PRODUCTS OF ABELIAN HOPFIAN GROUPS

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1. Introduction

1.1 Let $\mathfrak{N}_k$ be the variety of all nilpotent groups of class at most k. The purpose of this note is to prove the following

THEOREM 1. *Let $\mathfrak{V}$ be a variety of groups containing $\mathfrak{N}_2$, let A and B be torsion-free abelian hopfian groups and let P be the free $\mathfrak{V}$-product [2] of A and B. If P is residually torsion-free nilpotent, then P is hopfian.*

Let $\mathfrak{S}_k$ denote the variety of all soluble groups of derived length at most k. Since the free $\mathfrak{S}_k$-product of torsion-free abelian groups is residually torsion-free nilpotent ([2]) Theorem 1 applies and we have, therefore, the

COROLLARY 1 [3]. *For every $k \geqq 2$ the free $\mathfrak{S}_k$-product of any pair of torsion-free abelian hopfian groups is hopfian.*

Corollary 1 is somewhat surprisingly, no longer valid when $k = 1$; in this case P is the direct product of A and B, and A.L.S. Corner [3] has constructed a number of extraordinary counter-examples.

1.2 The proof of Theorem 1 is not difficult, depending essentially on an analysis of the centralizers of certain elements in the free $\mathfrak{N}_2$-product of torsion-free abelian groups.

2. A free $\mathfrak{N}_2$-product

2.1 Let S be the set of all quintuplets of rational numbers (r, s, t, u, v). We turn S into a group by defining

$$(r, s, t, u, v) \cdot (r^*, s^*, t^*, u^*, v^*) = (r+r^*, s+s^*, t+t^*, u+u^*-sr^*, v+v^*-tr^*).$$

The set of elements of the form

$$(r, 0, 0, 0, 0)$$

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[2] I.e. the free product F of A and B modulo the verbal subgroup of F determined by $\mathfrak{V}$ (cf. S. Moran [7]).

[3] This is a special case of a theorem announced in [1].

is a subgroup C of S isomorphic to the additive group Q of rational numbers. Similarly the set of elements of the form

$$(0, s, t, 0, 0)$$

is a subgroup D isomorphic to $Q \times Q$. (Indeed S is actually the free $\mathfrak{N}_2$-product of C and D.) S is readily seen to be nilpotent of class two.

3. Centralizers

3.1 Let A and B be torsion-free abelian groups and let P be their free $\mathfrak{N}_2$-product. We shall prove here that if $a \in A$ $(a \neq 1)$, $b \in B$ $(b \neq 1)$, $z \in Z$, the centre of P, then the centralizer $C(abz)$ of abz is locally cyclic modulo Z.

3.2 We begin with the following simple

LEMMA 1. *If b' $(b' \in B)$ centralizes abz, then $b' = 1$.*

PROOF. Consider the group S of **2.1**. Now assume $b' \neq 1$. By a characteristic property of divisible groups the mappings

$$a \to (1, 0, 0, 0, 0), \quad b' \to (0, 1, 0, 0, 0)$$

can be continued to homomorphisms of A into C and B into D, respectively, and hence to a homomorphism η of P into S.

But, remembering S is nilpotent of class two[4],

$$[(abz)\eta, b'\eta] = [a\eta, b'\eta] = (0, 0, 0, 1, 0) \neq 1.$$

This completes the proof of Lemma 1.

COROLLARY 1. *If b' centralizes a, $b' = 1$.*

3.3 Next we need

LEMMA 2. *If b' and b generate a free abelian subgroup of B of rank two, then $a'b'z'$ does not centralize abz for any choice of $a' \in A$, $z' \in Z$.*

PROOF. Consider again the group S. Choose a homomorphism η of P into S as in the proof of Lemma 1, which takes

$$a \text{ to } (1, 0, 0, 0, 0), \quad b \text{ to } (0, 1, 0, 0, 0), \quad b' \text{ to } (0, 0, 1, 0, 0).$$

Then, writing $a' = a^r$ where r may be rational (and therefore interpreted in the obvious way),

[4] If x, y are elements of a group we denote the commutator $x^{-1}y^{-1}xy$ of x and y by $[x, y]$.

