

Measure data elliptic problems with generalized Orlicz growth

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We study nonlinear measure data elliptic problems involving the operator of generalized Orlicz growth. Our framework embraces reflexive Orlicz spaces, as well as natural variants of variable exponent and double-phase spaces. Approximable and renormalized solutions are proven to exist and coincide for arbitrary measure datum and to be unique when for a class of data being diffuse with respect to a relevant nonstandard capacity. A capacity characterization of diffuse measures is provided.

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1. Introduction

Our objective is to study existence and uniqueness of two kinds of very weak solutions to nonlinear measure data problem

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, $n \geq 2$, μ is an arbitrary bounded measure on Ω , and $\mathcal{A} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ has growth prescribed by the means of an inhomogeneous function $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty)$ of an Orlicz growth with respect to the second variable. Special cases of the leading part of the operator \mathcal{A} include p -Laplacian, $p(x)$ -Laplacian, but we cover operators with Orlicz, double-phase growth, as well as weighted Orlicz or variable exponent double-phase one as long as it falls into the realm of Musielak–Orlicz spaces within the natural regime described in §2. The existence of renormalized solutions to general measure data problem and uniqueness for measure data is new even in the reflexive Orlicz case. It was also not known in two cases enjoying lately particular attention – double-phase and variable exponent double-phase ones.

Very weak solutions to measure data problems of the form (1.1) are already studied in depth in the classical setting of Sobolev spaces, that is when the growth of

the leading part of the operator is governed by a power function with the celebrated special case of p -Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. To give a flavour let us mention e.g. [12, 16, 17, 37, 38], where the existence is provided for various notions of very weak solutions for L^1 or measure data. Note that the notions in several cases coincide [37, 38, 54]. In general, it is possible to find a proper notion enjoying existence, but sharp assumptions on μ to ensure uniqueness for these type of problems are not known even when the operator \mathcal{A} exposes the mentioned standard p -growth. See counterexamples in [16] showing non-uniqueness for concentrated measures. The natural sufficient condition in the standard case is that μ is so diffuse that it does not charge the sets of proper capacity zero and the proof of uniqueness essentially employs its characterization of the form of theorem 1.1.

Analysis of problems exposing (p, q) -growth, where the operator is trapped between polynomials $|\xi|^p \lesssim \mathcal{A}(x, \xi) \cdot \xi \lesssim 1 + |\xi|^q$, are already classical topic investigated since [45, 56, 60, 67]. Nowadays, there is a great interest in analysis under nonstandard growth conditions that embraces more: problems with variable exponent growth used in modelling of electrorheological fluids [1, 65], thermistor models [71] or image processing [19], with double-phase growth good for description of composite materials [34], as well as Orlicz one – engaged in modelling of non-Newtonian fluids [46] and elasticity [7]. Studies on nonstandard growth problems form a solid stream in the modern nonlinear analysis [9, 20, 23, 25, 29, 32, 34, 39, 40, 50, 51, 61, 68]. The theory of existence of very weak solutions to problems with nonstandard growth and merely integrable data is under intensive investigation [3, 11, 31, 33, 47, 48, 66, 69]. For the study on Musielak–Orlicz growth L^1 -data elliptic equations we refer to [47] under growth restrictions on the conjugate of the modular function and to [48, 55], where existence is provided either in (all) reflexive spaces or when the growth of modular function is well-balanced (and the smooth functions are modularly dense, cf. also [4, 18]). Analogous parabolic study can be found in [26–28]. For measure data problems with Orlicz growth to our best knowledge we can refer only to [6] for some class of measures, to [13, 33, 43] for general measures in the reflexive case extended in [5, 24]. In [5, 24, 33], besides existence also regularity in the scale of Marcinkiewicz-type spaces is provided even for solutions to measure data problems, but therein the uniqueness is obtained only if the datum is integrable. In [25], precise regularity results are provided recently extended in [30]. On the other hand, existence of very weak solutions and uniqueness in the case of diffuse measures is studied in the variable exponent setting in [57, 59, 69]. Here, two kinds of very weak solutions are proven to exist and coincide for arbitrary measure datum.

We consider (1.1) involving the leading part of the operator governed by a function $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty)$ and, thereby, placing our analysis in an unconventional functional setting, where the norm is defined by the means of the functional

$$w \mapsto \int_{\Omega} \varphi(x, |Dw|) \, dx, \tag{1.2}$$

Let us make an overview of the special cases of the functional framework we capture. The operator can be governed by power function variable in space, namely $\varphi(x, s) = |s|^{p(x)}$, where $p : \Omega \rightarrow (1, \infty)$ is log-Hölder continuous, i.e. when there exists $c > 0$ such that $|p(x) - p(y)| \leq -c/\log(|x - y|)$ for $|x - y| < 1/2$, cf. [36]. Another model

example we cover is non-uniformly elliptic problems living in spaces with the double-phase energy, $\varphi(x, s) = |s|^p + a(x)|s|^q$, where $a \in C^{0,\alpha}(\Omega)$ is nonnegative and can vanish in some regions of $\Omega \subset \mathbb{R}^n$, while exponents satisfy $1 < p \leq q < \infty$ and are close in the sense that $\frac{q}{p} \leq 1 + \frac{\alpha}{n}$ necessary for density of smooth functions [4, 34]. What is more, we admit problems posed in the reflexive Orlicz setting, when φ is a doubling N -function $\varphi(x, s) = \varphi(s) \in \Delta_2 \cap \nabla_2$, including Zygmund-type spaces where $\varphi_{p,\alpha}(s) = s^p \log^\alpha(1 + s)$, $p > 1$, $\alpha \in \mathbb{R}$ or compositions and multiplications of functions from the family $\{\varphi_{\bar{p},\bar{\alpha}}\}_{\bar{p},\bar{\alpha}}$ with various parameters. More generally, under certain nondegeneracy and continuity conditions, given as (A0)–(A1) in §2, we capture also general case (1.2). The remaining examples we can give here cover all weighted reflexive Orlicz functionals with non-degenerating weights, double-phase functions with variable exponents $\varphi(x, s) = |s|^{p(x)} + a(x)|s|^{q(x)}$, double phase with Orlicz phases $\varphi(x, s) = \varphi_1(s) + a(x)\varphi_2(s)$ or multi-phase cases $\varphi(x, s) = \sum_{i=1}^k a_i(x)\varphi_i(s)$ (with appropriately regular weights) as long as conditions (A0)–(A1) are satisfied. We refer to [20] for a more detailed overview of differential equations and to [49] for the fundamental properties of the functional framework.

Diffuse measures

The natural property of a measure to ensure uniqueness of very weak solutions to (1.1) is that μ is diffuse with respect to a relevant capacity. In order to characterize such measures, let us denote by $\mathcal{M}_b(\Omega)$ the set of bounded measures on $\Omega \subset \mathbb{R}^n$ and by $W^{1,\varphi(\cdot)}(\Omega)$ the Musielak–Orlicz–Sobolev space. See §2.3 for the introduction to the functional setting and all assumptions and §2.5 for the capacity. By $\mathcal{M}_b^{\varphi(\cdot)}(\Omega)$ we mean the set of $\varphi(\cdot)$ -diffuse measures (or $\varphi(\cdot)$ -soft measures) consisting of such bounded measures $\mu_{\varphi(\cdot)}$ that do not charge sets of $\varphi(\cdot)$ -capacity zero (for every Borel set $E \subset \Omega$ such that $C_{\varphi(\cdot)}(E) = 0$ it holds that $\mu_{\varphi(\cdot)}(E) = 0$). One may think that a measure $\mu_{\varphi(\cdot)} \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$ is ‘absolutely continuous with respect to $C_{\varphi(\cdot)}$ ’. Our first result is the following theorem.

THEOREM 1.1 (Characterization of measures). *Suppose $\varphi \in \Phi_c(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Assume that φ satisfies $(aInc)_p$, $(aDec)_q$, (A0) and (A1). When $\mu \in \mathcal{M}_b(\Omega)$, then*

$$\mu \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega) \quad \text{if and only if} \quad \mu \in L^1(\Omega) + (W_0^{1,\varphi(\cdot)}(\Omega))',$$

i.e. there exist $f \in L^1(\Omega)$ and $G \in (L^{\tilde{\varphi}(\cdot)}(\Omega))^n$, such that $\mu = f - \operatorname{div} G$ in the sense of distributions.

REMARK 1.2. Let us note that upon our assumptions $t^p \lesssim \varphi(\cdot, t)$. If $p > n$ it holds that $\mathcal{M}_b(\Omega) \subset W^{-1,p'}(\Omega) \subset (W_0^{1,\varphi(\cdot)}(\Omega))'$. In this case, all measures are absolutely continuous with respect to the Lebesgue measure and, consequently, the result is really meaningful only for slowly growing functions φ .

REMARK 1.3. The decomposition of theorem 1.1

$$\mathcal{M}_b^{\varphi(\cdot)}(\Omega) \ni \mu = f - \operatorname{div} G \text{ with } f \in L^1(\Omega) \text{ and } G \in (L^{\tilde{\varphi}(\cdot)}(\Omega))^n$$

cannot be unique as $L^1(\Omega) \cap (W_0^{1,\varphi(\cdot)}(\Omega))' \neq \{0\}$. On the other hand, for every $\mu \in \mathcal{M}_b(\Omega)$ there exists a decomposition $\mu = \mu_{\varphi(\cdot)} + \mu_{\text{sing}}^+ - \mu_{\text{sing}}^-$ with some $\mu_{\varphi(\cdot)}$ which is absolutely continuous with respect to $\varphi(\cdot)$ -capacity, while $\mu_{\text{sing}}^+, \mu_{\text{sing}}^- \geq 0$ are singular with respect to the $C_{\varphi(\cdot)}$ (concentrated on some set of $\varphi(\cdot)$ -capacity zero) and this decomposition is unique up to sets of $\varphi(\cdot)$ -capacity zero, see lemma 3.2. Consequently, any $\mu \in \mathcal{M}_b(\Omega)$ admits a decomposition $\mu = f + \operatorname{div} G + \mu_{\text{sing}}^+ - \mu_{\text{sing}}^-$ in the sense of distributions, with some $f \in L^1(\Omega)$, $G \in (L^{\tilde{\varphi}(\cdot)}(\Omega))^n$, and $C_{\varphi(\cdot)}$ -singular $\mu_{\text{sing}}^+, \mu_{\text{sing}}^-$.

Let us point out a direct consequence of theorem 1.1 to the classical spaces.

COROLLARY 1.4 Orlicz case. Suppose $B : [0, \infty) \rightarrow [0, \infty)$ is a Young function, such that $B \in \Delta_2 \cap \nabla_2$. Then $\mu_B \in \mathcal{M}_b(\Omega)$ does not charge the sets of Sobolev B -capacity zero if and only if $\mu_B \in L^1(\Omega) + (W_0^{1,B}(\Omega))'$, that is there exist $f \in L^1(\Omega)$ and $G \in (L^{\tilde{B}}(\Omega))^n$, such that $\mu_B = f - \operatorname{div} G$. In particular, the special case of this result is the classical measure characterization [16]: if $p > 1$, then $\mu_p \in \mathcal{M}_b(\Omega)$ does not charge the sets of the Sobolev p -capacity zero if and only if $\mu_p \in L^1(\Omega) + W^{-1,p'}(\Omega)$ (there exist $f \in L^1(\Omega)$ and $G \in (L^{p'}(\Omega))^n$, such that $\mu_p = f - \operatorname{div} G$ in the sense of distributions).

COROLLARY 1.5 Variable exponent case. Suppose $p : \Omega \rightarrow (1, \infty)$ with $1 < p_- \leq p(\cdot) \leq p_+ < \infty$ is log-Hölder continuous and $p'(x) := p(x)/(p(x) - 1)$. Then $\mu_{p(\cdot)} \in \mathcal{M}_b(\Omega)$ does not charge the sets of Sobolev $p(\cdot)$ -capacity zero if and only if $\mu_{p(\cdot)} \in L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$, i.e. there exist $f \in L^1(\Omega)$ and $G \in (L^{p'(\cdot)}(\Omega))^n$, such that $\mu_{p(\cdot)} = f - \operatorname{div} G$ in the sense of distributions, cf. [70].

Measure-data problems

Assumptions Given $\varphi \in \Phi_c(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, such that φ satisfies (aInc) $_p$, (aDec) $_q$, (A0) and (A1), we shall study equation (1.1) where vector field \mathcal{A} satisfies the following conditions:

(A1) $\mathcal{A} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Carathéodory function, i.e. it is measurable with respect to the first variable and continuous with respect to the last one;

(A2) There exist numbers $c_1^\varphi, c_2^\varphi > 0$ and a function $0 \leq \gamma \in L^{\tilde{\varphi}(\cdot)}(\Omega)$, such that for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$ the following ellipticity and growth conditions are satisfied

$$c_1^\varphi \varphi(x, |\xi|) \leq \mathcal{A}(x, \xi) \cdot \xi \quad \text{and} \quad |\mathcal{A}(x, \xi)| \leq c_2^\varphi (1 + \gamma(x) + \varphi(x, |\xi|)/|\xi|).$$

(A3) \mathcal{A} is monotone, i.e. for a.e. $x \in \Omega$ and all $\eta \neq \xi \in \mathbb{R}^n$

$$(\mathcal{A}(x, \eta) - \mathcal{A}(x, \xi)) \cdot (\eta - \xi) > 0.$$

(A4) For a.e. $x \in \Omega$ it holds that $\mathcal{A}(x, 0) = 0$.

