Proceedings of the Edinburgh Mathematical Society (1992) **35**, 115–120 ©

AUTOMORPHISM GROUPS OF COVERING POSETS AND OF DENSE POSETS

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(Received 2nd March 1990)

Given a poset $(X, \leqq)$, the covering poset $(C(X), \leqq)$ consists of the set $C(X)$ of covering pairs, that is, pairs $(a,b)\in X^2$ with $a<b$ such that there is no $c\in X$ with $a<c<b$, partially ordered by $(a,b)\leqq(a',b')$ if and only if $(a,b)=(a',b')$ or $b\leqq a'$. There is a natural homomorphism ν from the automorphism group of $(X, \leqq)$ into the automorphism group of $(C(X), \leqq)$. It is shown that given groups G,H and a homomorphism α from G into H there exists a poset $(X, \leqq)$ and isomorphisms ϕ, ψ from G onto $\mathrm{Aut}(X, \leqq)$, respectively from H onto $\mathrm{Aut}(C(X), \leqq)$ such that $\phi\nu=\alpha\psi$. It is also shown that every group is isomorphic to the automorphism group of a poset all of whose maximal chains are isomorphic to the rationals.

1980 *Mathematics subject classification* (1985 Revision): Primary 06A10, Secondary 20B25, 20B27.

1. Introduction

Let $(X, \leqq)$ be a partially ordered set (in short, a poset). A pair (a,b) of elements of X is called a *covering pair* of $(X, \leqq)$ if $a<b$ and whenever $c\in X$ with $a\leqq c\leqq b$ then $c=a$ or $c=b$. On the set $C(X)$ of covering pairs of $(X, \leqq)$ a partial order can be defined by $(a,b)\leqq(a',b')$ if and only if $(a,b)=(a',b')$ or $b\leqq a'$, the poset $(C(X), \leqq)$ is called the *covering poset* of $(X, \leqq)$. A characterization of covering posets of finite posets was given in [**4**]. A poset $(X, \leqq)$ is called *dense* if $C(X)$ is empty.

For a class of mathematical structures, it is a well-known problem to ask whether every group is isomorphic to the automorphism group of a member of this class (see, for example, [**2**]). For posets, this was shown to be true by Garrett Birkhoff [**5**]. The maximal chains in the posets constructed there are (inversely) well-ordered, that is, they have no non-trivial automorphisms. It is thus interesting to see whether the same result holds in posets where the maximal chains have a high degree of homogeneity, for example, where they are all isomorphic to the rationals. In a similar spirit, we can consider a poset and its covering poset. Here we have two automorphism groups, and it is natural to ask the question in what way they are related.

2. Automorphism groups of dense posets

It is well known that there are just four isomorphism classes of countably infinite dense linearly ordered posets, namely the rationals with possibly a minimal and/or maximal element adjoined. Their automorphism group is very large. In order to represent every group as the automorphism group of a poset all of whose maximal chains are isomorphic to the rationals, we first represent the trivial group in this way. A poset $(X, \leqq)$ is called lower semilinear (see, for example, [1]) if

(i) whenever $x, y \in X$ then there exists $z \in X$ such that $z \leqq x$ and $z \leqq y$, and

(ii) whenever $x, y \in X$ are incomparable then there does not exist $z \in X$ with $x \leqq z$ and $y \leqq z$.

Let S be the set of all sequences $(x_1, k_1, x_2, k_2, \ldots, x_{n-1}, k_{n-1}, x_n)$ where $n \geqq 1$ is an integer, x_i is a rational number for $1 \leqq i \leqq n$ and k_i is a natural number for $1 \leqq i < n$. Clearly the set S is countable. Let $f: S \to \mathbb{N}$ be an injective mapping. Let $X_f = \{(x_1, k_1, \ldots, x_n) \in S \mid k_i = f(x_1, k_1, \ldots, x_i)$ for all i with $1 \leqq i \leqq n-1\}$. We define a partial order $\leqq_f$ on X_f as follows. Let $(x_1, k_1, \ldots, x_n) \leqq_f (y_1, h_1, \ldots, y_m)$ if and only if $n \leqq m$, $x_i = y_i$ and $k_i = h_i$ for all i with $1 \leqq i < n$, and $x_n \leqq y_n$.

