$ (reduced form), then*

(a) *n is odd;*

(b) *e is symmetric; that is, $\pm a_1 = \pm a_n$, $\pm a_2 = \pm a_{n-1}, \ldots$;*

(c) *When $n = 2m+1$ and $g \in S$, there exists $j \geqslant m+1$ so that $\pm a_j g = g$, $\sum_{i=1}^{j-1} \pm a_i g = 0$, and $\sum_{i=j+1}^{2m+1} \pm a_i g = 0$.*

PROOF. Since e is an identity, $eg = g$ for generator g, and so $\sum_{i=1}^{n} \pm a_i g = g$. Since summands reduce in pairs, n must be odd.

Consider e^{T}, that is, e written in reverse order : $e^{\mathrm{T}} = \pm a_n \dots \pm a_1$. For any generator g, any reduction in eg corresponds in a one-to-one fashion to a reduction in $e^{\mathrm{T}} g$, so we have $e^{\mathrm{T}} g = g$. So e^{T} is a left identity, and therefore $e = e^{\mathrm{T}}$, and we have proved (b).

Let $n = 2m+1$. For a generator g, $g = eg = \sum_{i=1}^{2m+1} \pm a_i g$. There must be some integer j such that $a_j g = g$, $\sum_{i=1}^{j-1} \pm a_i g = 0$, and $\sum_{i=j+1}^{2m+1} \pm a_i g = 0$. If $j \geqslant m+1$ we are done; if $j < m+1$, then using symmetry we have $a_{2m+2-j} g = g$, $\sum_{k=1}^{2m+1-j} \pm a_k g = 0$ and $\sum_{k=2m+3-j}^{2m+1} \pm a_k g = 0$.

The next lemma will establish a general reduction pattern in $N(S)$.

LEMMA 2. *Let w be a symmetric word of odd length (no summand is 0, but not necessarily reduced), say $w = x \pm c + x^{\mathrm{T}}$, that reduces to an element of S. If $w = x + c + x^{\mathrm{T}}$ then $x = 0$, and if $w = x - c + x^{\mathrm{T}}$ then $x - c = 0$.*

PROOF. We will use induction on the length of w.

If the length of w is 3, then $w = x - c + x$, and so $c = x$, and so $x - c = 0$.

Suppose $w = \pm s_1 \pm \dots \pm s_n - c \pm s_n \pm \dots \pm s_1$, each s_i, $c \in S$, and w reduces to some element of S. There is an integer $j \leqslant n$ so that $s_j - c = 0$ and $\sum_{k=j}^{m} \pm s_k - c = 0$. (If $j = 0$ we are done.) Then by the symmetry of w, we get that

$$w = \pm s_1 \pm \dots \pm s_{j-1} + 0 + s_j \pm \dots \pm s_1 = v.$$

The induction hypothesis applies to v, so $\pm s_1 \pm \dots \pm s_{j-1} = 0$. Then in w we have that $x - c = 0$.

In the case where $w = \pm s_1 \pm \dots \pm s_n + c \pm s_n \pm \dots \pm s_1$ the proof is similar and omitted.

Now we will show e fits the hypothesis of the previous lemmas and prove

THEOREM 3. *If $N(S)$ has two-sided identity e, then $e \in S$.*

PROOF. According to Lemma 1 we write $e = x \pm c + x^{\mathrm{T}}$ (reduced form).

Let $s \in S$. Then $s = es = xs \pm cs + x^{\mathrm{T}} s = (xs \pm cs + x^{\mathrm{T}} s)^{\mathrm{T}}$, where the summands belong to S and are hence not zero.

Applying Lemma 2, if the term $\pm c$ is positive, then $xs = 0$ for all $s \in S$, or if the term $\pm c$ is negative $(x - c)s = 0$ for all $s \in S$. Suppose $xs = 0$ for all $s \in S$, then it follows that $xe = 0$ and so $x = 0$, contrary to x being reduced. Similarly $(x - c)s = 0$ for all $s \in S$ gives a contradiction.

