

## EXPECTED NUMBER OF DEPARTURES IN $M/M/1$ AND $GI/G/1$ QUEUES

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### Abstract

For an initially empty  $M/M/1$  queue, it is shown that the transform of the expectation of the number of departures in the interval  $(0, t]$  is invariant under an interchange of arrival and service rates. However, in the  $GI/G/1$  queue with an initial single customer, the corresponding transform does not have this symmetric property.

TRANSFORM: EXPECTATION; DEPARTURE PROCESS

### 1. Introduction

A result of Takács (1962) is used to derive an expression for the Laplace–Stieltjes transform of the expected number of departures, in the interval  $(0, t]$ , for the  $M/M/1$  queue. It is then shown that for an initially empty queue, this expression is invariant under an interchange of arrival and service rates. This result is similar to that of Hubbard et al. (1986), who have shown that the probability of  $j$  departures in  $(0, t]$  has this invariant property. However, if the queue is not initially empty, this property does not hold.

As a partial generalization of this result, a duality theorem of Ali (1970) is invoked to show that for the  $GI/G/1$  queue, with a single customer initially, this invariant property again does not hold.

### 2. Departures from the $M/M/1$ queue

Consider the  $M/G/1$  queue with arrival rate  $\lambda$ . Let  $N(t)$  be the number of departures in the interval  $(0, t]$ . Theorem 16 of Takács (1962) states that

$$(2.1) \quad m(s) = \int_0^\infty e^{-st} dN(t) = \frac{\psi(s)}{1 - \psi(s)} \left\{ 1 - \frac{sU_0(\gamma(s))}{s + \lambda(1 - \gamma(s))} \right\}, \quad \text{Re}(s) > 0,$$

where  $z = \gamma(s)$  is the root of the equation

$$z = \psi(s + \lambda(1 - z))$$

with the smallest modulus,  $U_0(z) = E[z^{\xi(0)}]$  is the probability generating function of the initial queue size, and  $\psi(s)$  is the Laplace–Stieltjes transform of the service time distribution.

For the  $M/M/1$  queue with service rate  $\mu$  and  $\xi(0) = i$ ,  $\psi(s) = \mu/(\mu + s)$  and  $U_0(\gamma(s)) = [\gamma(s)]^i$ , so that (2.1) reduces to

$$(2.2) \quad m(s) = \frac{\mu}{s} \left( 1 - \frac{s[\gamma(s)]^i}{s + \lambda(1 - \gamma(s))} \right)$$

where  $\gamma(s)$  is the unique solution in  $|z| = 1$  satisfying

$$(2.3) \quad \lambda z^2 - (\lambda + \mu + s)z + \mu = 0.$$

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Case (a).  $i = 0$ .

From (2.2), we have

$$(2.4) \quad m(s) = \frac{\lambda\mu(1 - \gamma(s))}{s[s + \lambda(1 - \gamma(s))]}$$

where  $\gamma(s)$  is defined as above. It will be shown that (2.4) is unaltered by interchanging  $\lambda$  and  $\mu$ . This is equivalent to showing that

$$(2.5) \quad \frac{\lambda\mu(1 - \gamma(s))}{s[s + \lambda(1 - \gamma(s))]} = \frac{\lambda\mu(1 - \delta(s))}{s[s + \mu(1 - \delta(s))]}$$

where  $\delta(s)$  is the unique root in  $|z| = 1$  satisfying

$$(2.6) \quad \mu^2 - (\lambda + \mu + s)t + \lambda = 0.$$

It is clear that the roots of (2.3) and (2.6) are reciprocals of each other. Let the second root of (2.3) be  $\eta(s)$ , then

$$(2.7) \quad \gamma(s) + \eta(s) = (\lambda + \mu + s)/\lambda, \quad \gamma(s)\eta(s) = \mu/\lambda.$$

Further since  $\eta(s)$  is the unique root of (2.3) outside  $|z| = 1$ , we must have

$$(2.8) \quad \delta(s)\eta(s) = 1.$$

From (2.7) and (2.8), we obtain the following identities:

$$(2.9) \quad \begin{cases} \delta(s) - \gamma(s) = (\lambda - \mu)\delta(s)/\lambda \\ 1 + \delta(s)\gamma(s) = (\lambda + \mu + s)\delta(s)/\lambda \\ \delta(s) + \gamma(s) = (\lambda + \mu)\delta(s)/\lambda \end{cases}$$

whence we can readily show that (2.5) holds, so that the invariant result is established for  $i = 0$ .