Lemma 2.1. *$(X_f, \leqq_f)$ is a countable, dense, lower semilinear poset with no maximal or minimal elements and such that* $\mathrm{Aut}(X_f, \leqq_f) = \{1\}$.

Proof. It is obvious that $(X_f, \leqq_f)$ is countable and dense, and that it has no maximal or minimal elements. Let $(x_1, k_1, \ldots, x_n)$, $(y_1, h_1, \ldots, y_m) \in X_f$, and suppose that $n \leqq m$. Let $x = \min(x_1, y_1)$. Then $(x) \leqq_f (x_1, k_1, \ldots, x_n)$ and $(x) \leqq_f (y_1, h_1, \ldots, y_m)$. Suppose there exists $(z_1, g_1, \ldots, z_p)$ with $(x_1, k_1, \ldots, x_n) \leqq_f (z_1, g_1, \ldots, z_p)$ and $(y_1, h_1, \ldots, y_m) \leqq_f (z_1, g_1, \ldots, z_p)$. Then $m \leqq p$, $x_i = z_i$ and $k_i = g_i$ for $1 \leqq i < n$, $x_n \leqq z_n$, $y_i = z_i$ and $h_i = g_i$ for $1 \leqq i < m$, $y_m \leqq z_m$. Then either $n < m$ or $n = m$ and $x_n \leqq y_n$, in which case $(x_1, k_1, \ldots, x_n) \leqq_f (y_1, h_1, \ldots, y_m)$, or $n = m$ and $y_n \leqq x_n$, in which case $(y_1, h_1, \ldots, y_m) \leqq_f (x_1, k_1, \ldots, x_n)$. Thus $(X_f, \leqq_f)$ is semilinear. For $a \in X_f$, let $N(a)$ be the number of connected components of the subposet induced on $\{b \in X_f \mid a < b\}$. The connected components of the subposet induced on $\{b \in X_f \mid (x_1, k_1, \ldots, x_n) < b\}$ are just $C_0 = \{(y_1, h_1, \ldots, y_m) \in X_f \mid n \leqq m,\ y_j = x_j$ and $h_j = k_j$ for $1 \leqq j < n,\ x_n < y_n\}$ and $C_i = \{(y_1, h_1, \ldots, y_m) \in X_f \mid n < m,\ y_j = x_j$ for $1 \leqq j \leqq n$, $h_j = k_j$ for $1 \leqq j \leqq n-1,\ h_n = i\}$. Thus $N((x_1, k_1, \ldots, x_n)) = 1 + f(x_1, k_1, \ldots, x_n)$. As this number is invariant under automorphisms, and as it is distinct for all elements of X_f (as f is injective), it follows that $(X_f, \leqq_f)$ can have no non-trivial automorphisms. □

Proposition 2.2. *The number of isomorphism classes of countable, dense lower semilinear posets with no minimal or maximal elements and trivial automorphism group is* $2^{\aleph_0}$.

Proof. It is at most $2^{\aleph_0}$, as there are only $2^{\aleph_0}$ binary relations on a countable set. Every permutation π of $\{n \in \mathbb{N} \mid n$ even$\}$ can be extended to an injective mapping $\pi': S \to \mathbb{N}$ (if the number n is identified with the one-element sequence (n)). As for π_1, π_2 with $\pi_1 \neq \pi_2$ there exist even natural numbers n, m such that $n\pi_1 < m\pi_1$ and $m\pi_2 < n\pi_2$, it follows that $(X_{\pi_1'}, \leqq_{\pi_1'})$ is not isomorphic to $(X_{\pi_2'}, \leqq_{\pi_2'})$. As there are $2^{\aleph_0}$ permutations of $\{n \in \mathbb{N} \mid n$ even$\}$, the result follows. □

A subset S of a poset $(X, \leqq)$ is called *order-autonomous* if whenever $x, x' \in S$, $y \in X \backslash S$ then $x < y$ if and only if $x' < y$, and $y < x$ if and only if $y < x'$.