Special cases Of course, (A1)–(A4) with $\varphi \in \Phi_c(\Omega)$ satisfying (aInc) $_p$, (aDec) $_q$, (A0) and (A1) generalize not only classical conditions in the case when $\varphi(x, s) = s^p$:

$$c_1^p |\xi|^p \leq \mathcal{A}(x, \xi) \cdot \xi \quad \text{and} \quad |\mathcal{A}(x, \xi)| \leq c_2^p (1 + \gamma(x) + |\xi|^{p-1})$$

with $0 \leq \gamma \in L^{p'}(\Omega)$ with the special case of p -Laplacian. When $\varphi(x, s) = s^{p(x)}$ it covers

$$c_1^{p(\cdot)} |\xi|^{p(x)} \leq \mathcal{A}(x, \xi) \cdot \xi \quad \text{and} \quad |\mathcal{A}(x, \xi)| \leq c_2^{p(\cdot)} (1 + \gamma(x) + |\xi|^{p(x)-1})$$

with $0 \leq \gamma \in L^{p(\cdot)/(p(\cdot)-1)}(\Omega)$ with the special case of (possibly weighted) $p(x)$ -Laplacian. We allow for all $p : \Omega \rightarrow (1, \infty)$ under typical assumptions that $1 < p_- \leq p(x) \leq p_+$ and p is log-Hölder continuous. In the double-phase case $\varphi_{dp}(x, s) = s^p + a(x)s^q$, $0 \leq a \in C^{0,\alpha}(\Omega)$, $q/p \leq 1 + \alpha/n$, it covers non-uniformly elliptic operators satisfying

$$c_1^{(p,q)} |\xi|^p \leq \mathcal{A}(x, \xi) \cdot \xi \quad \text{and} \quad |\mathcal{A}(x, \xi)| \leq c_2^{(p,q)} (1 + \gamma(x) + |\xi|^{p-1} + a(x)|\xi|^{q-1})$$

with $0 \leq \gamma \in L^{\tilde{p}_{dp}(\cdot)}(\Omega)$. Finally, in Orlicz case when $B \in C^1([0, \infty))$ is a doubling N -function it also simplifies to typically considered conditions

$$c_1^B B(|\xi|) \leq \mathcal{A}(x, \xi) \cdot \xi \quad \text{and} \\ |\mathcal{A}(x, \xi)| \leq c_2^B (1 + \gamma(x) + B'(|\xi|)), \quad \text{with } 0 \leq \gamma \in L^{\tilde{B}}(\Omega).$$

To give more examples one can consider problems in weighted Orlicz, double phase with variable exponents, or multi-phase Orlicz cases, as long as $\varphi(x, s)$ is comparable to a function doubling with respect to the second variable and satisfy nondegeneracy conditions (A0)–(A1).

Problem Distributional solutions to equation $-\Delta_p u = \mu$ when p is small ($1 < p < 2 - 1/n$) do not necessarily belong to $W_{loc}^{1,1}(\Omega)$. The easiest example to give is the fundamental solution (when $\mu = \delta_0$). This restriction on the growth can be dispensed by the use of a weaker derivative. We make use of the symmetric truncation $T_k : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$T_k(s) = \begin{cases} s & |s| \leq k, \\ k \frac{s}{|s|} & |s| \geq k. \end{cases} \tag{1.3}$$

Note that as a consequence of [12, lemma 2.1] for every function u , such that $T_t(u) \in W_0^{1,\varphi(\cdot)}(\Omega)$ for every $t > 0$ there exists a (unique) measurable function $Z_u : \Omega \rightarrow \mathbb{R}^n$ such that

$$\nabla T_t(u) = \chi_{\{|u|<t\}} Z_u \quad \text{for a.e. in } \Omega \text{ and for every } t > 0. \tag{1.4}$$

With an abuse of notation, we denote Z_u simply by ∇u and call it a *generalized gradient*.

In order to introduce definitions of very weak solutions we define the space

$$\mathcal{T}_0^{1,\varphi(\cdot)}(\Omega) = \{u \text{ is measurable in } \Omega : T_t(u) \in W_0^{1,\varphi(\cdot)}(\Omega) \text{ for every } t > 0\}, \quad (1.5)$$

where $W_0^{1,\varphi(\cdot)}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in norm of $W^{1,\varphi(\cdot)}(\Omega)$. In fact, $u \in W_0^{1,\varphi(\cdot)}(\Omega)$ if and only if $u \in \mathcal{T}_0^{1,\varphi(\cdot)}(\Omega)$ and $Z_u \in L^{\varphi(\cdot)}(\Omega; \mathbb{R}^n)$. In the latter case, $Z_u = \nabla u$ a.e. in Ω .

Very weak solutions We define two kinds of very weak solutions to problem (1.1) under assumptions (A1)–(A4) involving a measure $\mu \in \mathcal{M}_b(\Omega)$.

Inspired by [14, 15, 33, 37], we define solutions that can be reached in the limit of solutions to approximate problems.

DEFINITION 1.6. A function $u \in \mathcal{T}_0^{1,\varphi(\cdot)}(\Omega)$ is called an approximable solution to problem (1.1) if u is an a.e. limit of a sequence of solutions $\{u_s\}_s \subset W_0^{1,\varphi(\cdot)}(\Omega)$ to

$$\int_\Omega \mathcal{A}(x, \nabla u_s) \cdot \nabla \phi \, dx = \int_\Omega \phi \, d\mu^s \quad \text{for any } \phi \in W_0^{1,\varphi(\cdot)}(\Omega) \cap L^\infty(\Omega), \quad (1.6)$$

when $\{\mu^s\} \subset C^\infty(\Omega)$ is a sequence of bounded functions that converges to μ weakly-* in the space of measures and such that

$$\limsup_{s \rightarrow 0} |\mu^s|(\overline{B}) \leq |\mu|(\overline{B}) \quad \text{for every } B \subset \Omega. \quad (1.7)$$

The definition seems *very weak* as we refrain from assuming any convergence of the gradients of (u_s) . Nonetheless, this is enough to show in the proofs that for fixed k also $\mathcal{A}(\cdot, \nabla(T_k u_s)) \rightarrow \mathcal{A}(\cdot, \nabla(T_k u))$ a.e. in Ω and thus it is justified to call u a *solution* (though in a *very weak* sense).

Having [38] and remark 1.3 we consider renormalized solutions defined as follows.

DEFINITION 1.7. A function $u \in \mathcal{T}_0^{1,\varphi(\cdot)}(\Omega)$ is called a renormalized solution to problem (1.1) with $\mu \in \mathcal{M}_b(\Omega)$, if

- (i) for every $k > 0$ one has $\mathcal{A}(x, \nabla(T_k u)) \in L^{\tilde{\varphi}(\cdot)}(\Omega)$;
- (ii) μ is decomposed to $\mu = \mu_{\varphi(\cdot)} + \mu_{\text{sing}}^+ - \mu_{\text{sing}}^-$, with $\mu_{\varphi(\cdot)} \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$ and nonnegative $\mu_{\text{sing}}^+, \mu_{\text{sing}}^- \in (\mathcal{M}_b(\Omega) \setminus \mathcal{M}_b^{\varphi(\cdot)}(\Omega)) \cup \{0\}$, then

$$\begin{aligned} & \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla u h'(u) \phi \, dx + \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \phi h(u) \, dx \\ &= \int_\Omega h(u) \phi \, d\mu_{\varphi(\cdot)}(x) + h(+\infty) \int_\Omega \phi \, d\mu_{\text{sing}}^+(x) - h(-\infty) \int_\Omega \phi \, d\mu_{\text{sing}}^-(x), \end{aligned} \quad (1.8)$$

holds for any $h \in W^{1,\infty}(\mathbb{R})$ having h' with compact support and for all $\phi \in C_0^\infty(\Omega)$, where $h(+\infty) := \lim_{r \rightarrow +\infty} h(r)$ and $h(-\infty) := \lim_{r \rightarrow +\infty} h(r)$ are well-defined as h is constant close to infinities.

Our main result reads as follows.

THEOREM 1.8. *Let $\varphi \in \Phi_c(\Omega)$ on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Suppose that φ satisfies $(aInc)_p$, $(aDec)_q$, $(A0)$ and $(A1)$, whereas a vector field $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies $(A1)$ – $(A4)$. When $\mu \in \mathcal{M}_b(\Omega)$, then the following claims hold true.*

- (i) *There exists an approximable solution to problem (1.1).*
- (ii) *There exists a renormalized solution to problem (1.1) satisfying (1.8) with measures such that $\text{supp } \mu_{\varphi(\cdot)} \subset \{|u| < \infty\}$, $\text{supp } \mu_{\text{sing}}^+ \subset \cap_{k>0} \{u > k\}$, and $\text{supp } \mu_{\text{sing}}^- \subset \cap_{k>0} \{u < -k\}$.*
- (iii) *A function $u \in T_0^{1,\varphi(\cdot)}(\Omega)$ is an approximable solution from (i) if and only if it is a renormalized solution from (ii).*

Uniqueness for approximable solution and renormalized solutions under additional assumption related to $\varphi(\cdot)$ -diffusivity of measure datum is provided in § 7.

As $h \equiv 1$ is an admissible choice in (1.8), we get the following remark.

REMARK 1.9. Under the assumptions of theorem 1.8 if u is an approximable (equivalently, renormalized) solution, then

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla \phi \, dx = \int_{\Omega} \phi \, d\mu \quad \text{for all } \phi \in C_0^\infty(\Omega),$$

so u is then a solution in the distributional sense (which in particular is proven to exist).

Moreover, for problems involving $\varphi(\cdot)$ -diffuse measures, by theorem 1.1 and proposition 6.1, we can formulate the following conclusion.

COROLLARY 1.10. *Under the assumptions of theorem 1.8 if u is an approximable (equivalently, renormalized) solution and $\mu \in (L^1(\Omega) + (W_0^{1,\varphi(\cdot)}(\Omega))') \cap \mathcal{M}_b(\Omega)$, then u exists, is unique, and satisfies*

$$\limsup_{k \rightarrow \infty} \int_{\{k < |u| < k+1\}} A(x, \nabla u) \cdot \nabla u \, dx = 0.$$

As a direct consequence of theorem 1.8 we retrieve the already classical existence results of [16, 38] involving p -Laplace operator, as well as variable exponent ones [69, 70]. We extend the existence results for problems in reflexive Orlicz spaces proven in [33] towards inhomogeneity of the spaces, as well as we extend the uniqueness result from L^1 to a class of diffuse measure data. It should be noted that renormalized solutions to general measure data problems with Orlicz growth were not studied so far. We also obtain the main goals of [47, 48] within a different and a bit more restrictive functional framework (and slightly different kind of control on the modular function), but allowing for essentially broader class of data and providing uniqueness. To our best knowledge no results on equivalence of very

weak solutions has been so far addressed in problems stated in generalized Orlicz spaces even in the L^1 -data case, for the p -Laplace case we refer to [38, 54]. Finding a setting where they essentially do not coincide would be interesting. Given an interest one may expect developing our main goals further towards anisotropic or non-reflexive settings cf. [5, 24, 48], as well as by involving lower-order terms in (1.1) as in [47], differential inclusions as in [41], or systems of equations.

There is some available information on the regularity of our very weak solutions following from comparison to solutions to problems with Orlicz growth. The conditions on $\varphi(\cdot)$ imply that there exists a Young function $B : [0, \infty) \rightarrow [0, \infty)$ such that $B(s) \leq \varphi(x, s)$ for a.a. $x \in \Omega$ and all $s \geq 0$. Then any of the very weak solutions of theorem 1.8 belongs to $\mathcal{T}_0^{1, \varphi(\cdot)}(\Omega) \subset \mathcal{T}_0^{1, B}(\Omega)$. Thus, we can get the same regularity of these solutions and their gradients expressed in Orlicz–Marcinkiewicz scale as in [33, theorem 3.2]; see [33, example 3.4] for applications with particular growth of B (including Zygmund-type ones). On the other hand, precise information on the local behaviour of solutions to problems with Orlicz growth obtained as a consequence of Wolff-potential estimates can be found in [25] depending on the scale of datum (in Orlicz versions of Lorentz, Marcinkiewicz and Morrey scales). Furthermore, [21] gives the Orlicz–Lorentz–Morrey-type regularity for gradients of solutions to problems involving related classes of measures, moreover, [22] describes the regularizing effect of the lower-order term (in the same scale). For Riesz potential estimates for such problems see [8]. For equations posed in the generalized Orlicz setting sharp conditions on a measure datum ensuring Hölder continuity of solutions has been recently provided via Wolff potential estimates [30].

The main ideas of the proofs follow several seminal papers including [12, 16, 17, 38] and involve analysis of fine convergence of solutions of some approximate problems. Nonetheless, the functional setting is far more demanding. In fact, we employ a lot of very recent results on structural properties of the generalized Orlicz spaces and nonstandard capacities, see e.g. [10, 39, 49, 52], and study properties of measures exposing certain capacity properties.

As for organization – after preliminary part, the measure characterization is proven in §3 and §4 is devoted to approximate problems. Approximable solutions are investigated in §5, while renormalized ones in §6. Uniqueness is proven in §7. The summary of the main proof is presented in §8.

2. Preliminaries

2.1. Notation

By Ω we always mean a bounded set in \mathbb{R}^n , $n \geq 2$. We shall make use of a Lipschitz continuous cut-off function $\psi_l : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\psi_l(r) := \min\{(l + 1 - |r|)^+, 1\}. \tag{2.1}$$

By $\mu_1 \ll \mu_2$ we denote we mean that μ_1 is absolutely continuous with respect to μ_2 .

We study spaces of functions defined in Ω , \mathbb{R} , or \mathbb{R}^n . $L^0(\Omega)$ denotes the set of measurable functions defined on Ω , $C_0(\Omega)$ are continuous functions taking value zero on $\partial\Omega$, while $C_b(\Omega)$ – continuous functions bounded on Ω ; $\mathcal{M}_b(\Omega)$ are Radon

measures with bounded total variation in Ω ; $\mathcal{M}_b^{\varphi(\cdot)}(\Omega)$ – bounded Radon measures diffuse with respect to $\varphi(\cdot)$ -capacity. If $\mu \in \mathcal{M}_b(\Omega)$, E is a Borel set included in Ω , the measure $\mu \llcorner E$ is defined by $(\mu \llcorner E)(B) = \mu(E \cap B)$ for any Borel set $B \subset \Omega$. If $\mu \in \mathcal{M}_b(\Omega)$ is such that $\mu = \mu \llcorner E$, then we say that μ is concentrated on E . In general, one cannot define the smallest set (in the sense of inclusion) where the measure is concentrated. By $L^1(\Omega, \mu)$ we denote classically functions with absolute value integrable with respect to μ , shortened to $L^1(\Omega)$ if μ is Lebesgue’s measure.

