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HOMEOMORPHISM AND ISOMORPHISM OF ABELIAN GROUPS

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An abelian topological group can be considered simply as an abelian group or as a topological space. The question considered in this article is whether the topological group structure is determined by these weaker structures. Denote homeomorphism, isomorphism, and homeomorphic isomorphism by $\approx$, $\cong$, and $=$, respectively. The principal results are these.

THEOREM 1. *If G_1 and G_2 are locally compact and connected, then $G_1 \approx G_2$ implies $G_1 = G_2$.*

THEOREM 2. (a) *There are compact connected G_1 and G_2 with $G_1 \cong G_2$ and $G_1 \neq G_2$.*

(b) *There are connected G_1 and G_2 with $G_1 \approx G_2$, $G_1 \cong G_2$, and $G_1 \neq G_2$.*

(c) *There are compact G_1 and G_2 with $G_1 \approx G_2$, $G_1 \cong G_2$, and $G_1 \neq G_2$.*

1. Proof of Theorem 1. The structure theorem for locally compact abelian groups (Chapter 2 of [**5**]) gives $G_i = \mathbf{R}^{n_i} \times K_i$, where n_i are non-negative integers and K_i are compact connected abelian groups. It is known that $H^1(G_i) \cong H^1(K_i) \cong \hat{K}_i$, where $\hat{K}_i$ is the Pontryagin dual group of K_i and the cohomology is Čech cohomology. The isomorphism $H^1(G_i) \cong H^1(K_i)$ is clear, since K_i is a deformation retract of G_i. The isomorphism $H^1(K_i) \cong \hat{K}_i$ is obtained in this manner (see [**1**, Chapters VIII–X]). $\hat{K}_i$ is discrete and torsion free, since K_i is compact and connected. Write $\hat{K}_i$ as a direct limit of its finitely generated subgroups, each of which is isomorphic to $\mathbf{Z}^n$, for some n. Then $K_i = (\hat{K}_i)\hat{}$ = the inverse limit of various tori $T^n (= \widehat{\mathbf{Z}^n} = (\hat{Z})^n)$. Since Čech cohomology is continuous on inverse limits and since $H^1(T)$ is naturally isomorphic to $\mathbf{Z}$, it follows that $H^1(K_i) \cong \hat{K}_i$. Therefore, $G_1 \approx G_2$ implies $\hat{K}_1 \cong \hat{K}_2$, which implies $K_1 = K_2$. The proof will be completed by a proof that $n_1 = n_2$, which is an immediate consequence of the next proposition.

PROPOSITION. *If K is a non-empty compact space and $P = \mathbf{R}^n \times K$, then the integer n is a topological invariant of P.*

Proof. R. J. Milgram kindly provided me with a proof (oral) that n is the smallest dimension for which the homology of $\bar{P} = P \cup (\infty)$ is non-vanishing. The proof below is based on showing this to be true for singular homology, but by a different method.

There is a projection π of $\bar{P}$ onto $S^n = \mathbf{R}^n \cup (\infty) : \pi(\infty) = \infty, \pi(x, k) = x$.

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Fix $k \in K$ and define $i : S^n \to \bar{P}$ by $i(\infty) = \infty$, $i(x) = (x, k)$ for $x \in \mathbf{R}^n$. Since πi is the identity on S^n, $\pi_* i_*$ is the identity on $H_n(S^n) = \mathbf{Z}$; in particular, $H_n(\bar{P}) \neq 0$.

To show that $H_m(\bar{P}) = 0$ for $m < n$, it is sufficient to consider $n \geqq 1$ and to show that every $f : S^m \to \bar{P}$, $m < n$, is homotopic to the constant map ∞. Define $C = f^{-1}(\infty) \subseteq S^m$ and $g = \pi f$. There is a continuous $h : S^m - C \to K$ such that $f(x) = (g(x), h(x))$ for $x \notin C$. It is sufficient to show that $g \simeq \infty$ relative to C, that is, that there is a continuous $G(x, t)$ for which $G(x, 0) = g(x)$, $G(x, 1) = \infty$, and $G(x, t) = \infty$ when $g(x) = \infty$. For then a homotopy $f \simeq \infty$ is given by $F(x, t) = (G(x, t), h(x))$ for $x \notin C$ and $F(x, t) = \infty$ for $x \in C$. Reducing the problem further, it is enough to find a homotopy of g, relative to C, to a map $g' : S^m \to S^n$ whose range omits a point of $\mathbf{R}^n$, since ∞ is a deformation retract of $S^n - x$, for any $x \in \mathbf{R}^n$.