Lemma 2.3. *Let $(Y, \leqq)$ be a poset, and let $X = Y \times X_f$, ordered lexicographically (that is, if (y,z), $(y',z') \in X$ then $(y,z) \leqq (y',z')$ if and only if $y < y'$, or $y = y'$ and $z \leqq_f z'$). Let $S \subseteq X$ be maximal with respect to being order-autonomous and the induced order on S being lower semilinear. Then there exists $y \in y$ such that $S = \{y\} \times X_f$.*

Proof. Let S be such a subset, and suppose there exist (y,z), $(y',z') \in S$ with $y \neq y'$. First suppose that $y < y'$. There exists $\bar{z} \in X_f$ such that $z, \bar{z}$ are incomparable. Then $(y,\bar{z}) < (y',z')$ and $(y,z) < (y',z')$, and $(y,\bar{z})$, (y,z) are incomparable. As S is lower semilinear, it follows that $(y,\bar{z}) \notin S$. But this a contradiction to S being order-autonomous. Thus we cannot have $y < y'$, and neither $y' < y$ by symmetry, hence y, y' must be incomparable. Then also (y,z), (y',z') are incomparable. As $(S, \leqq)$ is lower semilinear, there exists $(y'',z'') \in S$ with $(y'',z'') \leqq (y,z)$ and $(y'',z'') \leqq (y',z')$. Hence we must have $y'' < y$ and $y'' < y'$, but we have already shown that this leads to a contradiction. Therefore there exists $y \in Y$ such that $S \subseteq \{y\} \times X_f$. By maximality of S, we then get $S = \{y\} \times X_f$. □

We now can give the main result of this section.

Theorem 2.4. *Let G be a group. Then there exists a partially ordered set $(X, \leqq)$ with the following properties.*

(i) *Every maximal chain in $(X, \leqq)$ is isomorphic to the rationals.*

(ii) *G is isomorphic to* $\mathrm{Aut}(X, \leqq)$.

(iii) $|X| = \max(|G|, \aleph_0)$.

Proof. By [6] or [3], there exists a poset $(Y, \leqq)$ whose maximal chains have exactly two elements such that G is isomorphic to $\mathrm{Aut}(Y, \leqq)$. Moreover, Y can be taken to be finite if G is finite and such that $|Y| = |G|$ if G is infinite. Let $(X_f, \leqq_f)$ be as before, and let $X = Y \times X_f$, ordered lexicographically. If C is a maximal chain in $(X, \leqq)$ then there exist $y, y' \in Y$ with $y < y'$ and maximal chains D, D' in $(X_f, \leqq_f)$ such that $C = \{(y,z) \mid z \in D\} \cup \{(y',z') \mid z' \in D'\}$. As each of D, D' is isomorphic to the rationals, it follows that C is also isomorphic to the rationals. Also clearly $|X| = \max(|G|, \aleph_0)$. Let $\psi: G \to \mathrm{Aut}(Y, \leqq)$ be an isomorphism. Then it is easy to see that ϕ defined by $(y,z)(g\phi) = (y(g\psi), z)$ for $(y,z) \in X$, $g \in G$, is a monomorphism from G into $\mathrm{Aut}(X, \leqq)$.

By Lemma 2.3 it follows that $\bar{Y} = \{\{y\} \times X_f \mid y \in Y\}$ is the set of maximal order-autonomous lower semilinear subsets of X. It is clear that every automorphism of $(X, \leqq)$ induces a permutation on $\bar{Y}$. Furthermore $\bar{Y}$ is a partition of X, thus if $\alpha \in \mathrm{Aut}(X, \leqq)$ then there exist $\beta \in \mathrm{Aut}(Y, \leqq)$ and $\beta_y \in \mathrm{Aut}(X_f, \leqq_f)$ for $y \in Y$ such that $(y,z)\alpha = (y\beta, z\beta_y)$. By Lemma 2.1, we have $\beta_y = 1$ for all $y \in Y$. As ψ is an isomorphism, there exists $g \in G$ such that $\beta = g\psi$, hence $(y,z)\alpha = (y\beta, z) = (y(g\psi), z) = (y,z)(g\phi)$, thus $\alpha = g\phi$ and ϕ is surjective. □