The length of e must be 1. Hence $e \in S$.

The following example shows that we cannot weaken the result to left identities. Suppose we begin with the semigroup S given by $S = \{a, b, c\}$, and $x^2 = x$, for all

$x \in S$, *and* $xy = b$, for $x, y \in S$ and $x \neq y$. S has no left identity, but in $N(S)$ the expression $a-b+c$ is a left identity. Incidentally, $a-b+c$ is a right identity for generators but not for $N(S)$.

4. Left cancellation

We will now show that $N(S)$ is left cancellative when S is the free semigroup. This will be done by showing that no element of $N(S)$ is a divisor of zero, and then using the result that no divisors of zero implies left cancellation.

We embed S, the free semigroup, in a semigroup with identity $S^e = S \cup \{e\}$. Note that $N(S)$ is naturally embedded in $N(S^e)$.

LEMMA 4. *No word of length* 2 *in* $N(S^e)$ *is a left divisor of zero.*

PROOF. Suppose $wv = 0$, where w is reduced and of length 2, $w = \pm s_1 \pm s_2$. We proceed by induction on the length of v.

If v has length 1, it can be easily seen this leads to contradiction.

Assume that if v is a reduced word in $N(S^e)$ of length less than n then $wv = 0$ implies $v = 0$.

Let $v = \pm t_1 \pm \dots \pm t_n$ and suppose $wv = 0$.

$$wv = \pm(\pm s_1 \pm s_2)t_1 \pm \dots \pm(\pm s_1 \pm s_2)t_n = 0.$$

Since reduction can occur only between segments t_j and t_{j+1} of the same sign, we deduce that s_1 and s_2 have opposite signs.

Let k be a positive integer such that reduction occurs between the t_k and t_{k+1} segments, and furthermore let us assume both are positive. We have $s_2 t_k = s_1 t_{k+1}$. $|t_k| = |t_{k+1}|$ leads to a contradiction. Assume $|t_k| > |t_{k+1}|$. It follows that $|s_2| < |s_1|$: so $s_1 = s_2 s$ for some $s \in S$. Now $wv = s_2(s-e)v = 0$ and it follows that $(s-e)v = 0$ and so $\pm(st_1 - t_1) \pm \dots \pm (st_n - t_n) = 0$. Assuming all segments are positive leads to a contradiction.

Consider the first joint from the left which occurs between two segments of different signs (assume t_j is positive and t_{j+1} is negative),

$$uv = +(st_1 - t_1) + \dots + (st_j - t_j) + (t_{j+1} - st_{j+1}) + \dots.$$

There must exist $k > j+1$ such that $(s-e)(-t_{j+1} \pm \dots \pm t_k) = 0$, then by the induction hypothesis it follows that $-t_{j+1} \pm \dots \pm t_k = 0$, contrary to v being reduced.

LEMMA 5. *If* w *belongs to* $N(S^e)$ *and has odd length, then* w *is not a left divisor of zero.*

PROOF. We write $w = x + c + y$ (reduced) where the lengths x and y are the same,

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and we write $v = \pm t_1 \pm \ldots \pm t_m$. If $wv = 0$ then there exists j, $1 \leqslant j < m$, such that $\pm(c+y)t_j + (x+c)t_{j+1} = 0$ contrary to v being reduced.

The next lemma covers all remaining w, of even length. We omit the proof.

LEMMA 6. *If w (different from 0) belongs to $N(S^e)$ and has even length, then w is not a left divisor of zero.*

THEOREM 7. *$N(S^e)$ contains no divisors of zero and is therefore left cancellative.*

COROLLARY 8. (a). *If T is a subsemigroup of S^e, then $N(T)$ has no divisors of zero.*
(b) *If N is a subnear-ring of $N(S^e)$ then N has no divisors of zero.*

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