Case (b).  $i > 0$ .

In this case

$$m(s) = \frac{\mu}{s} \left( 1 - \frac{s[\gamma(s)]^i}{s + \lambda(1 - \gamma(s))} \right) = \frac{\mu(1 - [\gamma(s)]^i)}{s + \lambda(1 - \gamma(s))} + \frac{\lambda\mu(1 - \gamma(s))}{s + [s + \lambda(1 - \gamma(s))]}$$

Since the second term of the right-hand side is invariant, the first term cannot be and we have thus proved the following result.

*Theorem 1.* For the  $M/M/1$  queue, with  $i$  initial customers,  $m(s)$  is invariant under an interchange of arrival and service rates only for  $i = 0$ .

### 3. Departures from the $GI/G/1$ queue

For the  $GI/G/1$  queue, let the customers  $C_0, C_1, \dots$ , arrive at times  $T_0, T_1, \dots$ , and wait for times  $W_0 = 0, W_1, \dots$ , respectively before entering service. If  $R_n$  is the instant of serving of  $C_n$ , then  $R_n = T_n + W_n$ . It is shown in Ali (1970) that the distribution of  $R_n$  is symmetric with respect to the distributions of the interarrival and service times.

Let  $S_0, S_1, \dots$ , denote the service times of  $C_0, C_1, \dots$  respectively. If  $T'_n$  is the instant of departure of  $C_n$ , then  $T'_n = R_n + S_n$ . Clearly the distribution of  $T'_n$  cannot have the same symmetric property as that of  $R_n$ . Further if, as before,  $N(t)$  is the expected number of departures in  $(0, t]$ , renewal theory arguments indicate that

$$N(t) = \sum_{n=0}^{\infty} P(T'_n \leq t).$$

It follows then that  $m(s) = \int_0^{\infty} e^{-st} dN(t)$  is also not symmetric with respect to the distribution of the interarrival and service times. If we bear in mind that  $C_0$  represents the single initial customer, we have the following result.

**Theorem 2.** For the  $GI/G/1$  queue, with a single initial customer,  $m(s)$  is not symmetric with respect to the distributions of the interarrival and service times.

This generalizes the equivalent result for the  $M/M/1$  queue with  $i = 1$ .

**Corollary.** Result (b) of Theorem 1 with  $i = 1$  is a special case of Theorem 2.

#### 4. Comments

The joint distribution of the number of arrivals (number in the system) and departures in the  $M/M/1$  queue are discussed not only by Hubbard et al. (1986) but also in two other papers.

Greenberg and Greenberg (1966) derive the joint distribution of the number in the system at time  $t$  and the number of departures in  $(0, t]$ , assuming  $i(\geq 0)$  customers initially. This is then used to determine the distribution of the number of departures in  $(0, t]$ . In Boxma (1984) the joint distribution of the number of arrivals and departures in  $(0, t]$ , assuming  $x(\geq 0)$  customers initially, is found. From it, the distribution of the number of departures in  $(0, t]$  is then obtained.

The result of Hubbard et al. (1986), who assume an empty queue initially, is a special case of both Greenberg and Greenberg (1966) and Boxma (1984). In particular equation (5) in the first paper is equivalent to equation (41) in the second with  $i = 0$ , and equation (17) in the third with  $x = 0$ .

A similar equivalence of the three results relating to the distribution of the number of departures can also be established. By setting  $x = 0$  in equation (27) of Boxma (1984), it can be shown that the distribution of the number of departures in  $(0, t]$  is invariant under an interchange of  $\lambda$  and  $\mu$ . Boxma (1984) has not pointed out this invariance property explicitly.

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