When $\mu_k, \mu \in \mathcal{M}_b(\Omega)$, we say that $\mu_k \rightarrow \mu$ weakly- $*$ in the space of measures if

$$\lim_{k \rightarrow \infty} \int_{\Omega} \phi \, d\mu_k = \int_{\Omega} \phi \, d\mu \text{ for every } \phi \in C_0(\Omega).$$

LEMMA 2.1. *If $g_n : \Omega \rightarrow \mathbb{R}$ are measurable functions converging to g almost everywhere, then for each regular value t of the limit function g we have $\mathbb{1}_{\{t < |g_n|\}} \xrightarrow{n \rightarrow \infty} \mathbb{1}_{\{t < |g|\}}$ a.e. in Ω .*

Here, the term ‘regular value’ denotes a value t such that $g^{-1}(t)$ has measure zero.

LEMMA 2.2. *Suppose $w_n \rightarrow w$ in $L^1(\Omega)$, $v_n, v \in L^\infty(\Omega)$, and $v_n \rightarrow v$ a.e. in Ω . Then $w_n v_n \rightarrow wv$ in $L^1(\Omega)$.*

2.2. Generalized Orlicz functions

The framework we employ comes from the monograph [49]. For the classical treatment of the setting we refer to [62, 63], while for recent developments within the related functional settings see [4, 10, 18, 26, 35, 52].

A real-valued function is L -almost increasing, $L \geq 1$, if $Lf(s) \geq f(t)$ for $s > t$. L -almost decreasing is defined analogously.

DEFINITION 2.3. *We say that $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty]$ is a convex Φ -function, and write $\varphi \in \Phi_c(\Omega)$, if the following conditions hold:*

- (i) *For every $s \in [0, \infty)$ the function $x \mapsto \varphi(x, s)$ is measurable and for a.e. $x \in \Omega$ the function $s \mapsto \varphi(x, s)$ is increasing, convex, and left-continuous.*
- (ii) *$\varphi(x, 0) = \lim_{s \rightarrow 0^+} \varphi(x, s) = 0$ and $\lim_{s \rightarrow \infty} \varphi(x, s) = \infty$ for a.e. $x \in \Omega$.*

Furthermore, we say that $\varphi \in \Phi_c(\Omega)$ satisfies

- (aInc) $_p$ if there exists $L_p \geq 1$ such that $s \mapsto \varphi(x, s)/s^p$ is L_p -almost increasing in $[0, \infty)$ for every $x \in \Omega$,
- (aDec) $_q$ if there exists $L_q \geq 1$ such that $s \mapsto \varphi(x, s)/s^q$ is L_q -almost decreasing in $[0, \infty)$ for every $x \in \Omega$.

We write (aInc), if there exist $p > 1$ such that (aInc) $_p$ holds and (aDec) if there exist $q > 1$ such that (aDec) $_q$ holds. The corresponding conditions with $L = 1$ are denoted by (Inc) or (Dec).

We shall consider those $\varphi \in \Phi_c(\Omega)$, which satisfy the following set of conditions.

- (A0) There exists $\beta_0 \in (0, 1]$ such that $\varphi(x, \beta_0) \leq 1$ and $\varphi(x, 1/\beta_0) \geq 1$ for all $x \in \Omega$.

(A1) There exists $\beta_1 \in (0, 1)$, such that for every ball B with $|B| \leq 1$ it holds that

$$\beta_1 \varphi^{-1}(x, s) \leq \varphi^{-1}(y, s) \quad \text{for every } s \in [1, 1/|B|] \text{ and a.e. } x, y \in B \cap \Omega.$$

Condition (A0) is imposed in order to exclude degeneracy, while (A1) can be interpreted as local continuity.

We say that a function φ satisfies Δ_2 -condition (and write $\varphi \in \Delta_2$) if there exists a constant $c > 0$, such that for every $s \geq 0$ it holds $\varphi(x, 2s) \leq c(\varphi(x, s) + 1)$. When a function $\varphi \in \Phi_c(\Omega)$ satisfies $(\text{aInc})_p$ and $(\text{aDec})_q$, then $\varphi \in \Delta_2$.

The Young conjugate of $\varphi \in \Phi_c(\Omega)$ is the function $\tilde{\varphi} : \Omega \times [0, \infty) \rightarrow [0, \infty]$ defined as

$$\tilde{\varphi}(x, s) = \sup\{r \cdot s - \varphi(x, r) : r \in [0, \infty)\}.$$

The fact that Young conjugation is involute, i.e. $(\tilde{\tilde{\varphi}}) = \varphi$ is attributed to Fenchel and Moreau, see direct proof in [26, theorem 2.1.41]. Moreover, if $\varphi \in \Phi_c(\Omega)$, then $\tilde{\varphi} \in \Phi_c(\Omega)$. If $\tilde{\varphi} \in \Delta_2$, we say that φ satisfies ∇_2 -condition and denote it by $\varphi \in \nabla_2$. If $\varphi, \tilde{\varphi} \in \Delta_2$, then we call φ a doubling function. If $\varphi \in \Phi_c(\Omega)$ satisfies $(\text{aInc})_p$ and $(\text{aDec})_q$, then $\varphi, \tilde{\varphi} \in \Delta_2$, so within our framework φ is assumed to be doubling.

For $\varphi \in \Phi_c(\Omega)$, the following inequality of Fenchel–Young type holds true

$$rs \leq \varphi(x, r) + \tilde{\varphi}(x, s). \tag{2.2}$$

In fact, within our framework with, since φ is comparable to a doubling function there exist some constants depending only on φ for which we have

$$\tilde{\varphi}(x, \varphi(x, s)/s) \sim \varphi(x, s) \quad \text{for a.e. } x \in \Omega \text{ and all } s > 0. \tag{2.3}$$

2.3. Function spaces

We always deal with spaces generated by $\varphi \in \Phi_c(\Omega)$ satisfying $(\text{aInc})_p$, $(\text{aDec})_q$, (A0) and (A1). For $f \in L^0(\Omega)$ we define the modular $\varrho_{\varphi(\cdot), \Omega}$ by

$$\varrho_{\varphi(\cdot), \Omega}(f) = \int_{\Omega} \varphi(x, |f(x)|) dx.$$

When it is clear from the context we skip writing the domain.

Musielak–Orlicz space is defined as the set

$$L^{\varphi(\cdot)}(\Omega) = \left\{ f \in L^0(\Omega) : \lim_{\lambda \rightarrow 0^+} \varrho_{\varphi}(\lambda f) = 0 \right\}$$

endowed with the Luxemburg norm $\|f\|_{\varphi(\cdot)} = \inf\{\lambda > 0 : \varrho_{\varphi(\cdot)}(\frac{1}{\lambda}f) \leq 1\}$. For $\varphi \in \Phi_c(\Omega)$, the space $L^{\varphi(\cdot)}(\Omega)$ is a Banach space [49, theorem 2.3.13]. Moreover, the following Hölder inequality holds true

$$\|fg\|_{L^1(\Omega)} \leq 2\|f\|_{L^{\varphi(\cdot)}(\Omega)} \|g\|_{L^{\tilde{\varphi}(\cdot)}(\Omega)}. \tag{2.4}$$

If $a := \max\{L_p, L_q\}$ for constants from $(\text{aInc})_p$ and $(\text{aDec})_q$, then

$$\|w\|_{L^{\varphi(\cdot)}(\Omega)} \leq \max \left\{ (a\rho_{\varphi(\cdot); \Omega}(w))^{1/p}, (a\rho_{\tilde{\varphi}(\cdot); \Omega}(w))^{1/q} \right\}. \tag{2.5}$$

Sometimes it would be convenient for us to denote vector-valued functions integrable with the modular as $(L^{\varphi(\cdot)}(\Omega))^n$. Since there is no difference between

claiming $H = (H_1, \dots, H_n) \in (L^{\varphi(\cdot)}(\Omega))^n$ and $|H| \in L^{\varphi(\cdot)}(\Omega)$, we are not very careful with stressing it in the sequel. A function $f \in L^{\varphi(\cdot)}(\Omega)$ belongs to Musielak–Orlicz–Sobolev space $W^{1,\varphi(\cdot)}(\Omega)$, if its distributional partial derivatives $\partial_1 f, \dots, \partial_n f$ exist and belong to $L^{\varphi(\cdot)}(\Omega)$ too. Because of the growth conditions $W^{1,\varphi(\cdot)}(\Omega)$ is a separable and reflexive space. Moreover, smooth functions are dense there. As a zero-trace space $W_0^{1,\varphi(\cdot)}(\Omega)$ we mean the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi(\cdot)}(\Omega)$. In fact, due to [49, theorem 6.2.8] given a bounded domain Ω there exists a constant $c = c(n, \Omega) > 0$, such that for any $u \in W_0^{1,\varphi(\cdot)}(\Omega)$ it holds that

$$\|u\|_{L^{\varphi(\cdot)}(\Omega)} \leq c \|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}. \tag{2.6}$$

Moreover, [49, theorem 6.3.7] yields that

$$W_0^{1,\varphi(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\varphi(\cdot)}(\Omega), \tag{2.7}$$

where ‘ $\hookrightarrow\hookrightarrow$ ’ stands for a compact embedding.

REMARK 2.4 [49]. If $\varphi \in \Phi_c(\Omega)$ with Ω bounded, satisfies $(aInc)_p$, $(aDec)_q$, (A0) and (A1), then strong (norm) topology of $W^{1,\varphi(\cdot)}(\Omega)$ coincides with the modular topology. Moreover, smooth functions are dense in this space in both topologies. Thus, $W_0^{1,\varphi(\cdot)}(\Omega)$, under our assumptions, is a closure of $C_0^\infty(\Omega)$ with respect to the modular topology of gradients in $L^{\varphi(\cdot)}(\Omega)$.

Space $(W_0^{1,\varphi(\cdot)}(\Omega))'$ is considered endowed with the norm

$$\|H\|_{(W_0^{1,\varphi(\cdot)}(\Omega))'} = \sup \left\{ \frac{\langle H, v \rangle}{\|v\|_{W^{1,\varphi(\cdot)}(\Omega)}} : v \in W_0^{1,\varphi(\cdot)}(\Omega) \right\}.$$

2.4. The operator

Let us motivate that the growth and coercivity conditions from (A1)–(A4) imply the expected proper definition of the operator involved in problem (1.1). We consider the operator $\mathfrak{A}_{\varphi(\cdot)} : W_0^{1,\varphi(\cdot)}(\Omega) \rightarrow (W_0^{1,\varphi(\cdot)}(\Omega))'$ defined as

$$\mathfrak{A}_{\varphi(\cdot)}(v) = -\operatorname{div} \mathcal{A}(x, \nabla v),$$

that is acting

$$\langle \mathfrak{A}_{\varphi(\cdot)}(v), w \rangle := \int_{\Omega} \mathcal{A}(x, \nabla v) \cdot \nabla w \, dx \quad \text{for } w \in C_0^\infty(\Omega), \tag{2.8}$$

where $\langle \cdot, \cdot \rangle$ denotes dual pairing between reflexive Banach spaces $W^{1,\varphi(\cdot)}(\Omega)$ and $(W^{1,\varphi(\cdot)}(\Omega))'$ is well-defined. Note that when $v \in W^{1,\varphi(\cdot)}(\Omega)$ and $w \in C_0^\infty(\Omega)$,

growth condition (A2), Hölder’s inequality (2.4), equivalence (2.3) justify that

$$|\langle \mathfrak{A}_{\varphi(\cdot)}(v), w \rangle| \leq c \int_{\Omega} \frac{\varphi(x, |\nabla v|)}{|\nabla v|} |\nabla w| \, dx \leq c \left\| \frac{\varphi(\cdot, |\nabla v|)}{|\nabla v|} \right\|_{L^{\tilde{\varphi}(\cdot)}(\Omega)} \|\nabla w\|_{L^{\varphi(\cdot)}(\Omega)}.$$

Note that due to (2.3) the norm $\|\varphi(\cdot, |\nabla v|)/|\nabla v|\|_{L^{\tilde{\varphi}(\cdot)}(\Omega)} < c$ with c depending on $\|\nabla v\|_{L^{\varphi(\cdot)}}, p, q$ only. Therefore

$$|\langle \mathfrak{A}_{\varphi(\cdot)}(v), w \rangle| \leq c \|\nabla w\|_{L^{\varphi(\cdot)}(\Omega)} \leq c \|w\|_{W^{1, \varphi(\cdot)}(\Omega)}.$$

By density argument, the operator is well-defined on $W_0^{1, \varphi(\cdot)}(\Omega)$.

What is more, by (A1)–(A2) and [32, lemma 4.12] we have the following.

REMARK 2.5. For $u \in \mathcal{T}^{1, \varphi(\cdot)}(\Omega)$, such that for some $M, k_0 > 0$ it holds that $\varrho_{\varphi(\cdot), \Omega}(\nabla T_k u) \leq Mk$ for all $k > k_0$, there exists a continuous function $\zeta : [0, |\Omega|] \rightarrow [0, \infty)$, such that $\lim_{s \rightarrow 0^+} \zeta(s) = 0$ and for any measurable $E \subset \Omega$

$$\int_E |\mathcal{A}(x, \nabla u)| \, dx \leq \zeta(|E|),$$

where ‘ ∇ ’ is understood as in (1.4). In particular, $\mathcal{A}(\cdot, \nabla u) \in (L^1(\Omega))^n$.

