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THE WEIGHTED TURÁN TYPE INEQUALITY FOR GENERALISED JACOBI WEIGHTS

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We study the weighted Turán type inequality for generalised Jacobi weights, and give a complete positive answer to Zhou's conjecture.

Let H_n be the class of real algebraic polynomials of degree n, whose zeros all lie in the interval $[-1,1]$. Define

$$\|f\| = \max_{-1\leqslant x\leqslant 1}|f(x)|,$$

$$\|f\|_{L^p} = \left(\int_{-1}^{1}|f(x)|^p\,dx\right)^{1/p}, \quad 0 < p < \infty.$$

In 1939, Turán [**5**] established an inequality which was later referred as Turán's inequality. Precisely, Turán proved that for $f \in H_n$, $\|f'\| \geqslant C\sqrt{n}\|f\|$.

This inequality was studied quite extensively (interested readers could find useful information in a survey paper [**4**]). It was generalised to L^p spaces, $0 < p < \infty$ (see [**6, 7, 8, 10, 11, 12**]), and its optimal constants were estimated (see [**1, 2, 3, 6, 7, 8**]).

Note the following fact: If Φ_n is an orthogonal polynomial system on $[-1,1]$ with respect to a weight function $W(x)$, then all zeros of any function in Φ_n lie in the interval $(-1,1)$. To consider the potential application of Turán type inequality to orthogonal polynomial systems, we first should generalise it to the weighted case. For this reason, Zhou in[**9**] raises the following conjecture.

CONJECTURE. Let $f \in H_n$, then for $0 < p < \infty$ and some important weight functions $W(x)$ the inequality

$$\|f'W\|_{L^p} \geqslant C_W\sqrt{n}\|fW\|_{L^p}$$

holds for sufficiently large n, where the constant $C_W > 0$ depends upon $W(x)$ and p (in case $p \to 0$) only.

Xiao and Zhou [**9**] considered some general weight functions and the uniform norm to established the following

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THEOREM 1. *Let $W(x)$ be a nonnegative continuous piecewise monotone function on the interval $[-1,1]$. If $f \in H_n$, then there exists a positive constant C_W only depending upon $W(x)$ such that*

$$\|f'W\| \geqslant C_W\sqrt{n}\|fW\|$$

holds for sufficiently large n.

This paper will study the weighted Turán type inequality in general L^p spaces for $0 < p < \infty$. With a quite delicate approach, we shall prove the conjecture for a large group of important weights including Jacobi weights.

In the sequel, we always assume that a weight function $W(x) \geq 0$ satisfies $\int_{-1}^{1} W(x)\,dx < \infty$ and $W(x_1) \approx W(x_2)$ for any $-1 < x_1 < x_2 \leq 0$ and $|x_2-x_1| < 1+x_1$ or for any $0 < x_2 < x_1 \leq 1$ and $|x_2 - x_1| < 1 - x_1$. Such a function is called a Generalised Jacobi Weight. ($W(x_1) \approx W(x_2)$ means that there is a constant $M \geq 1$ (M depends on $W(x)$) such that $M^{-1}W(x_1) \leq W(x_2) \leq MW(x_1)$.)

We see that if $W(x)$ is a Generalised Jacobi Weight, then $W(x)$ may only have a zero or infinity point at the endpoints ± 1. It is easy to check that all Jacobi weights $W(x) = (1+x)^{\alpha}(1-x)^{\beta}$, $\alpha, \beta > -1$, are Generalised Jacobi Weights.

Now we give our main theorem.

THEOREM 2. *Let $W(x)$ be a Generalised Jacobi Weight, and let $0 < p < \infty$. If $f \in H_n$, then there exists a positive constant $C_{W,p}$ only depending upon $W(x)$ and p (in case $p \to 0$) such that, for n sufficiently large*

$$\left(\int_{-1}^{1} |f'(x)|^p W(x)\,dx\right)^{1/p} \geqslant C_{W,p}\sqrt{n}\left(\int_{-1}^{1} |f(x)|^p W(x)\,dx\right)^{1/p}. \tag{1}$$