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$$\begin{aligned}[(abz)\eta, (a'b'z')\eta] &= [a\eta, b'\eta][b\eta, s'\eta]\\ &= [(1, 0, 0, 0, 0), (0, 0, 1, 0, 0)][(0, 1, 0, 0, 0), (1, 0, 0, 0, 0)^r]\\ &= (0, 0, 0, 1, 0)(0, 0, 0, 0, r)\\ &= (0, 0, 0, 1, r).\end{aligned}$$

This proves Lemma 2.

3.4 Finally we arrive at the promised

PROPOSITION 1. $C(abz)/Z$ is locally cyclic.

PROOF. Let $a'b'z' \in C(abz)$. By Lemma 1 (and its analogue for a') either $a'b'z' \in Z$ or else

$$a' \neq 1, \quad b' \neq 1.$$

But by Lemma 2 (and its analogue for the subgroup generated by a and a')

$$gp(a, a') \text{ is cyclic}, \quad gp(b, b') \text{ is cyclic}.$$

We need only one of these remarks. Thus choose integers r and s so that

$$a^r(a')^s = 1 \qquad (r \neq 0 \neq s).$$

Then, obviously

$$(abz)^r(a'b'z')^s \in C(abz).$$

But Lemma 1 applies and so we find $(abz)^r(a'b'z')^s \in Z$; in other words abz and $a'b'z'$ generate a cyclic subgroup modulo Z and so the proof of the proposition is complete.

4. The proof of Theorem 1

4.1 Let P be the free $\mathfrak{V}$-product ($\mathfrak{V} \supseteq \mathfrak{N}_2$) of the torsion-free abelian hopfian groups A and B and let η be a homomorphism of P onto P. Let

$$P = P_1 \geqq P_2 \geqq P_3 \geqq \cdots$$

be the lower central series of P. Since $\mathfrak{V}$ contains $\mathfrak{N}_2$, P/P_3 is the free $\mathfrak{N}_2$-product of A and B. Notice that the centre of P/P_3 is simply P_2/P_3.

We consider first the case when both $A\eta$ and $B\eta$ are not locally cyclic modulo P_2. Look at $A\eta$. Since $A\eta P_2/P_2$ is not locally cyclic, $A\eta$ does not contain an element of the form $abz(a \neq 1, b \neq 1, a \in A, b \in B, z \in P_2)$ (Proposition 1). So, by Corollary 1, either

$$A\eta \leqq AP_2 \text{ or } A\eta \leqq BP_2.$$

Similarly

$$B\eta \leqq BP_2 \text{ or } B\eta \leqq AP_2.$$

As suggested by the order, we claim therefore that either $A\eta \leqq AP_2$ *and*

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$B\eta \leqq BP_2$ or $A\eta \leqq BP_2$ *and* $B\eta \leqq AP_2$ since P is certainly not generated by A modulo P_2. But, in the event that the latter possibility is in force, we may equally concern ourselves with η^2 where

$$A\eta^2 \leqq AP_2 \text{ and } B\eta^2 \leqq BP_2.$$

Now this means that, modulo P_2, η maps A isomorphically onto A and B isomorphically onto B. Since P is the free $\mathfrak{B}$-product of A and B, every pair of automorphisms of A and B respectively can simultaneously be extended to an automorphism of P. Thus we may assume that η induces the identity homomorphism of P modulo P_2. But then η is certainly monomorphic (P. Hall [5], Lemma 1).

4.2 We consider next the case where $A\eta P_2/P_2$ is not locally cyclic, but $B\eta P_2/P_2$ is locally cyclic. It then follows from the argument of **4.1** that we may assume that η leaves A fixed modulo P_2 and also that η leaves B fixed modulo AP_2.