2.5. Capacities

Understanding capacities is needed to describe pointwise behaviour of Sobolev functions. We employ the generalization of classical notions of capacities, cf. [2, 53, 64], as well as unconventional ones [42, 58] to the Musielak–Orlicz–Sobolev setting according to [10, 52].

For a set $E \subset \mathbb{R}^n$ we define

$$S_{1, \varphi(\cdot)}(E) := \{0 \leq v \in W^{1, \varphi(\cdot)}(\mathbb{R}^n) : v \geq 1 \text{ in an open set containing } E\}$$

and its generalized Orlicz capacity of Sobolev type (called later $W^{1, \varphi(\cdot)}$ -capacity) by

$$C_{\varphi(\cdot)}(E) = \inf_{v \in S_{1, \varphi(\cdot)}(E)} \left\{ \int_{\mathbb{R}^n} \varphi(x, v) + \varphi(x, |\nabla v|) \, dx \right\}.$$

We shall consider *generalized relative $\varphi(\cdot)$ -capacity* $\text{cap}_{\varphi(\cdot)}$. With this aim for every K compact in $\Omega \subset \mathbb{R}^n$ let us denote

$$\mathcal{R}_{\varphi(\cdot)}(K, \Omega) := \{v \in W^{1, \varphi(\cdot)}(\Omega) \cap C_0(\Omega) : v \geq 1 \text{ on } K \text{ and } v \geq 0\} \tag{2.9}$$

and set

$$\text{cap}_{\varphi(\cdot)}(K, \Omega) := \inf \{ \varrho_{\varphi(\cdot), K}(|\nabla v|) : v \in \mathcal{R}_{\varphi(\cdot)}(K, \Omega) \}.$$

For open sets $A \subset \Omega$ we define

$$\text{cap}_{\varphi(\cdot)}(A, \Omega) = \sup \left\{ \text{cap}_{\varphi(\cdot)}(K, \Omega) : K \subset A \text{ and } K \text{ is compact in } A \right\}$$

and finally, if $E \subset \Omega$ is an arbitrary set

$$\text{cap}_{\varphi(\cdot)}(E, \Omega) = \inf \left\{ \text{cap}_{\varphi(\cdot)}(A, \Omega) : E \subset A \text{ and } A \text{ is open in } \Omega \right\}.$$

This notion of capacity enjoys all fundamental properties of classical capacities [10, 52].

Let us pay some attention to sets of zero capacity. If B_R is a ball in \mathbb{R}^n , $E \subset B_R$ and $\text{cap}_{\varphi(\cdot)}(E, B_R) = 0$, then $|E| = 0$. Having bounded $\Omega \subset \mathbb{R}^n$ for a set $E \subset \Omega$ we have $\text{cap}_{\varphi(\cdot)}(E, \Omega) = 0$ if and only if $C_{\varphi(\cdot)}(E) = 0$. What is more, each set of $W^{1,\varphi(\cdot)}$ -capacity zero is contained in a Borel set of $W^{1,\varphi(\cdot)}$ -capacity zero. Countable union of sets of $W^{1,\varphi(\cdot)}$ -capacity zero has $W^{1,\varphi(\cdot)}$ -capacity zero.

Function u is called $C_{\varphi(\cdot)}$ -quasi-continuous if for every $\varepsilon > 0$ there exists an open set U with $C_{\varphi(\cdot)}(U) < \varepsilon$, such that f restricted to $\Omega \setminus U$ is continuous. We say that a claim holds $\varphi(\cdot)$ -quasi-everywhere if it holds outside a set of Sobolev $\varphi(\cdot)$ -capacity zero. A set $E \subset \Omega$ is said to be $C_{\varphi(\cdot)}$ -quasi-open if for every $\varepsilon > 0$ there exists an open set U such that $E \subset U \subset \Omega$ and $C_{\varphi(\cdot)}(U \setminus E) \leq \varepsilon$.

LEMMA 2.6. *For every Cauchy sequence in $W^{1,\varphi(\cdot)}(\Omega)$ (equivalently under our regime, with respect to the $W^{1,\varphi(\cdot)}(\Omega)$ -modular topology) of functions from $C(\mathbb{R}^n) \cap W^{1,\varphi(\cdot)}(\Omega)$ there is a subsequence which converges pointwise $C_{\varphi(\cdot)}$ -quasi-everywhere in Ω . Moreover, the convergence is uniform outside a set of arbitrary small capacity $C_{\varphi(\cdot)}$.*

In the sequel, we shall always identify u with its $C_{\varphi(\cdot)}$ -quasi-continuous representative.

LEMMA 2.7. *For each $u \in \mathcal{T}_0^{1,\varphi(\cdot)}(\Omega)$ there exists a unique $C_{\varphi(\cdot)}$ -quasi-continuous function $v \in \mathcal{T}_0^{1,\varphi(\cdot)}(\Omega)$ such that $u = v$ holds $C_{\varphi(\cdot)}$ -quasi-everywhere in Ω .*

As a direct consequence of lemma 2.7, we have the following observations.

LEMMA 2.8. *For a $C_{\varphi(\cdot)}$ -quasi-continuous function u and $k > 0$, the sets $\{|u| > k\}$ and $\{|u| < k\}$ are $C_{\varphi(\cdot)}$ -quasi-open.*

LEMMA 2.9. *For every $C_{\varphi(\cdot)}$ -quasi-open set $U \subset \Omega$ there exists an increasing sequence $\{v_n\}$ of nonnegative functions in $W_0^{1,\varphi(\cdot)}(\Omega)$ which converges to 1_U $C_{\varphi(\cdot)}$ -quasi-everywhere in Ω .*

LEMMA 2.10. *If $\mu_{\varphi(\cdot)} \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$ and $u \in W_0^{1,\varphi(\cdot)}(\Omega)$, then $C_{\varphi(\cdot)}$ -quasi-continuous representative \hat{u} of u is measurable with respect to $\mu_{\varphi(\cdot)}$. If additionally $u \in L^\infty$, then $\hat{u} \in L^\infty(\Omega, \mu_{\varphi(\cdot)}) \subset L^1(\Omega, \mu_{\varphi(\cdot)})$.*

3. Measure characterization

In order to prove theorem 1.1 let us concentrate on the continuity of $\mu \in (L^1(\Omega) + (W_0^{1,\varphi(\cdot)}(\Omega))') \cap \mathcal{M}_b(\Omega)$ with respect to the generalized capacity. Note that for a nonnegative measure having decomposition $\mu = f - \text{div } G \in (L^1(\Omega) +$

$(W_0^{1,\varphi(\cdot)}(\Omega))' \cap \mathcal{M}_b(\Omega)$ with $f \in L^1(\Omega)$ and $G \in (L^{\tilde{\varphi}(\cdot)}(\Omega))^n$ and for an arbitrary set $E \subset \Omega$, we have

$$\begin{aligned} \mu(E) &\leq \int_E f \phi \, dx + \int_E G \cdot \nabla \phi \, dx \\ &\leq \|f\|_{L^1(E)} \|\phi\|_{L^\infty(E)} + \|G\|_{(L^{\tilde{\varphi}(\cdot)}(\Omega))^n} \|\nabla \phi\|_{(L^{\varphi(\cdot)}(\Omega))^n} \end{aligned}$$

for every $\phi \in W^{1,\varphi(\cdot)}(\Omega)$. In general, it is possible that a set has zero measure, but positive capacity. This is excluded if the measure enjoys the above decomposition.

LEMMA 3.1. *If $\mu \in (L^1(\Omega) + (W_0^{1,\varphi(\cdot)}(\Omega))') \cap \mathcal{M}_b(\Omega)$ and a set $E \subset \Omega$ is such that $\text{cap}_{\varphi(\cdot)}(E, \Omega) = 0$, then $\mu(E) = 0$.*

Proof. By the assumption μ can be represented with the use of $f \in L^1(\Omega)$ and $G \in (L^{\tilde{\varphi}(\cdot)}(\Omega))^n$, such that $\mu = f - \text{div } G$ in the sense of distributions. Let us fix arbitrary (small) $\varepsilon > 0$. We consider a sequence of truncations $\{T_\ell f\}_\ell$. Note that $\{T_\ell f\}_\ell$ converges to f strongly in $L^1(\Omega)$, so we can choose k large enough for $\|T_k f - f\|_{L^1(\Omega)} < \varepsilon/2$.

Note that for any $\tilde{\varepsilon} > 0$ there exists an open set $A \supset E$ with $\text{cap}_{\varphi(\cdot)}(A, \Omega) < \tilde{\varepsilon}$. Parameter $\tilde{\varepsilon}$ will be chosen in a few lines. Let us fix a compact set $K \subset A$. By definition of $\text{cap}_{\varphi(\cdot)}$ there exists a sequence $\{\phi_j\}_j \subset C_0^\infty(A)$ of functions such that

$$K \subset \{\phi_j = 1\}, \quad 0 \leq \phi_j \leq 1 \text{ and for } j \geq j_0 \quad \|\nabla \phi_j\|_{(L^{\varphi(\cdot)})^n(A)} \leq 2\tilde{\varepsilon}.$$

Indeed to restrict to ϕ_j such that $0 \leq \phi_j \leq 1$ let us point out that the map $t \mapsto \min\{t, 1\}$ is Lipschitz, so $\tilde{f} := \min\{f, 1\} \in \mathcal{R}_{\varphi(\cdot)}(K, \Omega)$ and $\varrho_{\varphi(\cdot), K}(|\nabla \tilde{f}|) \leq \varrho_{\varphi(\cdot), K}(|\nabla f|)$. On the other hand, we substituted the modular used in definition of $\text{cap}_{\varphi(\cdot)}$ with the norm. This is justified by the fact that the sequence realizing the infimum converges also in norm topology, which follows from the doubling growth of φ via (2.5).

We note that

$$|\mu|(K) \leq \left| \int_A \phi_j \, d\mu \right| = \left| \int_A f \phi_j \, dx + \int_A G \cdot \nabla \phi_j \, dx \right|.$$

Due to Hölder inequality (2.4) and then the Poincaré inequality (2.6), we have

$$\begin{aligned} |\mu|(K) &\leq \int_A |T_k f - f| |\phi_j| \, dx + \int_A |T_k f| |\phi_j| \, dx + \int_A |G \cdot \nabla \phi_j| \, dx \\ &\leq \|T_k f - f\|_{L^1(A)} \|\phi_j\|_{L^\infty(A)} + 2\|T_k f\|_{L^{\tilde{\varphi}(\cdot)}(A)} \|\phi_j\|_{L^{\varphi(\cdot)}(A)} \\ &\quad + 2\|G\|_{(L^{\tilde{\varphi}(\cdot)}(A))^n} \|\nabla \phi_j\|_{(L^{\varphi(\cdot)}(A))^n} \\ &\leq \|T_k f - f\|_{L^1(A)} + 2(\|T_k f\|_{L^{\tilde{\varphi}(\cdot)}(A)} + \|G\|_{(L^{\tilde{\varphi}(\cdot)}(\Omega))^n}) \|\nabla \phi_j\|_{(L^{\varphi(\cdot)}(A))^n}. \end{aligned}$$

We pick $\tilde{\varepsilon} = \varepsilon / (8(2k + \|G\|_{(L^{\tilde{\varphi}(\cdot)}(\Omega))^n}))$. Then $|\mu|(K) < \varepsilon$ with arbitrary $\varepsilon > 0$, so $|\mu|(K) = 0$. Therefore

$$|\mu|(E) \leq |\mu|(A) = \sup\{|\mu|(K) : K \subset A, K \text{ compact}\} = 0,$$

which ends the proof. □

We are in position to prove theorem 1.1. We take basic ideas from [16] with classical growth. Similar reasoning in variable exponent setting is given in [69].

Proof of theorem 1.1. Lemma 3.1 provides the implication: if μ belongs to $(L^1(\Omega) + (W_0^{1,\varphi(\cdot)}(\Omega))' \cap \mathcal{M}_b(\Omega))$, then $\mu \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$. Therefore, we concentrate now on the essentially more demanding converse, that is, if $\mu_{\varphi(\cdot)} \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$, then $\mu_{\varphi(\cdot)} \in L^1(\Omega) + (W_0^{1,\varphi(\cdot)}(\Omega))'$.

Step 1. Initial decomposition. We will show that for a nonnegative $\mu_{\varphi(\cdot)} \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$ there exists a nonnegative measure $\gamma^{\text{meas}} \in (W_0^{1,\varphi(\cdot)}(\Omega))'$ and nonnegative Borel measurable function $h \in L^1(\Omega, \gamma^{\text{meas}})$ such that $d\mu_{\varphi(\cdot)} = h d\gamma^{\text{meas}}$.