Triangulate S^m and define L to be the subcomplex consisting of all simplices σ which meet C, together with all faces of such σ. Let M be the subcomplex of all simplices which do not meet C, together with all their faces. Then $S^m = |L| \cup |M|$ and C is contained in the interior of $|L|$. We can assume the triangulation is so fine that $x \in |L| \Rightarrow |g(x)| > 2$, where $|\infty| = \infty$. Since $g(|M|)$ is compact in $\mathbf{R}^n$, we can subdivide M sufficiently so that if σ is any simplex of the subdivision N, $g(|\sigma|)$ has diameter less than 1.

Define $g' : S^m \to R^n \cup (\infty)$ as follows. Let $g'(x) = g(x)$ for every $x \in |L|$ and every vertex x of N. For any other x choose a simplex σ of N containing x; let ρ be the largest face of σ contained in $|L|$ and τ be the complementary face. By appropriately numbering the vertices of σ, we can write

$$\sigma = \langle v_0, \ldots, v_p \rangle, \qquad \tau = \langle v_0, \ldots, v_q \rangle, \qquad \rho = \langle v_{q+1}, \ldots, v_p \rangle.$$

Let $(\lambda_0, \ldots, \lambda_p)$ be the barycentric coordinates of x. If ρ is empty, put $g'(x) = \lambda_0 g(v_0) + \ldots + \lambda_p g(v_p)$. Otherwise, let $\lambda = \lambda_0 + \ldots + \lambda_q < 1$ and let y be the point of ρ with coordinates $(\lambda_{q+1}/(1 - \lambda), \ldots, \lambda_p/(1 - \lambda))$. Define

$$g'(x) = \frac{\lambda_0}{\lambda} g(v_0) + \ldots + \frac{\lambda_q}{\lambda} g(v_q) + (1 - \lambda) g(y).$$

It is easy to see that g' is continuous. A homotopy $g \simeq g'$, relative to C, is given by $H(x, t) = \infty$ for $x \in C$, $H(x, t) = tg'(x) + (1 - t)g(x)$ for $x \notin C$.

If $x \in |L|$, $|g'(x)| = |g(x)| > 2$. If σ is a simplex of N, then $g'(\sigma)$ has diameter less than 1, being contained in the convex hull of $g(|\sigma|)$. If $|\sigma|$ meets $|L|$, this means $|g'(x)| > 1$ for all $x \in |\sigma|$. On the rest of S^m, which is the rest of N, g' is piecewise linear; so its range is contained in a finite number of m-dimensional subspaces of $\mathbf{R}^n$. Therefore, $g'(S^m)$ omits a point of $\mathbf{R}^n$ of norm less than 1.

2. Proof of Theorem 2. Let $\mathbf{Q}$ be the rational numbers with the discrete topology. $\hat{\mathbf{Q}}$, the dual of $\mathbf{Q}$, is compact, connected, and torsion free, since $\mathbf{Q}$ is discrete, torsion free, and has no subgroup of finite index. Since $\mathbf{Q}$ is the direct limit of groups isomorphic to $\mathbf{Z}$, $\hat{\mathbf{Q}}$ is the inverse limit of groups isomorphic to the circle T. Since T is divisible, $\hat{\mathbf{Q}}$ is divisible. Thus, $\hat{\mathbf{Q}}$ is a torsion-free divisible group containing $c(=2^{\aleph_0})$ elements. Hence, $\hat{\mathbf{Q}}$ is isomorphic to the c-dimensional vector space over $\mathbf{Q}$. It immediately follows that $\hat{\mathbf{Q}} \cong \hat{\mathbf{Q}} \oplus \hat{\mathbf{Q}}$. By Theorem 1 $\hat{\mathbf{Q}} \not\approx \hat{\mathbf{Q}} \oplus \hat{\mathbf{Q}}$ since the dual groups $\mathbf{Q}$ and $\mathbf{Q} \oplus \mathbf{Q}$ are not isomorphic. This proves 2(a).