3. Automorphism groups of covering posets

Let $(X, \leqq)$ be a poset and g an automorphism of $(X, \leqq)$. *For* $(x,y) \in C(X)$ define $(x,y)(gv) = (xg, yg)$. It is easy to see that gv is an automorphism of $(C(X), \leqq)$, and that the mapping $v: \mathrm{Aut}(X, \leqq) \to \mathrm{Aut}(C(X), \leqq)$ is a homomorphism. It is not hard to see that, in general, v needs to be neither injective nor surjective. Thus in generalization of the problem of representing groups as automorphism groups of structures, here it is natural to ask whether every triple consisting of two groups and a homomorphism between them can be represented.

Theorem 3.1. *Let G, H be groups and $\alpha: G \to H$ a homomorphism. Then there exist a poset $(X, \leqq)$ and isomorphisms $\phi: G \to \mathrm{Aut}(X, \leqq)$ and $\psi: H \to \mathrm{Aut}(C(X), \leqq)$ such that $\phi v = \alpha\psi$, where $v: \mathrm{Aut}(X, \leqq) \to \mathrm{Aut}(C(X), \leqq)$ is the natural homomorphism defined above.*

Proof. Let U be the image of α in H. Let X be the disjoint union of $H \times (H \cup \{a,b\})$, H, $G \times X_f$ and $G \times G \times X_f$. Let $\leqq_G$, $\leqq_H$ be inverse well-orderings on G, H respectively, such that in both cases the neutral element is the maximal element with respect to this order. We define a partial order on X as follows. Let $y \leqq y$ for all $y \in X$. Let $(g,a) \leqq (g,h)$ for all $g, h \in H$. For $g, h, h' \in H$ let $(g,h) \leqq (g,h')$ whenever $h \leqq_H h'$. Let $(g,a) \leqq (g',b)$ for all $g, g' \in H$, and let $(g,h) \leqq (g',b)$ if and only if $h \leqq_H gg'^{-1}$ for $g, g', h \in H$. For $y \in H \times (H \cup \{a,b\})$ and $h \in H$ let $y \leqq h$ if and only if $y \leqq (h,b)$. For (g,x), $(g,x') \in G \times X_f$ let $(g,x) \leqq (g,x')$ if and only if $x \leqq_f x'$. For $g, g', h, h' \in G$, $x, x' \in X_f$ let $(g,h,x) \leqq (g,h',x')$ whenever $h <_G h'$ or $h = h'$ and $x \leqq_f x'$, and let $(g,h,x) \leqq (g',x')$ if and only if $h \leqq_G gg'^{-1}$. Finally, for $y \in (G \times X_f) \cup (G \times G \times X_f)$ and $u \in U$ let $y \leqq u$ if and only if there exists $(g,x) \in G \times X_f$ such that $y \leqq (g,x)$ and $g\alpha = u$. It is not hard to see that this defines a partial order on X.

Define $\phi: G \to \mathrm{Sym}(X)$ as follows. Let $(H,x)(g\phi) = (hg, x)$, $(h,h',x)(g\phi) = (hg, h', x)$, $k(g\phi) = k(g\alpha)$, $(k,s)(g\phi) = (k(g\alpha), s)$ for $g, h, h' \in G$, $x \in X_f$, $k \in H$, $s \in H \cup \{a,b\}$. It is not hard to see that ϕ is a monomorphism from G into $\mathrm{Aut}(X, \leqq)$. Note that $X_1 = U \cup (G \times X_f) \cup (G \times G \times X_f)$ is the set of all elements of X which are contained in a dense maximal chain of $(X, \leqq)$, thus X_1 is setwise invariant under $\mathrm{Aut}(X, \leqq)$. As $U = \mathrm{Max}(X_1, \leqq)$ it follows that both U and $X_2 = X_1 \backslash U$ are setwise invariant under $\mathrm{Aut}(X, \leqq)$.