For any $\tilde{u} \in W^{1,\varphi(\cdot)}(\Omega)$ we can find its $C_{\varphi(\cdot)}$ -quasi-continuous representative denoted by u (lemma 2.7). We define a functional $\mathcal{F} : W_0^{1,\varphi(\cdot)}(\Omega) \rightarrow [0, \infty]$ by

$$\mathcal{F}[u] = \int_{\Omega} u_+ d\mu_{\varphi(\cdot)}$$

and observe that it is convex and lower semicontinuous on a separable space $W_0^{1,\varphi(\cdot)}(\Omega)$. Thus, \mathcal{F} can be expressed as a supremum of a countable family of continuous affine functions [26, lemma 2.1.11]. By its very definition $(W_0^{1,\varphi(\cdot)}(\Omega))'$ consists of all linear functionals on $W_0^{1,\varphi(\cdot)}(\Omega)$. Therefore, there exist sequences of functions $\{\xi_n\}_n \subset (W_0^{1,\varphi(\cdot)}(\Omega))'$ and numbers $\{a_n\}_n \subset \mathbb{R}^n$ such that

$$\mathcal{F}[u] = \sup_{n \in \mathbb{N}} \{ \langle \xi_n, u \rangle - a_n \} \quad \text{for all } u \in W_0^{1,\varphi(\cdot)}(\Omega).$$

Then, for any $s > 0$, $s\mathcal{F}[u] = \mathcal{F}[su] \geq s\langle \xi_n, u \rangle - a_n$ for every n . When we divide it by s and let $s \rightarrow \infty$ we obtain that $\mathcal{F}[u] \geq \langle \xi_n, u \rangle$ for all $u \in W_0^{1,\varphi(\cdot)}(\Omega)$. As $\mathcal{F}[0] = 0$ we infer that $a_n \geq 0$. Therefore, $\mathcal{F}[u] \geq \sup_{n \in \mathbb{N}} \langle \xi_n, u \rangle \geq \sup_{n \in \mathbb{N}} \{ \langle \xi_n, u \rangle - a_n \} = \mathcal{F}[u]$ and, in turn,

$$\mathcal{F}[u] = \sup_{n \in \mathbb{N}} \langle \xi_n, u \rangle. \tag{3.1}$$

This means that for all $\phi \in C_0^\infty(\Omega)$ we have

$$\langle \xi_n, \phi \rangle \leq \sup_{n \in \mathbb{N}} \langle \xi_n, \phi \rangle = \mathcal{F}[\phi] = \int_{\Omega} \phi_+ d\mu_{\varphi(\cdot)} \leq \|\mu_{\varphi(\cdot)}\|_{\mathcal{M}_b(\Omega)} \|\phi\|_{L^\infty(\Omega)}.$$

By the same arguments for $-\varphi$ we get

$$|\langle \xi_n, \phi \rangle| \leq \|\mu_{\varphi(\cdot)}\|_{\mathcal{M}_b(\Omega)} \|\phi\|_{L^\infty(\Omega)}$$

implying that $\xi_n \in (W_0^{1,\varphi(\cdot)}(\Omega))' \cap \mathcal{M}_b(\Omega)$. By the Riesz representation theorem there exists nonnegative $\xi_n^{\text{meas}} \in \mathcal{M}_b(\Omega)$, such that

$$\langle \xi_n, \phi \rangle = \int_{\Omega} \phi d\xi_n^{\text{meas}} \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Note that

$$\xi_n^{\text{meas}} \leq \mu_{\varphi(\cdot)} \quad \text{and} \quad \|\xi_n^{\text{meas}}\|_{\mathcal{M}_b(\Omega)} \leq \|\mu_{\varphi(\cdot)}\|_{\mathcal{M}_b(\Omega)}. \tag{3.2}$$

Let us define

$$\eta = \sum_{n=1}^{\infty} \frac{\xi_n}{2^n (\|\xi_n\|_{(W_0^{1,\varphi(\cdot)}(\Omega))'} + 1)} \tag{3.3}$$

and observe that the series is absolutely convergent in $(W_0^{1,\varphi(\cdot)}(\Omega))'$. Therefore, for $\phi \in C_0^\infty(\Omega)$ we can write

$$\begin{aligned} |\langle \eta, \phi \rangle| &\leq \sum_{n=1}^{\infty} \frac{|\langle \xi_n, \phi \rangle|}{2^n (\|\xi_n\|_{(W_0^{1,\varphi(\cdot)}(\Omega))'} + 1)} \\ &\leq \sum_{n=1}^{\infty} \frac{\|\xi_n^{\text{meas}}\|_{\mathcal{M}_b(\Omega)}}{2^n} \|\phi\|_{L^\infty(\Omega)} \leq \|\mu_{\varphi(\cdot)}\|_{\mathcal{M}_b(\Omega)} \|\phi\|_{L^\infty(\Omega)} \end{aligned}$$

and so $\eta \in (W_0^{1,\varphi(\cdot)}(\Omega))' \cap \mathcal{M}_b(\Omega)$ too. We denote

$$\eta^{\text{meas}} = \sum_{n=1}^{\infty} \frac{\xi_n^{\text{meas}}}{2^n (\|\xi_n\|_{(W_0^{1,\varphi(\cdot)}(\Omega))'} + 1)}$$

and note that this is a series of positive elements that is absolutely convergent in $\mathcal{M}_b(\Omega)$. Moreover, $\xi_n^{\text{meas}} \ll \eta^{\text{meas}}$ and thus for every n there exists a nonnegative function $h_n \in L^1(\Omega, d\eta^{\text{meas}})$ such that $d\xi_n^{\text{meas}} = h_n d\eta^{\text{meas}}$. Having (3.1) we deduce that

$$\begin{aligned} \langle \mu_{\varphi(\cdot)}, \phi \rangle &= \int_{\Omega} \phi d\mu_{\varphi(\cdot)} = \sup_{n \in \mathbb{N}} \int_{\Omega} \phi d\xi_n^{\text{meas}} \\ &= \sup_{n \in \mathbb{N}} \int_{\Omega} h_n \phi d\eta^{\text{meas}} \quad \text{for any } \phi \in C_0^\infty(\Omega). \end{aligned} \tag{3.4}$$

On the other hand, (3.2) ensures that $h_n \eta^{\text{meas}} \leq \mu_{\varphi(\cdot)}$. This means that for any measurable set $E \subset \Omega$ and every n we have

$$\int_E h_n d\eta^{\text{meas}} \leq \mu_{\varphi(\cdot)}(E). \tag{3.5}$$

We denote $h_{\max}^k = \max\{h_1(x), \dots, h_k(x)\}$ and

$$E^{j,k} = \{x \in E : h_{\max}^j(x) > h_i(x) \text{ for every } i = 1, \dots, k-1\}. \tag{3.6}$$

Then $E^{j,k}$ for $i = 1, \dots, k$ are pairwise disjoint and $E = \cup_{j=1}^k E^{j,k}$, so

$$\int_E h_{\max}^k(x) d\eta^{\text{meas}} \leq \sum_{j=1}^k \int_{E^{j,k}} h_{\max}^k(x) d\eta^{\text{meas}} \leq \sum_{j=1}^k \mu_{\varphi(\cdot)}(E^{j,k}) = \mu_{\varphi(\cdot)}(E). \tag{3.7}$$

Let us pass with $k \rightarrow \infty$ and take $h(x) = \sup_{l \in \mathbb{N}} h_l(x)$. We infer that for any measurable set $E \subset \Omega$

$$\int_E h d\eta^{\text{meas}} \leq \mu_{\varphi(\cdot)}(E). \tag{3.8}$$

According to (3.4), for every nonnegative $\phi \in C_0^\infty$ we have

$$\int_\Omega \phi \, d\mu_{\varphi(\cdot)} = \sup_{l \in \mathbb{N}} \int_\Omega h_l \phi \, d\eta^{\text{meas}} \leq \int_\Omega h \phi \, d\eta^{\text{meas}} \leq \int_\Omega \phi \, d\mu_{\varphi(\cdot)}$$

that is $d\mu_{\varphi(\cdot)} = h \, d\eta^{\text{meas}}$. Since $\mu_{\varphi(\cdot)}(\Omega) \in \mathcal{M}_b(\Omega)$, we deduce that $h \in L^1(\Omega, d\eta^{\text{meas}})$ and the aim of this step is achieved with $\gamma^{\text{meas}} = \eta^{\text{meas}} \in (W_0^{1,\varphi(\cdot)}(\Omega))'$.

Step 2. *Auxiliary sequence of measures.* We take an increasing sequence of sets $\{K_i\}_i$ compact in Ω , such that $\cup_{i=1}^\infty K_i = \Omega$. We set

$$\tilde{\mu}_i = T_i(h\mathbb{1}_{K_i})\gamma^{\text{meas}} \quad \text{for every } i \in \mathbb{N}.$$

Then $\{\tilde{\mu}_i\}_i$ is an increasing sequence of positive measures in $(W_0^{1,\varphi(\cdot)}(\Omega))'$ with supports compact in Ω . Let us denote

$$\mu_0 = \tilde{\mu}_0 \quad \text{and} \quad \mu_i = \tilde{\mu}_i - \tilde{\mu}_{i-1} \quad \text{for every } i \in \mathbb{N}.$$

Then $\sum_{i=1}^k \mu_i = T_k(h\mathbb{1}_{K_k})\gamma^{\text{meas}} \in \mathcal{M}_b(\Omega)$. Since $\mu_i \geq 0$, we have $\sum_{i=1}^\infty \|\mu_i\|_{\mathcal{M}(\Omega)} < \infty$. Furthermore, $\mu_{\varphi(\cdot)} = \sum_{i=1}^\infty \mu_i$ and this series is absolutely convergent in $\mathcal{M}_b(\Omega)$.

Step 3. *Construction of decomposition.* Suppose $\varrho \in C_0^\infty(B(0, 1))$ is a standard mollifier (nonnegative and symmetric function with $\int_{\mathbb{R}^n} \varrho \, dx = 1$) and set $\varrho_k(x) = k^n \varrho(kx)$. We consider mollification

$$\mu_{i,k}^\varrho(x) = \int_{\mathbb{R}^n} \varrho_k(x - y) \, d\mu_i(y)$$

For k large enough, we can decompose $\mu_i = f_i + w_i$ with

$$f_i = \mu_{i,k_i}^\varrho \in C_0^\infty(\Omega) \quad \text{and} \quad w_i = \mu_i - \mu_{i,k_i}^\varrho \in (W_0^{1,\varphi(\cdot)}(\Omega))'$$

by choosing for every i sufficiently large k_0^i such that for $k_i > k_0^i$, μ_{i,k_i}^ϱ belongs to $C_0^\infty(\Omega)$, so we restrict our attention to such k_i . Therefore, we get – up to a subsequence – convergence of $\{f_i\}_i = \{\mu_{i,k_i}^\varrho\}_i$ to $\mu_{\varphi(\cdot)}$ in measure and for every i we have $\|f_i\|_{L^1(\Omega)} \leq \|\mu_{\varphi(\cdot)}\|_{\mathcal{M}_b(\Omega)}$. By step 2 the series $\sum_{i=1}^\infty f_i$ is convergent in $L^1(\Omega)$ and there exists its limit $f^0 = \sum_{i=1}^\infty f_i \in L^1(\Omega)$. As for convergence of w_i we observe first that due to [49, lemma 6.4.5] we get convergence of $\{\mu_{i,k}^\varrho\}_k$ to μ_i in $(W_0^{1,\varphi(\cdot)}(\Omega))'$ as $k \rightarrow \infty$. We note that the series $\sum_{i=1}^\infty w_i$ converges in $(W_0^{1,\varphi(\cdot)}(\Omega))'$ and there exists its limit $w^0 = \sum_{i=1}^\infty w_i \in (W_0^{1,\varphi(\cdot)}(\Omega))'$. Therefore, the three following series converge in the sense of distributions

$$\sum_{i=1}^\infty \mu_i = \mu_{\varphi(\cdot)}, \quad \sum_{i=1}^\infty f_i = f^0 \quad \text{and} \quad \sum_{i=1}^\infty w_i = w^0$$

and, consequently, $\mu_{\varphi(\cdot)} = f^0 + w^0$.

Step 4. *Summary.* Let us recall that the proof starts with justification that for a nonnegative measure $\mu \in L^1(\Omega) + (W_0^{1,\varphi(\cdot)}(\Omega))' \subset \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$. Step 3 provides the reverse implication.

If the measure was not nonnegative, we can write that $\mu_{\varphi(\cdot)} = ((\mu_{\varphi(\cdot)})_+ + (\mu_{\varphi(\cdot)})_-)$ and repeat the above reasoning separately for its positive and negative part. Note that by the monotonicity of capacity, if $\text{cap}_{\varphi(\cdot)}(A, \Omega) = 0$, then $(\mu_{\varphi(\cdot)})_+(A) = 0 = (\mu_{\varphi(\cdot)})_-(A)$ ($\varphi(\cdot)$ -capacity can be exhausted over Borel sets included in A), see [10]. Note that if $\mu_{\varphi(\cdot)}$ is nonnegative, then $f \geq 0$. Hence, for a signed measure $\mu \in \mathcal{M}_b(\Omega)$ we infer that $\mu \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$ if and only if $\mu \in (L^1(\Omega) + (W_0^{1,\varphi(\cdot)}(\Omega))') \cap \mathcal{M}_b(\Omega)$, that is when there exists $f \in L^1(\Omega)$ and $G \in (L^{\tilde{\varphi}(\cdot)}(\Omega))^n$, such that $\mu_{\varphi(\cdot)} = f - \text{div } G$ in the sense of distributions. Hence, the proof of the capacity characterization is completed. \square

To conclude remark 1.3 we need the following decomposition lemma. Its proof is essentially the one of [44, lemma 2.1], but we find it valuable to present it for the sake of completeness.

LEMMA 3.2. *Suppose Ω is a bounded set in \mathbb{R}^n and \mathfrak{M} is a family of its measurable subsets. Then for every $\mu \in \mathcal{M}_b(\Omega)$ there exist a decomposition $\mu = \mu_{ac} + \mu_{\text{sing}}$, such that*

- (a) $\mu_{ac}(D) = 0$ for all sets $D \subset \mathfrak{M}$ with $C_{\varphi(\cdot)}(D) = 0$,
- (b) $\mu_{\text{sing}} = \mu \mathbb{1}_N$ for some set $N \subset \mathfrak{M}$ with $C_{\varphi(\cdot)}(N) = 0$.

Moreover, such decomposition is unique up to sets of $W^{1,\varphi(\cdot)}$ -capacity zero.