For 2(b) consider $G_1 = L^1(0, 1)$ and $G_2 = L^2(0, 1)$. If G_1 were equal to G_2 as topological groups, then they would be equal as rational vector spaces, since they are torsion free. Continuity would then yield $G_1 = G_2$ as real topological vector spaces, which is obviously false. However, $G_1 \cong G_2$ as real vector spaces, hence as groups. And $G_1 \approx G_2$ by a theorem of Mazur [**4**].

For 2(c) it will be convenient to have a summary of the basic theory of p-adic groups. Details can be found in [**3**], especially § 16. A subgroup H of G is *pure* if and only if for every $n > 0$ every element of H which can be divided by n in G can already be divided by n in H. A subset is pure if and only if the subgroup generated by it is pure. A subset S of G is *independent* if and only if for every $T \subseteq S$ the subgroups generated by T and by $S - T$ have only 0 in common.

Fix a prime number p. The *p-adic topology* on G is obtained by letting $\{p^n G : n \geqq 0\}$ be the system of basic neighborhoods of 0. This topology is metrizable in the following manner. Define

$$h(x) = \sup\{n : x \text{ can be divided by } p^n \text{ in } G\},$$

called the *height* of x in G; p^nG is the set of all elements of height at least n. The distance function $d(x, y) = [1 + h(x - y)]^{-1}$ induces the p-adic topology. From now on assume G has no elements of infinite height; then d is a genuine metric. In fact, d is non-Archimedean: $d(x, y) \leqq \max\{d(x, y), d(y, z)\}$; so g_n is Cauchy if and only if $d(g_n, g_{n+1}) \to 0$. If H is pure in G, then $p^nH = H \cap p^nG$; so the inclusion $H \subseteq G$ is a homeomorphic isomorphism.

Let G^* be the abstract completion of G as a metric group. G^* has no elements of infinite height, and G is pure in G^*. Thus, the p-adic metric on G^* coincides with the natural extension of the metric on G, and the inclusion $G \subseteq G^*$ is an isometric isomorphism. More generally, if G is a pure subgroup of a complete group C, then the inclusion $G \subseteq C$ extends to a (unique) homeomorphic isomorphism of G^* onto $\bar{G}$, the closure of G in C. Since d is non-Archimedean, every series $\sum_0^\infty p^j x_j$ converges in C. The closure of G consists of all sums $\sum_0^\infty p^j g_j$, $g_j \in G$.

The *p-adic integers* $\mathbf{Z}_p$ is obtained as the completion of $\mathbf{Z}$ and is a ring since $\mathbf{Z}$ is a ring. $\mathbf{Z}_p$ is homeomorphic to Cantor's middle-third set (see [**2**, § 2–15]), and $\hat{\mathbf{Z}}_p$ is $\mathbf{Z}(p^\infty)$, the subgroup of the circle consisting of all the (p^n)th roots of unity, for all n. It is often convenient to express $\mathbf{Z}_p$ as the set of all formal sums $\sum_0^\infty a_j p^j$, $0 \leqq a_j < p$, with addition and multiplication as for the integers (finite sums) in base p. A p-adically complete group is natural $\mathbf{Z}_p$-module, via

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$(\sum a_j p^j)x = \sum a_j(p^j x)$. Note that

$$-1 = (p-1)(1-p)^{-1} = (p-1)(1+p+p^2+\ldots).$$

Let G be complete and S be a maximal pure independent subset; that is, no subset of G properly containing S is both pure and independent. Then the subgroup H generated by S is dense in G; so $G \cong H^*$. Furthermore, if we partition S as $S_1 \cup S_2$ and let G_i be the closed subgroup of G generated by S_i, then $G = G_1 \oplus G_2$, an internal direct sum. That is, disjoint pure subgroups have disjoint closures.