By Lemma 2.3, it follows that every maximal order-autonomous lower semilinear subset of $(X_2, \leqq)$ is of the form $\{g\} \times X_f$ for some $g \in G$, or $\{(g,h)\} \times X_f$ for $(g,h) \in G \times G$. As in Theorem 2.4, it follows that every automorphism of $(X_2, \leqq)$ is induced by an automorphism of $G \cup (G \times G)$ (with the natural order induced on a partition by order-autonomous subsets) and vice versa. As the image of G under ϕ operates regularly on G, in order to show that ϕ is surjective, it is sufficient to show that if τ is an automorphism of $(X, \leqq)$ which fixes 1 then τ is the identity. Let $\tau \in \mathrm{Aut}(X, \leqq)$ such that τ fixes 1. As the order on $G \cup (G \times G)$ is just the order constructed by Birkhoff in [5] for the group G, it follows that τ fixes $G \cup (G \times G)$, and thus also X_2 pointwise. It is then clear that τ fixes U pointwise. As (u,a) is the unique element covered by u for $u \in U$, it follows that $(u,a)\tau = (u,a)$ for all $u \in U$. It is also clear that τ has to fix $H \times (H \cup \{a\})$

setwise. However, the order induced on $H \times (H \cup \{a\})$ is also just the order constructed by Birkhoff [5] for the group H, and as $(1,a)\tau=(1,a)$, it follows that τ fixes $H \times (H \cup \{a\})$ pointwise. It is then obvious that τ is the identity on X. Hence ϕ is surjective.

For $g \in H$ define $r(g)=((g,a),g)$. If $g,h \in H$ and h is not the smallest element of $(H, \leqq_H)$ let $s(g,h)=((g,h'),(g,h))$ where h' is the unique element of H such that $(h',h) \in C(H, \leqq_H)$. If $(H, \leqq_H)$ has a smallest element g_0, let $s(g,g_0)=((g,a),(g,g_0))$. For $g,h \in H$ let $t(g,h)=((g,gh^{-1}),(h,b))$. Note that $C(X)=\{r(g) \mid g \in H\} \cup \{s(g,h), t(g,h) \mid g,h \in H\}$. Define $\psi: H \to \mathrm{Sym}(C(X))$ as follows. For $g,h,h' \in H$ let $r(h)(g\psi)=r(hg)$, $s(h,h')(g\psi)=s(hg,h')$, and $t(h,h')(g\psi)=t(hg,h'g)$. It is not hard to see that ψ is a homomorphism from G into $\mathrm{Aut}(C(X), \leqq)$, and it is clear that $\phi v=\alpha\psi$. It thus remains to prove that ψ is surjective.

Note that every maximal chain of $(C(X), \leqq)$ is of the form $M(g,h)=\{r(h),t(g,h)\} \cup \{s(g,h') \mid h' \in H, h' \leqq_H gh^{-1}\}$ for some $g,h \in H$. We say that a maximal chain M of $(C(X), \leqq)$ is of type 1 if there is no maximal chain $M' \neq M$ with $|M \setminus M'| \leqq 2$. Let $k \in H$. Suppose we have defined all types h of maximal chains for all $h \in H$ with $k <_H h$. We say that a maximal chain M is of type k if M is not of type h for any $h \in H$ with $k <_H h$, and whenever $M' \neq M$ is a maximal chain with $|M \setminus M'| \leqq 2$ then M' is of type h for some $h \in H$ with $k <_H h$. We note that M is of type k for $k \in H$ if and only if there exists $g \in H$ such that $M=M(g,k^{-1}g)$. Also it is clear that the type of a maximal chain is invariant under automorphisms.

The image of H under ψ operates regularly on $\{r(g) \mid g \in H\}$. Hence, in order to show that ψ is surjective, it is sufficient to prove that every automorphism of $(C(X), \leqq)$ which fixes $r(1)$ is the identity. Let $\sigma \in \mathrm{Aut}(C(X), \leqq)$ with $r(1)\sigma=r(1)$. As $M(1,1)$ is the unique maximal chain of type 1 which contains $r(1)$, it follows that σ leaves $M(1,1)$ setwise invariant. Clearly $M(1,1)$ is inversely well-ordered, thus σ leaves $M(1,1)$ pointwise invariant. As $M(1,k^{-1})$ is the unique maximal chain of type k which intersects $M(1,1) \setminus \{r(1)\}$ non-trivially, and as $r(k)$ is the unique maximal element of this chain, it follows that σ fixes $r(k)$. But $M(k,k)$ is the unique maximal chain of type 1 which contains $r(k)$, hence, as before, σ fixes every element of $M(k,k)$, that is, σ fixes $\{r(g) \mid g \in H\} \cup \{s(g,h) \mid g,h \in H\} \cup \{t(g,g) \mid g \in H\}$ pointwise. Finally, $t(g,h)$ is covered by $r(h)$ and covers $s(g,gh^{-1})$, and it is the only element of $C(X)$ with this property, thus $t(g,h)$ is also fixed by σ, hence σ is the identity. Therefore ψ is surjective. □

