A PROPERTY OF GROUPS WITH NO CENTRAL FACTORS

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1. Let $\mathfrak{C}_1$ denote the class of all groups with no non-trivial central factors. We prove the following theorem.

THEOREM. There exist non-trivial locally solvable $\mathfrak{C}_1$ groups; but there is no non-trivial locally k-step polynilpotent $\mathfrak{C}_1$ group for any integer k.

It is well known that a minimal normal subgroup of a locally solvable group is abelian. Thus no non-trivial locally solvable group can be pluperfect - the class of all perfect groups in which every subnormal subgroup is also perfect. On the other hand there are examples (see [2] or [3]) of locally nilpotent perfect groups. The above result thus shows (i) that $\mathfrak{C}_1$ groups lie strictly between pluperfect groups and perfect groups, and (ii) that the class of all locally solvable groups is decidedly larger than the class of all locally k-step polynilpotent groups, for all integers k.

Result (i) is well known and easy to verify even for finite groups. But (ii) is not so easy to show. It was first established by D.H. McLain in [4] where he proved that locally solvable groups satisfying max n - the maximal condition for normal subgroups - need not be finitely generated whereas every locally k-step polynilpotent group satisfying max n is finitely generated for all integers k.

It is perhaps interesting to note the similar role played by $\mathfrak{C}_1$ groups and groups satisfying max n. Theorems 1 and 2 of [1] give another example of this phenomenon. Since every simple locally solvable group is abelian, any non-trivial locally solvable group satisfying max n has a non-trivial abelian quotient and hence no such group is perfect. In particular the class of locally solvable $\mathfrak{C}_1$ groups and locally solvable groups satisfying max n are mutually exclusive.

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2. Notations and Definitions. A group G is k-step poly-nilpotent if and only if there is a series

$$1 = G_o \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_k = G$$

such that for all $i = 0, 1, \cdots, k-1$, G_{i+1}/G_i is nilpotent. H/K is called a central factor of G if $K \trianglelefteq G$ and H/K lies in the centre of G/K, or, equivalently, if $[H, G] = \{[h, g]: h \in H, g \in G\} \leq K$. A group G is perfect if and only if it coincides with its derived subgroup $G' = [G, G]$. The centre of G is denoted by $Z(G)$. Finally we refer the reader to [5, Section 9.2] for the definition and basic properties of the standard wreath product $H \wr K$ of two groups H and K.

3. Proof of the Theorem. The following construction, due to P. Hall, serves our purpose in proving the first part of the Theorem. Let $p_1, p_2, \cdots$ be an infinite sequence of distinct odd primes. Define $G_1 = \langle t_1 \rangle$; $G_{n+1} = G_n \wr \langle t_{n+1} \rangle = B_n \langle t_{n+1} \rangle$ for $n = 1, 2, \ldots,$ where t_i is of order p_i for all i, and G_n is embedded in G_{n+1} as a direct factor of the base group B_n. Let $G = \bigcup_{n=1}^{\infty} G_n$. We assert that the derived group G' of G has no non-trivial central factors.

Note that G is a locally finite solvable group and for all integers n, $B'_n \leq G''_{n+1}$ [2, Lemma 5] so that $G' = G''$. Also the subgroups B_n have all their sylow subgroups abelian, so that $B'_n \cap Z(B_n) = 1$ [6, Theorem 4.1]. If L/M is a non-trivial central factor of G', then $[G', L] \leq M$ and we can choose an element $x \in L$, $x \notin M$. Since $G' = \bigcup_{n=1}^{\infty} B'_n$, $x \in B'_m$ for some m. Now $B_m \triangleleft G_{m+1}$ and $M \triangleleft G'$ so that $B'_m \cap M \triangleleft G'_{m+1}$. Also the sylow subgroups of $G'_{m+1}/B'_m \cap M$ are again abelian, and hence

$$(1) \qquad (G''_{m+1}/B'_m \cap M) \cap (B'_m \cap L/B'_m \cap M) = 1$$

since $[G'_{m+1}, B'_m \cap L] \leq B'_m \cap M$. But $x \in B'_m \cap L \leq G''_{m+1}$ and $x \notin B'_m \cap M$. This contradicts (1) and we conclude that G'

has no non-trivial central factor.

To prove the second half of the Theorem, we need the following result.

LEMMA. $G \in \mathfrak{C}_1$ <u>if and only if for every</u> $x \in G$,

$$x \in D_x = \{[x,g]: g \in G\} .$$

<u>Proof</u>. If $x \in D_x$ for all x in G, then $L = [L, G]$ for every subgroup L of G so that G has no non-trivial central factors. Conversely, $x \notin D_x$ implies that $< x, D_x >/D_x$ is a non-trivial central factor of G so that $G \notin \mathfrak{C}_1$.

Suppose that G is a non-trivial locally k-step polynilpotent $\mathfrak{C}_1$ group. Choose any element $x \neq 1$ in G. Then $x \in D_x$ and we can choose a finite subset S_1 of G satisfying $x \in S_1$ and $x \in \{[x,s]: s \in S_1\}$. By induction let us assume the existence of a finite subset S_r of G satisfying $S_{r-1} \subseteq S_r$ and $S_{r-1} \subseteq \{[u,v]; u \in S_{r-1}, v \in S_r\}$. Since S_r is a finite set, and for every $v \in S_r$ there is a finite set T_v satisfying $v \in T_v$ and $v \in \{[v,t]: t \in T_v\}$, the union S_{r+1} of the sets T_v, $v \in S_r$ is finite, $S_r \subseteq S_{r+1}$ and $S_r \subseteq \{[v,w]: v \in S_r, w \in S_{r+1}\}$. Our construction of the sets S_r ensures that $< S_r >$ is not r-step polynilpotent. To begin with, $< S_1 >$ is not nilpotent since $1 \neq x \in \{[x,s]: s \in S_1\}$. Assume that $< S_{r-1} >$ is not (r-1)-step polynilpotent. Let M be the normal closure of S_{r-1} in $< S_r >$, so that M is not (r-1)-step polynilpotent. If $H = < S_r >$ is r-step polynilpotent, then there exists a subgroup $L \lhd H$ with H/L nilpotent and L (r-1)-step polynilpotent. Since M is not (r-1)-step polynilpotent, $1 \neq LM/L \leq H/L$. But $M = [M, H]$ and hence H/L is not nilpotent, a contradiction. Thus we conclude that $< S_r >$ is not r-step polynilpotent, and by choosing $r > k$ we get the required contradiction.

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