Proof. We fix an arbitrary sequence $D_1 \subset D_2 \subset \dots \subset \mathfrak{M}$ of sets with $C_{\varphi(\cdot)}(D_i) = 0$, such that

$$\lim_{i \rightarrow \infty} \mu(D_i) = \alpha := \sup\{\mu(D) : D \in \mathfrak{M} \text{ and } C_{\varphi(\cdot)}(D) = 0\} < \infty.$$

Denote $D_\infty = \bigcup_{i=1}^\infty D_i$ and note that $D_\infty \in \mathfrak{M}$, $C_{\varphi(\cdot)}(D_\infty) = 0$ and $\mu(D_\infty) = \alpha$. Let us observe that $\mu(D \setminus D_\infty) = 0$ for every $D \in \mathfrak{M}$ with $C_{\varphi(\cdot)}(D) = 0$. By setting

$$\mu_{ac} = \mathbb{1}_{\mathbb{R}^n \setminus D_\infty} \mu \quad \text{and} \quad \mu_{\text{sing}} = \mathbb{1}_{D_\infty} \mu$$

we get the decomposition of the desired properties. Due to [10, proposition 7] countable sum of sets of $W^{1,\varphi(\cdot)}$ -capacity zero is of $W^{1,\varphi(\cdot)}$ -capacity zero, so the uniqueness of the decomposition $C_{\varphi(\cdot)}$ -quasi-everywhere is evident. \square

4. Approximate problems

This section is devoted to analysis of approximate problems with general datum.

In the case of a measure $\mu \in \mathcal{M}_b(\Omega)$ we shall consider an approximate sequence of bounded functions $\{\mu^s\} \subset C^\infty(\Omega)$ that converges to μ weakly-* in the space of measures and satisfies (1.7). We study solutions to

$$\begin{cases} -\text{div } \mathcal{A}(x, \nabla u_s) = \mu^s & \text{in } \Omega, \\ u_s = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

It is known that there exists at least one distributional solution $u_s \in W^{1,\varphi(\cdot)}(\Omega)$ to (4.1), see the proof by Galerkin approximation in [47, § 5.1.1] under more general

growth conditions embracing our case. In fact, since smooth functions are dense in the space where the solutions live i.e. $W_0^{1,\varphi(\cdot)}(\Omega)$, we can test the equation by the solution itself to get energy estimates and, consequently, the distributional solutions u_s are weak solutions.

REMARK 4.1. Note that requiring regularity of μ^s is not a restriction. In fact, the proof requires only $\mu^s \in (W_0^{1,\varphi(\cdot)}(\Omega))' \cap \mathcal{M}_b(\Omega)$.

PROPOSITION 4.2 (Basic a priori estimates). *Let Ω be bounded open domain in \mathbb{R}^n , $\mathcal{A} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy (A1)–(A4), $\varphi \in \Phi_c(\Omega)$ satisfy (aInc) $_p$, (aDec) $_q$, (A0), and (A1) and $\mu_{\varphi(\cdot)} \in \mathcal{M}_b(\Omega)$. Then, for a weak solutions u_s to (4.1) and $k > 1$, we have*

$$\int_{\Omega} \varphi(x, |\nabla T_k(u_s)|) \, dx \leq \bar{c}_1 k, \tag{4.2}$$

$$\int_{\Omega} \tilde{\varphi}(x, |\mathcal{A}(x, \nabla T_k(u_s))|) \, dx \leq \bar{c}_2 k, \tag{4.3}$$

with constants $\bar{c}_1 = 2\|\mu\|_{\mathcal{M}_b(\Omega)}/c_1^\varphi, \bar{c}_2 = c(c_1^\varphi, c_2^\varphi, c_{\Delta_2}(\varphi), q, \|\mu\|_{\mathcal{M}_b(\Omega)}, \|1 + \gamma\|_{L^{\tilde{\varphi}(\cdot)}(\Omega)}) > 0$. Furthermore, for some $\bar{c}_3 = \bar{c}_3(\Omega, n) > 0$

$$|\{|u_s| \geq k\}| \leq \bar{c}_3 k^{1-p}. \tag{4.4}$$

Since constants in the above estimates do not depend on s , we can infer what follows.

REMARK 4.3. Note that within our doubling regime this implies that for any fixed $k > 0$ the sequence $\{\nabla T_k(u_s)\}_s$ is uniformly bounded in $L^{\varphi(\cdot)}(\Omega)$, $\{|\mathcal{A}(x, \nabla T_k(u_s))|\}_s$ is uniformly bounded in $L^{\tilde{\varphi}(\cdot)}(\Omega)$ and the set $\{|u_s| \geq k\}$ for increasing k is shrinking uniformly in s .

Proof of proposition 4.2. To get (4.2), we use first (A2) $_1$, (4.1) tested by $T_k(u_s) \in W_0^{1,\varphi(\cdot)}(\Omega)$, and the above remark, in the following way

$$\begin{aligned} c_1^\varphi \int_{\Omega} \varphi(x, |\nabla T_k(u_s)|) \, dx &\leq \int_{\Omega} \mathcal{A}(x, \nabla T_k(u_s)) \cdot \nabla T_k(u_s) \, dx \\ &= \int_{\Omega} \mathcal{A}(x, \nabla u_s) \cdot \nabla T_k(u_s) \, dx \\ &= \int_{\Omega} T_k(u_s) \, d\mu^s \leq 2k \|\mu^s\|_{\mathcal{M}_b(\Omega)}. \end{aligned}$$

We conclude the last inequality above because of the assumed properties of μ^s .

In order to get (4.3), we use $(A2)_2$, doubling growth (2.3), and finally (4.2) to conclude that for any $k > 1$ we have

$$\begin{aligned} \int_{\Omega} \tilde{\varphi}(x, |\mathcal{A}(x, \nabla T_k(u_s))|) \, dx &\leq \int_{\Omega} \tilde{\varphi}\left(x, 1 + \gamma(x) + \varphi(x, |\nabla T_k(u_s)|)/|\nabla T_k(u_s)|\right) \, dx \\ &\leq \frac{1}{2} \left(\int_{\Omega} \tilde{\varphi}\left(x, 2(1 + \gamma(x))\right) \, dx + \int_{\Omega} \tilde{\varphi}\left(x, \varphi(x, 2|\nabla T_k(u_s)|)/|\nabla T_k(u_s)|\right) \, dx \right) \\ &\leq c \left(\int_{\Omega} \tilde{\varphi}(x, 1 + \gamma(x)) \, dx + \int_{\Omega} \varphi(x, |\nabla T_k(u_s)|) \, dx \right) \leq ck, \end{aligned} \tag{4.5}$$

where $c = c(c_1^\varphi, c_2^\varphi, c_{\Delta_2}(\varphi), q, \|\mu\|_{\mathcal{M}_b(\Omega)}, \|1 + \gamma\|_{L^{\tilde{\varphi}(\cdot)}(\Omega)})$.

To get (4.4) we start with observing that $|\{|u_s| \geq k\}| = |\{T_k(|u_s|) \geq k\}|$. Then by Tchebyshev inequality, Poincaré inequality and (4.2) as follows

$$\begin{aligned} |\{|u_s| \geq k\}| &\leq \int_{\Omega} \frac{|T_k(u_s)|^p}{k^p} \, dx \leq \frac{c}{k^p} \int_{\Omega} |\nabla T_k(u_s)|^p \, dx \\ &\leq \frac{c}{k^p} \int_{\Omega} \varphi(x, |\nabla T_k(u_s)|) \, dx \leq ck^{1-p} \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

□

5. Approximable solutions

Let us find the fundamental properties of limits of approximate problems.

PROPOSITION 5.1 (Existence of approximable solutions and convergences). *Let Ω be bounded open domain in \mathbb{R}^n , $\mathcal{A} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy (A1)–(A4), $\varphi \in \Phi_c(\Omega)$ satisfy $(aInc)_p$, $(aDec)_q$, (A0) and (A1), and $\mu \in \mathcal{M}_b(\Omega)$. Then there exists at least one approximable solution u (see definition 1.6). Namely, up to a subsequence $\{u_s\}_s$ consisting of solutions to (4.1), there exists a function $u \in \mathcal{T}_0^{1, \varphi(\cdot)}(\Omega)$, such that when $s \rightarrow 0$ and $k > 0$ is fixed we have*

$$u_s \rightarrow u \quad \text{a.e. in } \Omega, \tag{5.1}$$

$$T_k(u_s) \rightarrow T_k(u) \quad \text{strongly in } L^{\varphi(\cdot)}(\Omega), \tag{5.2}$$

$$\nabla T_k(u_s) \rightharpoonup \nabla T_k(u) \quad \text{weakly in } (L^{\varphi(\cdot)}(\Omega))^n, \tag{5.3}$$

$$\mathcal{A}(x, \nabla T_k(u_s)) \rightarrow \mathcal{A}(x, \nabla T_k(u)) \quad \text{a.e. in } \Omega, \tag{5.4}$$

$$\mathcal{A}(x, \nabla T_k(u_s)) \rightharpoonup \mathcal{A}(x, \nabla T_k(u)) \quad \text{weakly in } (L^{\tilde{\varphi}(\cdot)}(\Omega))^n \tag{5.5}$$

$$\mathcal{A}(x, \nabla T_k(u_s)) \rightarrow \mathcal{A}(x, \nabla T_k(u)) \quad \text{strongly in } (L^1(\Omega))^n. \tag{5.6}$$

Moreover, for $k \rightarrow \infty$

$$\mathcal{A}(x, \nabla T_k(u)) \rightarrow \mathcal{A}(x, \nabla u) \quad \text{strongly in } (L^1(\Omega))^n. \tag{5.7}$$

Proof. Having (4.2) we get that $\{T_k(u_s)\}_s$ is uniformly bounded in $W_0^{1, \varphi(\cdot)}(\Omega)$. By recalling the Banach–Alaoglu theorem in the reflexive space, we infer that there

exists a (non-relabelled) subsequence of $\{u_s\}$ and function $u \in \mathcal{T}_0^{1,\varphi(\cdot)}$ such that for $s \rightarrow 0$ we have (5.3). Note that in the general case we would have here weak- $*$ convergence, but our space is reflexive and these notions of convergences coincide. Since embedding (2.7) is compact, up to a non-relabelled subsequence, for $s \rightarrow 0$ also (5.2). Consequently, due to the Dunford–Pettis theorem, up to an (again) non-relabelled subsequence $\{T_k(u_s)\}_s$ is a Cauchy sequence in measure and (5.1) holds. By the same arguments, due to (4.3), there exists $\mathcal{A}_k^\infty \in (L^{\tilde{\varphi}(\cdot)}(\Omega))^n$ such that for $s \rightarrow 0$

$$\mathcal{A}(x, \nabla T_k(u_s)) \rightharpoonup \mathcal{A}_k^\infty \quad \text{weakly in } (L^{\tilde{\varphi}(\cdot)}(\Omega))^n \quad \text{for every } k > 0. \tag{5.8}$$

The effort will be put now in identification of the limit function

$$\mathcal{A}_k^\infty = \mathcal{A}(x, \nabla T_k(u)) \quad \text{a.e. in } \Omega, \quad \text{for every } k > 0 \tag{5.9}$$

and proving that u obtained in this procedure is a very weak solution. Recall that \mathcal{A} is continuous with respect to the last variable, we have the convergence (5.2) and what remains to prove is fine behaviour of $\{\nabla u_s\}_s$. In order to show that $\{\nabla u_s\}_s$ is a Cauchy sequence in measure we set $\epsilon > 0$ and $m, n \in \mathbb{N}$ arbitrary (large). Given any $t, \tau, r > 0$, one has that

$$\begin{aligned} |\{|\nabla u_l - \nabla u_m| > t\}| &\leq |\{|\nabla u_l| > \tau\}| + |\{|\nabla u_m| > \tau\}| + |\{|u_l| > \tau\}| \\ &\quad + |\{|u_m| > \tau\}| + |\{|u_l - u_m| > r\}| + E, \end{aligned} \tag{5.10}$$

where

$$E = |\{|u_l - u_m| \leq r, |u_l| \leq \tau, |u_m| \leq \tau, |\nabla u_l| \leq \tau, |\nabla u_m| \leq \tau, |\nabla u_l - \nabla u_m| > t\}|. \tag{5.11}$$

Note that (4.4) and arguments of the proof of [32, lemma 4.12] enable to justify uniform integrability of $\{g(|\nabla u_k|)\}_k$ for some continuous and increasing function g . Therefore, we can choose for any $\epsilon > 0$ a number τ_ϵ large enough so that for $\tau > \tau_\epsilon$ we obtain

$$|\{|\nabla u_l| > \tau\}| < \epsilon, \quad |\{|\nabla u_m| > \tau\}| < \epsilon, \quad |\{|u_l| > \tau\}| < \epsilon, \quad \text{and } |\{|u_m| > \tau\}| < \epsilon. \tag{5.12}$$

From now on we restrict ourselves to $\tau > \tau_\epsilon$. On the other hand, since $\{u_l\}$ is a Cauchy sequence in measure,

$$|\{|u_l - u_m| > r\}| < \epsilon, \text{ if } l, m, r \text{ are sufficiently large.} \tag{5.13}$$

What remains to prove is that there exists $\delta_{\tau,\epsilon} > 0$, such that for every $\delta < \delta_{\tau,\epsilon}$, we get

$$|E| < \epsilon. \tag{5.14}$$

Let us define a set

$$S = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi| \leq \tau, |\eta| \leq \tau, |\xi - \eta| \geq t\},$$

which is compact. Consider the function $\psi : \Omega \rightarrow [0, \infty)$ given by

$$\psi(x) = \inf_{(\xi, \eta) \in S} [(\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)) \cdot (\xi - \eta)].$$

Monotonicity assumption (A3) and the continuity of the function $\xi \mapsto \mathcal{A}(\cdot, \xi)$ a.e. in S ensure that $\psi \geq 0$ in Ω . Furthermore, (A4) implies that $|\{\mathcal{A}(x, 0) = 0\}| = 0$. Moreover,

$$\begin{aligned} \int_S \psi(x) \, dx &\leq \int_S (\mathcal{A}(x, \nabla u_l) - \mathcal{A}(x, \nabla u_m)) \cdot (\nabla u_l - \nabla u_m) \, dx & (5.15) \\ &\leq \int_{\{|u_l - u_m| \leq r\}} (\mathcal{A}(x, \nabla u_l) - \mathcal{A}(x, \nabla u_m)) \cdot (\nabla u_l - \nabla u_m) \, dx \\ &= \int_\Omega (\mathcal{A}(x, \nabla u_l) - \mathcal{A}(x, \nabla u_m)) \cdot (\nabla T_r(u_l - u_m)) \, dx \\ &= \int_\Omega T_r(u_l - u_m) \, d\mu^s(x) \leq 2r \|\mu\|_{\mathcal{M}_b(\Omega)}, \end{aligned}$$

where the last but one equality follows on making use of the test function $T_r(u_l - u_m)$ and in the corresponding equation with l replaced by m , and subtracting the resultant equations. Estimate (5.15) and the properties of the function ψ ensure that, if s is chosen sufficiently small, then (5.14) holds. From inequalities (5.10), (5.12), (5.14) and (5.13), we infer that $\{\nabla u_s\}_s$ is a Cauchy sequence in measure.