We finally note that if G and H are finite, and we also want X to be finite, the situation is somewhat different. Let $(X, \leqq)$ be a finite poset and let S be the set of isolated points of $(X, \leqq)$ (that is, the set of all those elements which are incomparable with all other elements). It is easy to see that in this case the kernel of v is isomorphic to the symmetric group on S, and $\mathrm{Aut}(X, \leqq)$ is just the direct product of $\mathrm{Aut}(X \setminus S, \leqq)$ and this kernel. In particular, if $(X, \leqq)$ has no isolated points then v is injective.

Theorem 3.2. *Let G be a finite group and H a subgroup of G. Then there exist a finite poset $(X, \leqq)$ with no isolated points and isomorphisms $\phi: G \to \mathrm{Aut}(C(X), \leqq)$ and $\psi: H \to \mathrm{Aut}(X, \leqq)$ such that $\psi v=\iota\phi$, where $v: \mathrm{Aut}(X, \leqq) \to \mathrm{Aut}(C(X), \leqq)$ is the natural homomorphism and $\iota: H \to G$ is the inclusion mapping.*

Proof. First assume that G is cyclic of order 2 and that $H=\{1\}$. Let $X=\{1,2,3,4,5,6\}$ with $1<3<5$, $2<4<6$ and $2<5$. It is easy to see that $(X,\leqq)$ has the desired properties. We now can assume that $G=H$ or $|G\backslash H|>1$. Let X be the disjoint union of $G\times(G\cup\{a,b\})$, H and, if $G\neq H$, also $\{\infty\}$. Let $\leqq_0$ be a linear ordering on G such that 1 is the maximal element with respect to this order. We define a partial order on X as follows. Let $x\leqq x$ for all $x\in X$. Let $(g,a)\leqq(g,h)$ for all $h\in G$. For $g,h,h'\in G$ let $(g,h)\leqq(g,h')$ whenever $h\leqq_0 h'$. Let $(g,a)\leqq(g',b)$ for all $g,g'\in G$, and let $(g,h)\leqq(g',b)$ if and only if $h\leqq_0 gg'^{-1}$. For $x\in G\times(G\cup\{a,b\})$ and $h\in H$ let $x\leqq h$ if and only if $x\leqq(h,b)$, and let $x\leqq\infty$ if and only if there exists $g\in G\backslash H$ such that $x\leqq(g,b)$. It is not hard to see that this defines a partial order on X. The rest of the proof follows in a similar way as the proof of Theorem 3.1, and shall therefore be left to the reader. □

REFERENCES

1. S. A. Adeleke, M. A. E. Dummett and Peter M. Neumann, On a question of Frege's about right-ordered groups, *Bull. London Math. Soc.* **19** (1987), 513–521.

2. László Babai, On the abstract group of automorphisms, in: *Combinatorics* (H. N. V. Temperley (ed.), Cambridge Univ. Press, Cambridge, 1981), 1—40.

3. L. Babai and D. Duffus, Dimension and automorphism groups of lattices, *Algebra Universalis* **12** (1981), 279–289.

4. Gerhard Behrendt, Covering posets, *Discrete Math.* **71** (1988), 189–195.

5. Garrett Birkhoff, Sobre los grupos de automorfismos, *Rev. Un. Mat. Argentina* **11** (1946), 155–157.

6. R. Frucht, Lattices with a given abstract group of automorphisms, *Canad. J. Math.* **2** (1950), 417–419.

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