Denote by $-1 \leqslant x_1 < x_2 < \cdots < x_k \leqslant 1$ all the distinct zeros of $f \in H_n$ and by l_i the multiplicity of x_i, $1 \leqslant i \leqslant k$. Let α_j be the maximum point of $|f(x)|$ between (x_j, x_{j+1}), $1 \leqslant j < k$. Obviously, for $x \in [x_j, \alpha_j]$ (or $x \in [\alpha_j, x_{j+1}]$) $f(x)$ is increasing (or decreasing). Set

$$m(x) = \frac{f'(x)}{f(x)} = \sum_{i=1}^{k} \frac{l_i}{x - x_i},$$

$$d_j = |m'(\alpha_j)|^{-1}.$$

For $x \in [-1,1]$, it is easy to show that

$$|m'(x)| = \sum_{i=1}^{k} \frac{l_i}{(x-x_i)^2} \geqslant \frac{n}{4}, \tag{2}$$

$$|m'(x)| \geqslant (x - x_i)^{-2}, \quad 1 \leqslant i \leqslant k,$$

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so that

$$\sqrt{d_j} \leqslant \min_{1\leqslant i\leqslant k}\{|\alpha_j - x_i|, 2n^{-1/2}\}. \tag{3}$$

We estimate $m(x)$. In the sequel, assume all the inequalities hold for sufficiently large n if not specified.

Lemma 1. *If $x \in \left(x_j, \alpha_j - \left(\sqrt{d_j}/8\right)\right] \cup \left[\alpha_j + \left(\sqrt{d_j}/8\right), x_{j+1}\right)$, then*

$$\left|m(x)\right| \geqslant \frac{2}{25}\frac{1}{\sqrt{d_j}}, \tag{4}$$

and if $x \in \left[\alpha_j - \left(\sqrt{d_j}/4\right), \alpha_j + \left(\sqrt{d_j}/4\right)\right]$, then

$$\left|m(x)\right| \leqslant \frac{4}{9}\frac{1}{\sqrt{d_j}}. \tag{5}$$

Proof: For $x \in \left[\alpha_j - \left(\sqrt{d_j}/4\right), \alpha_j + \left(\sqrt{d_j}/4\right)\right]$, $i = 1, 2, \cdots, k$, from (3) we have

$$\frac{3}{4}|x_i - \alpha_j| \leqslant |x_i - \alpha_j| - \left(\sqrt{d_j}/4\right) \leqslant |x_i - x| \leqslant |x_i - \alpha_j| + \frac{\sqrt{d_j}}{4} \leqslant \frac{5}{4}|x_i - \alpha_j|,$$

thus by summing up all the terms we get

$$\frac{16}{25}\left|m'(\alpha_j)\right| \leqslant \left|m'(x)\right| \leqslant \frac{16}{9}\left|m'(\alpha_j)\right|.$$

Because $m'(x) < 0$, $m(\alpha_j) = 0$, we have

$$\left|m\left(\alpha_j \pm \frac{\sqrt{d_j}}{4}\right)\right| = \left|\int_{\alpha_j \pm \left(\sqrt{d_j}/4\right)}^{\alpha_j} m'(x)\,dx\right| \leqslant \frac{16}{9}\left|m'(\alpha_j)\right|\frac{\sqrt{d_j}}{4} = \frac{4}{9}d_j^{-1/2},$$

$$\left|m\left(\alpha_j \pm \left(\sqrt{d_j}/8\right)\right)\right| = \left|\int_{\alpha_j \pm \left(\sqrt{d_j}/8\right)}^{\alpha_j} m'(x)\,dx\right| \geqslant \frac{16}{25}\left|m'(\alpha_j)\right|\frac{\sqrt{d_j}}{8} = \frac{2}{25}d_j^{-1/2}.$$

Noting that for $x \in (x_j, x_{j+1})$, $m(x)$ is decreasing and $m(\alpha_j) = 0$, from the above inequalities we obtain Lemma 1. ∎

We divide the proof of Theorem 2 into the following three lemmas.

Lemma 2. *For $j = 1, 2, \cdots, k-1$, we have*

$$\int_{x_j}^{\alpha_j} \left|f'(x)\right|^p W(x)\,dx \geqslant \frac{1}{2M^2}\left(\frac{1}{50}\right)^p n^{p/2} \int_{x_j}^{\alpha_j} \left|f(x)\right|^p W(x)\,dx,$$

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where $M \geq 1$ is the constant appearing in the definition of Generalised Jacobi Weights.