LEMMA. *Let $G = \Pi_1^\infty G_j$, where each G_j is either $\mathbf{Z}_p$ or $\mathbf{Z}(p^n)$, the cyclic group of order p^n, for some n; assume that the order of G_j tends to ∞ as $j \to \infty$. Define $e_n \in G$ by $e_n(j) = 1$ for $j = n$ and 0 for $j \neq n$. Then $S_0 = \{e_n : n \geqq 1\}$ is pure and independent. Let S be a maximal pure independent set containing S_0; let H and K be the subgroups generated by S_0 and $S - S_0$, respectively. Then*

(1) *$\bar{H}$, the p-adic closure of H, contains the torsion subgroup of G;*
(2) *$K \cong \sum_c \oplus \mathbf{Z}$, the (weak) direct sum of c copies of $\mathbf{Z}$;*
(3) *$G = \bar{H} \oplus \bar{K}$, an internal direct sum;*
(4) *$\bar{K} \cong K^* \cong \mathbf{Z}_p^{\aleph_0}$.*

Proof. S_0 is clearly pure and independent. Since G is complete, we know by the general theory that $H^* \cong \bar{H}$, $K^* \cong \bar{K}$, and $\bar{H} \oplus \bar{K} = G$. G_n is the closed subgroup generated by e_n; therefore,

$$\bar{H} = \overline{\sum \oplus G_j} = \left\{ g = \sum_0^\infty p^n g_n, g_n \in \sum \oplus G_j \right\}.$$

It is easy to see that this last group is

$$\{g : h(g(j)) \to \infty \text{ as } j \to \infty\}.$$

If $mg = 0$, then $g(j) = 0$ when $G_j = \mathbf{Z}_p$ and $mg(j)$ is divisible p^n when $G_j = \mathbf{Z}(p^n)$. Since $n \to \infty$ as $j \to \infty$, $g \in \bar{H}$.

Since $\bar{K}$ is independent of $\bar{H}$, $\bar{K}$ is torsion free. Therefore, $K \cong \sum \oplus \mathbf{Z}$, with the number of copies of $\mathbf{Z}$ being the cardinal of $S - S_0$. Since S is infinite, $H + K$ has the same cardinal as S. Since $H + K$ is dense in G, $(H+K)/pG$ is dense in G/pG, which is discrete. Therefore $(H+K)/pG = G/pG$. Now it is easy to see that $G/pG \cong \mathbf{Z}(p)^{\aleph_0} \cong \sum_c \oplus \mathbf{Z}(p)$, the latter isomorphism being a $\mathbf{Z}(p)$-vector space isomorphism. Therefore there are c elements in $H + K$, hence in S, hence in $S - S_0$.

We already know $\bar{K} \cong K^*$, so we must finally see that $K^* \cong \mathbf{Z}_p^{\aleph_0}$. Apply the foregoing discussion to the case where every $G_j = \mathbf{Z}_p$. Then

$$H \cong \sum_{\aleph_0} \oplus \mathbf{Z} \quad \text{and} \quad K \cong \sum_c \oplus \mathbf{Z};$$

thus, $K \cong K + H$. So $K^* \cong (K+H)^* \cong \overline{(K+H)} = \bar{K} + \bar{H} = G = \mathbf{Z}_p^{\aleph_0}$.

The proof of Theorem 2c can now be completed. Let $G_1 = \Pi_1^\infty \mathbf{Z}(p^n)$ with the product topology and $G_2 = G_1 \oplus \mathbf{Z}_p$. G_1 and G_2 are each homeomorphic to Cantor's middle-third set (see [**2**, § 2–15]). By the lemma $G_1 \cong \bar{H} \oplus \mathbf{Z}_p^{\aleph_0}$; so

$$G_2 = G_1 \oplus \mathbf{Z}_p \cong \bar{H} \oplus \mathbf{Z}_p^{\aleph_0} \oplus \mathbf{Z}_p \cong \bar{H} \oplus \mathbf{Z}_p^{\aleph_0} \cong G_1.$$

However, $\hat{G}_1 = \sum \oplus \mathbf{Z}(p^n)$ and $\hat{G}_2 = \hat{G}_1 \oplus \mathbf{Z}(p^\infty)$. $\mathbf{Z}(p^\infty)$ is a divisible subgroup of $\hat{G}_2$, and $\hat{G}_1$ has no divisible elements other than 0. Hence $\hat{G}_1 \ncong \hat{G}_2$; so $G_1 \ncong G_2$.

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