To conclude that the function u obtained in (5.3) and (5.2) is a desired approximable solution, we observe that it belongs to the class $\mathcal{T}_0^{1, \varphi(\cdot)}(\Omega)$, and that $\nabla u_s \rightarrow \nabla u$ a.e. in Ω (up to subsequences), where ∇u is understood in the sense of (1.4). Since $\{\nabla u_s\}$ is a Cauchy sequence in measure, there exist a subsequence (still indexed by s) and a measurable function $W : \Omega \rightarrow \mathbb{R}^n$ such that $\nabla u_s \rightarrow W$ a.e. in Ω . To motivate that $\nabla u = W$ and

$$\chi_{\{|u| < k\}} W \in (L^{\varphi(\cdot)}(\Omega))^n \quad \text{for every } k > 0 \tag{5.16}$$

it suffices to recall (5.3). Indeed, then for each fixed $k > 0$, there exists a subsequence of $\{u_s\}$, still indexed by s , such that

$$\lim_{s \rightarrow \infty} \nabla T_k(u_s) = \lim_{s \rightarrow \infty} \chi_{\{|u_s| < k\}} \nabla u_s = \chi_{\{|u| < k\}} W \quad \text{a.e. in } \Omega, \tag{5.17}$$

and $\lim_{s \rightarrow \infty} \nabla T_k(u_s) = \nabla T_k(u)$ weakly in $(L^{\varphi(\cdot)}(\Omega))^n$. Therefore, $\nabla T_k(u) = \chi_{\{|u| < k\}} W$ a.e. in Ω , whence (5.16) follows. Then, due to (A1) also (5.9) holds, that is we have (5.4) and (5.5). Due to remark 2.5 we get uniform integrability of $\{\mathcal{A}(x, \nabla T_k u)\}_k$, so Lebesgue’s monotone convergence theorem justifies (5.7), where the limit is in $(L^1(\Omega))^n$ by lemma 2.10. By (4.3) and Vitali’s convergence theorem we infer (5.6). □

6. Renormalized solutions

Our aim now is to analyse the measures generated by truncations of approximable solutions.

PROPOSITION 6.1. *If u is an approximable solution under assumptions of proposition 5.1 and $\mathfrak{A}_{\varphi(\cdot)}$ is given by (2.8), then for every $k > 0$ we have $\lambda_k := \mathfrak{A}_{\varphi(\cdot)}(T_k u) \in \mathcal{M}_b(\Omega) \cap (W_0^{1,\varphi(\cdot)}(\Omega))'$ and*

$$\int_{\{|u|<k\}} \mathcal{A}(x, \nabla u) \cdot \nabla \phi \, dx = \int_{\Omega} \phi \, d\lambda_k \quad \text{for every } \phi \in W_0^{1,\varphi(\cdot)}(\Omega) \cap L^\infty(\Omega). \tag{6.1}$$

Then for $k \rightarrow \infty$ we have

$$\mathfrak{A}_{\varphi(\cdot)}(T_k u) \rightharpoonup \mathfrak{A}_{\varphi(\cdot)}(u) \quad \text{weakly-}^* \text{ in the space of measures.} \tag{6.2}$$

Moreover, for every $k > 0$ it holds $|\mathfrak{A}_{\varphi(\cdot)}(T_k u)|(\{|u| > k\}) = 0$ and for every $\phi \in C_0(\Omega)$ we have

$$\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_{\{k-\delta \leq u \leq k\}} \mathcal{A}(x, \nabla u) \cdot \nabla u \, \phi \, dx = \int \phi \, d\nu_k^+, \tag{6.3}$$

$$\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_{\{-k \leq u \leq -k+\delta\}} \mathcal{A}(x, \nabla u) \cdot \nabla u \, \phi \, dx = \int \phi \, d\nu_k^- \tag{6.4}$$

with $\nu_k^+ = \mathfrak{A}_{\varphi(\cdot)}(T_k u) \llcorner \{u = k\}$ and $\nu_k^- = \mathfrak{A}_{\varphi(\cdot)}(T_k u) \llcorner \{u = -k\}$.

Proof. We prove first weak- \ast convergence of measures generated by truncations of solutions and then their further properties.

Step 1. $\lambda_k \in \mathcal{M}_b(\Omega)$ and $\mathfrak{A}_{\varphi(\cdot)}(T_k u) \rightharpoonup \mathfrak{A}_{\varphi(\cdot)}(u)$ weakly- \ast in the space of measures.

For $k > \delta > 0$ we define a Lipschitz functions $h_\delta, \sigma_\delta^+, \sigma_\delta^- : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} h_\delta(r) = 1 & \text{if } |r| \leq k - \delta, \\ |h'_\delta(r)| = \frac{1}{\delta} & \text{if } k - \delta \leq |r| \leq k, \\ h_\delta(r) = 0 & \text{if } |r| \geq k. \end{cases} \quad \begin{cases} \sigma_\delta^+(r) = 0 & \text{if } r \leq k - \delta, \\ (\sigma_\delta^+)'(r) = \frac{1}{\delta} & \text{if } k - \delta \leq r \leq k, \\ \sigma_\delta^+(r) = 1 & \text{if } r \geq k, \end{cases}$$

and $\sigma_\delta^-(r) = \sigma_\delta^+(-r)$. We note that if $\{u_s\}$ is an approximate sequence from definition 1.6 solving (4.1) with μ^s being bounded and smooth function, $\phi \in C_0^\infty(\Omega)$, then $h_\delta(u_s)\phi, \sigma_\delta^+(u_s), \sigma_\delta^-(u_s) \in W_0^{1,\varphi(\cdot)}(\Omega) \cap L^\infty(\Omega)$ are admissible test functions in (1.6). By testing (4.1) by $h_\delta(u_s)\phi$ we get

$$\begin{aligned} \int_{\Omega} h_\delta(u_s) \mathcal{A}(x, \nabla u_s) \cdot \nabla \phi \, dx &= \int_{\Omega} \phi \mu^s h_\delta(u_s) \, dx - \int_{\Omega} h'_\delta(u_s) \mathcal{A}(x, \nabla u_s) \cdot \nabla u_s \, \phi \, dx \\ &= \int_{\Omega} \phi \, d\lambda_\delta^s + \int_{\Omega} \phi \, d\gamma_\delta^{s,+} - \int_{\Omega} \phi \, d\gamma_\delta^{s,-}, \end{aligned}$$

where

$$\begin{aligned} \lambda_\delta^s &= \mu^s h_\delta(u_s), \\ \nu_\delta^{s,+} &= \frac{1}{\delta} \mathbb{1}_{\{k-\delta \leq u_s \leq k\}} \mathcal{A}(x, \nabla u_s) \cdot \nabla u_s, \end{aligned} \tag{6.5}$$

$$\nu_\delta^{s,-} = \frac{1}{\delta} \mathbb{1}_{\{-k \leq u_s \leq -k+\delta\}} \mathcal{A}(x, \nabla u_s) \cdot \nabla u_s. \tag{6.6}$$

Observe that

$$\lambda_\delta^s, \nu_\delta^{s,+}, \nu_\delta^{s,-} \in L^1(\Omega).$$

Indeed,

$$\|\lambda_\delta^s\|_{L^1(\Omega)} \leq \int_\Omega |\mu^s| |h_\delta(u_s)| \, dx \leq \int_\Omega |\mu^s| \, dx \leq 2\|\mu\|_{\mathcal{M}_b(\Omega)}.$$

To estimate $\|\nu_\delta^{s,+}\|_{L^1(\Omega)}$ and $\|\nu_\delta^{s,-}\|_{L^1(\Omega)}$, we test (4.1) against $\sigma_\delta^+(u_s)$ (respectively $\sigma_\delta^-(u_s)$) and obtain

$$\begin{aligned} \|\nu_\delta^{s,+}\|_{L^1(\Omega)} &\leq \frac{1}{\delta} \int_{\{k-\delta \leq u_s \leq k\}} \mathcal{A}(x, \nabla u_s) \cdot \nabla u_s \, dx \\ &= \int_\Omega \mu^s \sigma_\delta^+(u_s) \, dx \leq 2\|\mu\|_{\mathcal{M}_b(\Omega)}, \end{aligned} \tag{6.7}$$

$$\begin{aligned} \|\nu_\delta^{s,-}\|_{L^1(\Omega)} &\leq \frac{1}{\delta} \int_{\{-k \leq u_s \leq -k+\delta\}} \mathcal{A}(x, \nabla u_s) \cdot \nabla u_s \, dx \\ &= \int_\Omega \mu^s \sigma_\delta^-(u_s) \, dx \leq 2\|\mu\|_{\mathcal{M}_b(\Omega)}. \end{aligned} \tag{6.8}$$

In the end we have that

$$\left\| -\operatorname{div} \left(h_\delta(u_s) \mathcal{A}(x, \nabla u_s) \right) \right\|_{L^1(\Omega)} \leq 6\|\mu\|_{\mathcal{M}_b(\Omega)}.$$

Due to remark 2.5 and (A2) we get uniform integrability of $\{\mathcal{A}(x, \nabla(T_k(u_s)))\}_k$, so Lebesgue’s monotone convergence theorem justifies we can let $\delta \rightarrow 0$ getting

$$|h_\delta(u_s) \mathcal{A}(x, \nabla u_s)| \rightarrow |\mathcal{A}(x, \nabla(T_k(u_s)))| \quad \text{strongly in } L^1(\Omega).$$

Therefore $\mathfrak{A}_{\varphi(\cdot)}(T_k u_s) \in \mathcal{M}_b(\Omega)$ and $\|\mathfrak{A}_{\varphi(\cdot)}(T_k u_s)\|_{\mathcal{M}_b(\Omega)} \leq 6\|\mu\|_{\mathcal{M}_b(\Omega)}$, where the bound is uniform with respect to s and k . Consequently, the use of proposition 5.1 enables to infer that also that $\mathfrak{A}_{\varphi(\cdot)}(T_k u) \in \mathcal{M}_b(\Omega)$, $\|\mathfrak{A}_{\varphi(\cdot)}(T_k u)\|_{\mathcal{M}_b(\Omega)} \leq 6\|\mu\|_{\mathcal{M}_b(\Omega)}$, and – finally – (6.2). By remark 2.4 we can extend the family of admissible test functions to get (6.1) and conclusion that $\mathfrak{A}_{\varphi(\cdot)}(T_k u) \in \mathcal{M}_b(\Omega) \cap (W_0^{1,\varphi(\cdot)}(\Omega))'$.

Step 2. Existence of a diffuse measure $\vartheta \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$, such that

$$\vartheta \llcorner \{|u| < k\} = \mathfrak{A}_{\varphi(\cdot)}(T_1 u) \llcorner \{|u| < k\} \text{ for every } k > 0 \text{ and every } l \geq k.$$

Lemma 2.10 ensures that $\phi \in W_0^{1,\varphi(\cdot)}(\Omega) \cap L^\infty(\Omega)$ belongs to $L^1(\Omega, \vartheta)$ with any $\vartheta \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$.

Note that $\lambda_1 \llcorner \{|u| < k\} = \lambda_k \llcorner \{|u| < k\}$ for every $l \geq k > 0$. Since the set $\{|u| < k\}$ is $C_{\varphi(\cdot)}$ -quasi-open, lemmas 2.8 and 2.9 ensure that there exists an increasing sequence $\{w_j\}$ of nonnegative functions in $W_0^{1,\varphi(\cdot)}(\Omega)$ which converges to $\mathbb{1}_{\{|u| < k\}}$ $C_{\varphi(\cdot)}$ -quasi-everywhere in Ω . Then $w_j = 0$ a.e. in $\{|u| \geq k\}$. If $\psi \in C_0^\infty$, then

$\phi = w_j \psi \in W_0^{1,\varphi(\cdot)}(\Omega) \cap L^\infty(\Omega)$ is an admissible test function in (6.1), so for every $l \geq k$ we get

$$\begin{aligned} \int_{\Omega} w_j \psi \, d\lambda_k &= \int_{\{|u| \leq k\}} \mathcal{A}(x, \nabla u) \cdot \nabla(w_j \psi) \, dx \\ &= \int_{\{|u| \leq l\}} \mathcal{A}(x, \nabla u) \cdot \nabla(w_j \psi) \, dx = \int_{\Omega} w_j \psi \, d\lambda_l. \end{aligned}$$

Passing to the limit with $j \rightarrow \infty$ we get

$$\int_{\{|u| < k\}} \psi \, d\lambda_k = \int_{\{|u| < k\}} \psi \, d\lambda_l \quad \text{for every } \psi \in C_0^\infty,$$

so of course $\lambda_l \llcorner \{|u| < k\} = \lambda_k \llcorner \{|u| < k\}$. Consequently, there exists a unique (up to sets of $W^{1,\varphi(\cdot)}$ -capacity zero) Borel measure ϑ , such that $\vartheta \llcorner \{|u| = +\infty\} = 0$ and $\vartheta \llcorner \{|u| < k\} = \lambda_l \llcorner \{|u| < k\}$ for every $k > 0$ and every $l \geq k$. As λ_k vanishes on every set of zero capacity $C_{\varphi(\cdot)}$, so does ϑ . By (6.2) the measures $|\lambda_k|$ are uniformly bounded with respect to k , so $\{|\vartheta|(\{|u| < k\})\}_k$ is bounded. In turn, $|\vartheta|(\Omega) < \infty$ and – finally – we infer that $\vartheta \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$.