PROOF: From (4) we have

$$\int_{x_j}^{\alpha_j-(\sqrt{d_j}/4)} |f'(x)|^p W(x)\,dx = \int_{x_j}^{\alpha_j-(\sqrt{d_j}/4)} |f(x)|^p |m(x)|^p W(x)\,dx$$
$$\geqslant \Bigl(\frac{2}{25}\sqrt{d_j}^{-1}\Bigr)^p \int_{x_j}^{\alpha_j-(\sqrt{d_j}/4)} |f(x)|^p W(x)\,dx$$
$$\geqslant \Bigl(\frac{1}{25}\Bigr)^p n^{p/2} \int_{x_j}^{\alpha_j-(\sqrt{d_j}/4)} |f(x)|^p W(x)\,dx. \quad \text{(by (2))} \tag{6}$$

For the interval $\bigl[\alpha_j - (\sqrt{d_j}/4), \alpha_j\bigr]$ we consider three cases.

CASE 1. $\alpha_j \leqslant 0$.

In this case, for $x \in \bigl[\alpha_j - (\sqrt{d_j}/4), \alpha_j\bigr] \subset [x_j, 0]$, we see that $0 \leqslant \alpha_j - x \leqslant (\sqrt{d_j}/4) \leqslant \alpha_j - (\sqrt{d_j}/4) - x_j \leq x + 1$ (by (3)), thus have $W(x) \approx W(\alpha_j)$.

SUBCASE 1.1. $\bigl|f\bigl(\alpha_j - (\sqrt{d_j}/4)\bigr)\bigr| \geqslant |f(\alpha_j)|/2$.

Then, for $x \in \bigl[\alpha_j - (\sqrt{d_j}/4), \alpha_j\bigr]$, $|f(x)| \geqslant |f(\alpha_j)|/2$, applying (4), in a similar way to (6) we have

$$\int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j-(\sqrt{d_j}/8)} |f'(x)|^p W(x)\,dx \geq \int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j-(\sqrt{d_j}/8)} |f(x)|^p |m(x)|^p W(x)\,dx$$
$$\geq \frac{1}{M}\Bigl(\frac{1}{50}\Bigr)^p n^{p/2} |f(\alpha_j)|^p W(\alpha_j) \frac{\sqrt{d_j}}{8}$$
$$\geq \frac{1}{2M^2}\Bigl(\frac{1}{50}\Bigr)^p n^{p/2} \int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j} |f(x)|^p W(x)\,dx. \tag{7}$$

SUBCASE 1.2. $\bigl|f\bigl(\alpha_j - (\sqrt{d_j}/4)\bigr)\bigr| < |f(\alpha_j)|/2$.

We first assume $0 < p < 1$. Now that $p - 1 < 0$, from (5) we have

$$\int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j} |f'(x)|^p W(x)\,dx$$
$$= \int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j} |f(x)|^{p-1} |m(x)|^{p-1} |f'(x)| W(x)\,dx$$
$$\geqslant \Bigl(\frac{4}{9}\sqrt{d_j}^{-1}\Bigr)^{p-1} \frac{W(\alpha_j)}{M} |f(\alpha_j)|^{p-1} \int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j} |f'(x)|\,dx$$

$$\geqslant \left(\frac{4}{9}\sqrt{d_j}^{-1}\right)^{p-1} \frac{W(\alpha_j)}{M}\left|f(\alpha_j)\right|^{p-1}\left|f(\alpha_j) - f\left(\alpha_j - \frac{\sqrt{d_j}}{4}\right)\right|$$

$$\geq \frac{9}{2}\left(\frac{4}{9}\sqrt{d_j}^{-1}\right)^{p} M^{-1}|f(\alpha_j)|^p W(\alpha_j)\frac{\sqrt{d_j}}{4}$$

$$(8) \qquad \geqslant M^{-2}\left(\frac{2}{9}\right)^p n^{p/2}\int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j}|f(x)|^p W(x)\,dx.$$

In case $1 \leq p < \infty$, by Hölder's inequality we have

$$\int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j}|f'(x)|^p W(x)\,dx \geqslant \frac{1}{M}W(\alpha_j)\int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j}|f'(x)|^p\,dx$$

$$\geqslant \frac{1}{M}W(\alpha_j)\left(\frac{\sqrt{d_j}}{4}\right)^{-p+1}\left(\int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j}|f'(x)|\,dx\right)^p$$

$$\geqslant \frac{1}{M}\left(\frac{\sqrt{d_j}}{4}\right)^{-p+1}\left(\frac{|f(\alpha_j)|}{2}\right)^p W(\alpha_j)$$

$$\geqslant \frac{1}{M}n^{p/2}|f(\alpha_j)|^p W(\alpha_j)\frac{\sqrt{d_j}}{4}$$

$$(8') \qquad \geqslant \frac{1}{M}n^{p/2}\int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j}|f(x)|^p W(x)\,dx.$$