Now embed A and B in minimal torsion-free divisible groups $\bar{A}$ and $\bar{B}$ respectively (see, for example, L. Fuchs [4]). Let R be the free $\mathfrak{B}$-product of $\bar{A}$ and $\bar{B}$. It is not difficult to prove that A and B generate in R their free $\mathfrak{B}$-product (isomorphic to) P and that R is residually torsion-free nilpotent (see, for example, the argument on pages 364 and 365 of [2]). Now for each i let

$$R(i) = R/T_i,$$

where T_i is the inverse image of the torsion-subgroup of R/R_i. As R is residually torsion-free nilpotent, the T_i have intersection 1. Notice that $R(i)$ is a torsion-free nilpotent divisible group since it is Cernikov complete (see, for example, A. G. Kurosh [6], vol. 2, p. 233). Consider now

$$A\eta T_i/T_i \text{ and } B\eta T_i/T_i.$$

Since $R(i)$ is a torsion-free divisible nilpotent group, both $A\eta T_i/t_i$ and $B\eta T_i/T_i$ can be enlarged to minimal torsion-free divisible abelian groups (see, for example, A. G. Kurosh [6], vol. 2, p. 256). So by a characteristic property of divisible groups the mappings which are induced by η in AT_i/T_i and BT_i/T_i can be extended to homomorphisms α and β say from $\bar{A}T_i/T_i$ and $\bar{B}T_i/T_i$ respectively into $R(i)$. But $R(i)$ is clearly the "freest" group in $\mathfrak{B}$ which is torsion-free and nilpotent of class at most $i-1$, generated by (copies of) $\bar{A}$ and $\bar{B}$. So α and β can be simultaneously extended to a homomorphism γ of $R(i)$ into $R(i)$. But γ leaves $\bar{A}T_i/T_i$ fixed modulo the derived group of $R(i)$ and also $\bar{B}T_i/T_i$ fixed modulo the group generated by $\bar{A}T_i/T_i$ and the derived group of $R(i)$. Thus the images of $\bar{A}T_i/T_i$ and $\bar{B}T_i/T_i$ under γ generate $R(i)$ modulo its derived group. Hence, remembering $R(i)$ is nilpotent, $(\bar{A}T_i/T_i)\gamma$ and $(\bar{B}T_i/T_i)\gamma$ generate $R(i)$ and so γ

is onto. Now pick an element $b \neq 1$, $b \in \bar{B}T_i/T_i$. Then $b\gamma = ba$ modulo the derived group of $R(i)$, where $a \in \bar{A}T_i/T_i$. Embed ba^{-1} in a copy of the rationals inside $R(i)$. Then the mapping δ_2 defined by

$$b\delta_2 = ba^{-1}$$

can be continued to a homomorphism of $\bar{B}T_i/T_i$ into $R(i)$. Since $\bar{B}T_i/T_i$ is in fact isomorphic to the additive group of rationals, $\delta_2\gamma$ leaves $\bar{B}T_i/T_i$ fixed modulo the derived group of $R(i)$. Now let δ_1 be the restriction of γ to $\bar{A}T_i/T_i$ and extend δ_1 and δ_2 to a homomorphism δ of $R(i)$ into $R(i)$. Now δ is onto as before and $\gamma\delta$ leaves $R(i)$ fixed modulo its derived group. So $\gamma\delta$ is an automorphism by Lemma 1 of P. Hall [5]. So γ is one-to-one and hence η must be one-to-one as the T_i intersect trivially. This completes the second part of the analysis.

4.3 Suppose finally that $A\eta$ and $B\eta$ are both locally cyclic modulo P_2. It follows that both A and B are locally cyclic. Hence P is itself residually a poly-locally-infinite-cyclic group and therefore clearly hopfian.

The completes the proof of the theorem.

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