Step 3. $\mathfrak{A}_{\varphi(\cdot)}(T_k u) \llcorner \{|u| > k\} = 0$.

Lemma 2.7 gives that u is $C_{\varphi(\cdot)}$ -quasi-continuous, thus the set $\{|u| > k\}$ is $C_{\varphi(\cdot)}$ -quasi-open. Fix arbitrary open $V \subset \Omega$. By lemma 2.9, there exists an increasing sequence $\{\widehat{w}_j\}$ of nonnegative functions in $W_0^{1,\varphi(\cdot)}(\Omega)$ which converges to $\mathbb{1}_{V \cap \{|u| < k\}}$ $C_{\varphi(\cdot)}$ -quasi-everywhere in Ω . Then $\widehat{w}_j = 0$ a.e. in $\{|u| \leq k\}$ and we can test (6.1) against $w_j \in W_0^{1,\varphi(\cdot)}(\Omega) \cap L^\infty(\Omega)$. We obtain

$$\int_{\Omega} w_j \, d\lambda_k = \int_{\{|u| \leq k\}} \mathcal{A}(x, \nabla u) \cdot \nabla(w_j) \, dx = 0.$$

Letting $j \rightarrow \infty$ we get that $(\lambda_k \llcorner \{|u| > k\})(V) = 0$. Since V was arbitrary open set, we have what was claimed.

Step 4. Limits. Since we have (6.7) and (6.8), we get (6.3) and (6.4) for any $\phi \in C_0(\Omega)$, with some nonnegative $\nu_k^+, \nu_k^- \in \mathcal{M}_b(\Omega)$. They have the form given in the claim, because $\mathfrak{A}_{\varphi(\cdot)}(T_k u) \in \mathcal{M}_b(\Omega) \cap (W_0^{1,\varphi(\cdot)}(\Omega))'$ has properties proven in steps 3 and 4. □

PROPOSITION 6.2 (Existence of renormalized solutions). *Let Ω be bounded open domain in \mathbb{R}^n , $\mathcal{A} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy (A1)–(A4), $\varphi \in \Phi_c(\Omega)$ satisfy (aInc)_p, (aDec)_q, (A0) and (A1), and $\mu \in \mathcal{M}_b(\Omega)$. Then there exists at least one renormalized solution to (1.1) (definition 1.7).*

Proof. By proposition 5.1 there exists an approximable solution $u \in \mathcal{T}_0^{1,\varphi(\cdot)}(\Omega)$ to (1.1). We shall show that actually it is also a renormalized solution. Due to proposition 6.1, measure μ can be seen as the weak- $*$ limit of $\{\lambda_k\}$, which are expressed

as

$$\lambda_k = \mathfrak{A}_{\varphi(\cdot)}(T_k u) = \vartheta \llcorner \{|u| < k\} + \nu_k^+ - \nu_k^-$$

with $\vartheta \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega)$, $\nu_k^+, \nu_k^- \in (\mathcal{M}_b(\Omega) \setminus \mathcal{M}_b^{\varphi(\cdot)}(\Omega)) \cup \{0\}$ being such that $\nu_k^+ = \nu_k^+ \llcorner \{u = k\}$ and $\nu_k^- = \nu_k^- \llcorner \{u = -k\}$. Given $h \in W^{1,\infty}(\mathbb{R})$ having h' with compact support, $\phi \in C_0^\infty(\Omega)$, and arbitrary $k > 0$, function $h(T_{k+1}(u))\phi \in W_0^{1,\varphi(\cdot)}(\Omega) \cap L^\infty(\Omega)$, so we can test equation (6.1) to get

$$\int_{\{|u| \leq k\}} \mathcal{A}(x, \nabla u) \cdot \nabla (h(T_{k+1}(u))\phi) \, dx = \int_{\{|u| \leq k+1\}} h(u)\phi \, d\lambda_k \tag{6.9}$$

$$= \int_{\{|u| < k\}} h(u)\phi \, d\vartheta + h(k) \int_{\Omega} \phi \, d\nu_k^+ - h(-k) \int_{\Omega} \phi \, d\nu_k^-. \tag{6.10}$$

We need to justify letting $k \rightarrow \infty$. We start with the left-hand side of (6.9) by having a look on

$$\mathcal{A}(x, \nabla u) \cdot \nabla (h(u)\phi) = \mathcal{A}(x, \nabla u) \cdot \nabla u (h'(u)\phi) + \mathcal{A}(x, \nabla u) \cdot \nabla \phi h(u).$$

If we prove that both terms on the right-hand side in the last display are integrable, Lebesgue’s monotone convergence theorem will give the desired conclusion. Recall that $u \in \mathcal{T}_0^{1,\varphi(\cdot)}(\Omega)$ and satisfy (4.2), so by proposition 4.2 and lemma 2.10, $\mathcal{A}(\cdot, \nabla u) \in (L^1(\Omega))^n$. Moreover, h' is bounded and $\text{supp } h' \subset [-M, M]$ for some $M > 0$, so

$$\mathcal{A}(\cdot, \nabla u) \cdot \nabla u h'(u) = \mathcal{A}(\cdot, \nabla T_M u) \cdot \nabla (T_M u) h'(u)$$

is integrable by (4.2). For the second term we see that

$$\|\mathcal{A}(x, \nabla u) \cdot \nabla \phi h(u)\|_{L^1(\Omega)} \leq \|\mathcal{A}(x, \nabla u)\|_{L^1(\Omega)} \|\nabla \phi\|_{L^\infty(\Omega)} \|h\|_{L^\infty(\Omega)},$$

so it suffices to use the same arguments as before. Therefore, (6.9) becomes the left-hand side of (1.8) in the limit. By remark 1.3 the following decomposition

$$\begin{aligned} \mu &= \mu_{\varphi(\cdot)} + \mu_{\text{sing}}^+ - \mu_{\text{sing}}^-, \quad \mu_{\varphi(\cdot)} \in \mathcal{M}_b^{\varphi(\cdot)}(\Omega), \quad 0 \leq \mu_{\text{sing}}^+, \\ \mu_{\text{sing}}^- &\in (\mathcal{M}_b(\Omega) \setminus \mathcal{M}_b^{\varphi(\cdot)}(\Omega)) \cup \{0\} \end{aligned}$$

is unique up to sets of $W^{1,\varphi(\cdot)}$ -capacity zero. By (6.2) it holds that $\vartheta \llcorner \{|u| < k\} \rightarrow \mathfrak{A}_{\varphi(\cdot)}(u)$. Note that it is also (6.2) to justify testing against $W_0^{1,\varphi(\cdot)}(\Omega) \cap L^\infty(\Omega)$ -function. To conclude we use Lebesgue’s dominated convergence theorem in (6.10). To motivate the convergence of the first term we note that we can split the first term to positive and negative part, whose majorants are integrable due to lemma 2.10. For the remaining two terms it suffices to recall that h is bounded and constant in infinities. By (6.3) one has $\nu_k^+ \rightarrow \mu_{\text{sing}}^+$ with $\text{supp } \mu_{\text{sing}}^+ \subset \cap_{k>0} \{u > k\}$, and by (6.4) also $\nu_k^- \rightarrow \mu_{\text{sing}}^-$ with $\text{supp } \mu_{\text{sing}}^- \subset \cap_{k>0} \{u < -k\}$. □

7. Uniqueness in problems with diffuse measure data

The previous results worked for a general measure data problems. Here, we restrict to a class of diffuse measure data to provide uniqueness.

PROPOSITION 7.1 (Uniqueness of approximable solutions). *Let all assumptions of proposition 5.1 be satisfied. Assume that f, G are obtained as limits of approximate sequences $\{f_s^i\}$ in $C^\infty(\Omega)$, $i = 1, 2$, satisfying*

$$f_s^i \rightarrow f \text{ in } L^1(\Omega) \text{ and } \|f_s^i\|_{L^1(E)} \nearrow \|f\|_{L^1(E)} \text{ for measurable } E \subset \Omega \tag{7.1}$$

and $\{G_s^i\}$ in $C^\infty(\Omega)$, $i = 1, 2$, such that

$$G_s^i \rightarrow G \text{ strongly in } (L^{\tilde{\varphi}(\cdot)}(\Omega))^n \text{ and } \varrho_{\tilde{\varphi}, E}(|G_s^i|) \leq 2\varrho_{\tilde{\varphi}, E}(|G|) \tag{7.2}$$

on measurable $E \subset \Omega$. If v^i is an approximable solution defined as an a.e. limit of weak solutions v_s^i to the approximate problems

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla v_s^i) = f_s^i - \operatorname{div} G_s^i & \text{in } \Omega, \\ v_s^i = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.3}$$

then $v^1 = v^2$ a.e. in Ω .

Proof. We note first that the problem is well posed as $\mu^{i,s} = f_s^i - \operatorname{div} G_s^i \rightharpoonup \mu_{\varphi(\cdot)}$ weakly-* in the space of measures, $i = 1, 2$. We fix arbitrary $t, l > 0$, use $\phi = T_t(T_l(v_s^1) - T_l(v_s^2)) \in W_0^{1, \varphi(\cdot)}(\Omega) \cap L^\infty(\Omega)$ as a test function in both (1.6) and subtract the equations to obtain for every $s > 0$

$$\begin{aligned} L_s &= \int_{\{|T_l(v_s^1) - T_l(v_s^2)| \leq t\}} (\mathcal{A}(x, \nabla v_s^1) - \mathcal{A}(x, \nabla v_s^2)) \cdot (\nabla v_s^1 - \nabla v_s^2) \, dx \\ &= \int_{\Omega} (f_s^1 - f_s^2) T_t(T_l(v_s^1) - T_l(v_s^2)) \, dx \\ &\quad + \int_{\Omega} (G_s^1 - G_s^2) \cdot \nabla T_t(T_l(v_s^1) - T_l(v_s^2)) \, dx = R_s^1 + R_s^2. \end{aligned} \tag{7.4}$$

The right-hand side above tends to 0. Indeed, the convergence of R_s^1 holds because $|T_t(T_l v_s^1 - T_l v_s^2)| \leq t$ and for $s \rightarrow 0$ we have $f_s^1 - f_s^2 \rightarrow 0$ in $L^1(\Omega)$. As for R_s^2 it

suffices to note that

$$\begin{aligned}
 |R_s^2| &= \left| \int_{\{|T_l(v_s^1) - T_l(v_s^2)| \leq t\}} (G_s^1 - G_s^2) \cdot \nabla T_l v_s^1 \, dx \right. \\
 &\quad \left. - \int_{\{|T_l(v_s^1) - T_l(v_s^2)| \leq t\}} (G_s^1 - G_s^2) \cdot \nabla T_l(v_s^2) \, dx \right| \\
 &\leq \left| \int_{\Omega} (G_s^1 - G_s^2) \cdot \nabla T_l(v_s^1) \, dx \right| + \left| \int_{\Omega} (G_s^1 - G_s^2) \cdot \nabla T_l(v_s^2) \, dx \right| \\
 &\leq 2 \|G_s^1 - G_s^2\|_{(L^{\tilde{\varphi}(\cdot)}(\Omega))^n} \|\nabla T_l(v_s^1)\|_{(L^{\varphi(\cdot)}(\Omega))^n} \\
 &\quad + 2 \|G_s^1 - G_s^2\|_{(L^{\tilde{\varphi}(\cdot)}(\Omega))^n} \|\nabla T_l(v_s^2)\|_{(L^{\varphi(\cdot)}(\Omega))^n} \\
 &\leq c \|G_s^1 - G_s^2\|_{(L^{\tilde{\varphi}(\cdot)}(\Omega))^n},
 \end{aligned}$$

where we used that weak convergence of the $\{\nabla T_l(v_s^j)\}_s$ ($j = 1, 2$) in $(L^{\varphi(\cdot)}(\Omega))^n$, which in particular implies uniform boundedness of $\{\|\nabla T_l(v_s^j)\|_{L^{\varphi(\cdot)}(\Omega)}\}_s$ ($j = 1, 2$) and recalled that the strong convergence of $(G_s^1 - G_s^2) \rightarrow 0$ in $(L^{\tilde{\varphi}(\cdot)}(\Omega))^n$. The left-hand side of (7.4) is nonnegative due to the monotonicity of \mathcal{A} . Moreover, as $R_s^1 + R_s^2 \rightarrow 0$, we get

$$\begin{aligned}
 0 &\leq \int_{\{|T_l v^1 - T_l v^2| \leq t\}} (\mathcal{A}(x, \nabla v^1) - \mathcal{A}(x, \nabla v^2)) \cdot (\nabla v^1 - \nabla v^2) \, dx \\
 &\leq \limsup_{s \rightarrow 0} L_s = \limsup_{s \rightarrow 0} (R_s^1 + R_s^2) = 0.
 \end{aligned}$$

Consequently, $\nabla v^1 = \nabla v^2$ a.e. in $\{|T_l(v^1) - T_l(v^2)| \leq t\}$ for every $t, l > 0$, and so

$$\nabla v^1 = \nabla v^2 \quad \text{a.e. in } \Omega. \tag{7.5}$$

Given the boundary value also $v^1 = v^2$ a.e. in Ω . □

8. Main proof

Proof of theorem 1.8. Existence of approximable solutions is provided in proposition 5.1. Proposition 6.2 yields that an approximable solution is a renormalized solutions. Proposition 6.1 actually localizes the support of singular measures. Approximable solutions can be achieved from renormalized ones by a choice of $h = T_k$. □

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