From (7), (8) and (8′), we obtain that

$$(9) \qquad \int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j}|f'(x)|^p W(x)\,dx \geqslant \frac{1}{2M^2}\left(\frac{1}{50}\right)^p n^{p/2}\int_{\alpha_j-(\sqrt{d_j}/4)}^{\alpha_j}|f(x)|^p W(x)\,dx.$$

CASE 2. $x_j \geqslant 0$.

In this case, $x \in \left[\alpha_j - (\sqrt{d_j}/4), \alpha_j\right] \subset [x_j, \alpha_j] \subset [0,1]$, thus $\alpha_j - x \leqslant (\sqrt{d_j}/4) \leqslant x_{j+1} - \alpha_j \leqslant 1 - \alpha_j$ (by (3)), so $W(x) \approx W(\alpha_j)$. With a similar way to Case 1, we reach (9) as well.

CASE 3. $x_j < 0 < \alpha_j$.

When $\alpha_j \leqslant (\sqrt{d_j}/4)$, we see that $\left[\alpha_j - (\sqrt{d_j}/4), \alpha_j\right] \subset \left[-(\sqrt{d_j}/4), (\sqrt{d_j}/4)\right] \subset \left[-(1/2\sqrt{n}), (1/2\sqrt{n})\right] \subset [-1/2, 1/2]$. From the definition of Generalised Jacobi Weights, for $x \in \left[\alpha_j - (\sqrt{d_j}/4), \alpha_j\right]$, we surely have $W(x) \approx W(\alpha_j)$.

When $\alpha_j > (\sqrt{d_j}/4)$, then $\left[\alpha_j - (\sqrt{d_j}/4), \alpha_j\right] \subset [0, \alpha_j] \subset [0, x_{j+1}] \subset [0,1]$. for $x \in \left[\alpha_j - (\sqrt{d_j}/4), \alpha_j\right]$, $\alpha_j - x \leqslant (\sqrt{d_j}/4) \leqslant x_{j+1} - \alpha_j < 1 - \alpha_j$ (see (3)), thus we also have the relation $W(x) \approx W(\alpha_j)$.

A similar argument to Case 1 leads to (9) in case 3.

Combining the conclusions of the above three cases with (6), we have finished the proof of Lemma 2. ∎

By the same technique, we have

Lemma 3. *For $j = 1, 2, \cdots, k-1$, we have*

$$\int_{\alpha_j}^{x_{j+1}} |f'(x)|^p W(x)\,dx \geqslant \frac{1}{2M^2}\left(\frac{1}{50}\right)^p n^{p/2} \int_{\alpha_j}^{x_{j+1}} |f(x)|^p W(x)\,dx.$$

If $-1 < x_1$ or $x_k < 1$, for $x \in [-1, x_1)$ or $x \in (x_k, 1]$, it is easy to see

$$|m(x)| = \left|\sum_{i=1}^{k} \frac{l_i}{x - x_i}\right| \geqslant n/2,$$

thus we have the next lemma.

Lemma 4. *If $f(-1) \neq 0$, then*

$$\int_{-1}^{x_1} |f'(x)|^p W(x)\,dx \geqslant \left(\frac{n}{2}\right)^p \int_{-1}^{x_1} |f(x)|^p W(x)\,dx;$$

if $f(1) \neq 0$, then

$$\int_{x_k}^{1} |f'(x)|^p W(x)\,dx \geqslant \left(\frac{n}{2}\right)^p \int_{x_k}^{1} |f(x)|^p W(x)\,dx.$$

From Lemma 2 to Lemma 4, the proof of the inequality (1) is completed. Therefore we have finished Theorem 2.

Remark 1. In fact, the constant $C_{W,p}$ in Theorem 2 can be taken as $\left(\left(2^{-1}M^{-2}\right)\right)^{1/p}\Big)/50$ for $0 < p < 1$, and $M^{-2}/100$ for $1 \leq p < \infty$.

Remark 2. With $f(x) = \left(1 - x^2\right)^{[n/2]}$, we see that the order $n^{1/2}$ in Theorem 2 can not be improved for Jacobi weights $(1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$.

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