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A DESCENT THEOREM FOR HERMITIAN K-THEORY

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1. Let KO and KU respectively denote the real and complex periodic K-theory spectra [**1**, Part III]. Let KSC denote the spectrum representing self-conjugate K-theory [**2**, G]. Thus we have a fibring

$$(1.1)\qquad KSC \to KU \xrightarrow{1-T} KU$$

where T is induced by complex conjugation on the unitary group.

The following result is due to R. Wood [**1**, p. 206] and, I believe, to D. W. Anderson.

1.2. PROPOSITION. *Let $\eta \in \pi_1^S(S^0) \simeq Z/2$ generate the stable one-stem. Then there are weak equivalences of spectra*

a) $KO \wedge \left(S^0 \underset{\eta}{\cup} e^2\right) \simeq KU$

$\left(\textit{note } \Sigma^2\left(S^0 \underset{\eta}{\cup} e^2\right) \simeq \mathbf{C}P^2\right)$ *and*

b) $KO \wedge \left(S^0 \underset{\eta^2}{\cup} e^3\right) \simeq KSC.$

Proof. The reduced Hopf bundle on $\mathbf{C}P^2$ gives

$$h:S^0 \underset{\eta}{\cup} e^2 \to KU.$$

Let $c:KO \to KU$ denote complexification and form

$$\Phi:KO \wedge \left(S^0 \underset{\eta}{\cup} e^2\right) \xrightarrow{c \wedge h} KU \wedge KU \xrightarrow{\text{mult}} KU.$$

The following diagram homotopy commutes.

$$(1.2)\qquad \begin{array}{ccc} KO \simeq KO \wedge S^0 & \xrightarrow{c} & KU \\ \downarrow \alpha & & \downarrow 1 \\ KO \wedge \Sigma^{-2}\mathbf{C}P^2 & \xrightarrow[\Phi]{} & KU \end{array}$$

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Applied to X, the cofibration sequence of c takes the form [**3**, (3.4)]

$$(1.3)\qquad \dots \to KO^m(X) \xrightarrow{(\eta\cdot -)} KO^{m-1}(X) \xrightarrow{c} KU^{m-1}(X) \to \dots$$

By (1.2), Φ clearly induces an isomorphism from (1.3) to the cofibration sequence of α, which proves (a).

The proof of (b) is similar, replacing (1.3) by the sequence of [**3**, (3.6)] which relates $KO^*(X)$ and $KSC^*(X)$.

2. Now we turn to algebraic K- and L-theory. For the latter, the reader is referred to [**12** and **13**]. For the former, [**4**, **19**, **8**, **21**, **22**, **29**, **30**] are suitable references.

2.1. a) Throughout the remainder of this note, l will be a prime, S a separately closed field or, more generally, a strictly Henselian local ring in which $2l$ is invertible. Let A denote a commutative S-algebra with an anti-involution, $x \mapsto \bar{x}$, such that $\overline{sx} = s(\bar{x})$ $(x \in A, s \in S)$.

b) In our proofs, we will use [**29**, Theorem 4.1] which requires that A satisfy certain (very commonplace) cohomological conditions, as follows: Suppose that A (or Spec(A)) is separated, noetherian and regular of finite Krull dimension and that there is a uniform bound on the l-torsion étale cohomological dimension of all residue fields of A (even at non-closed points).

2.2. We briefly recall a construction of [**29**, Part 1.33 et seq.] (see also [**22**, IV, Part 1]). Let F be a presheaf of fibrant spectra on the étale site of a scheme, X. For example, F might be an ${}_{\epsilon}L$- or K-theory spectrum, possibly with coefficients mod l^ν.

There exists a spectrum $\mathbf{H}^{\bullet}_{et}(X; F)$, an augmentation map

$$\eta: F(X) \to \mathbf{H}^{\bullet}_{et}(X; F)$$

and a spectral sequence of cohomological type

$$(2.3)\qquad E_2^{s,t} = H^s_{et}(X; \pi_t(F)) \Rightarrow \pi_{t-s}(\mathbf{H}^{\bullet}_{et}(X; F)).$$

When $X = \operatorname{Spec} A$, this spectral sequence converges strongly under the conditions of Parts 2.1 (a)/(b).

$\mathbf{H}^{\bullet}_{et}(X; F)$ commutes with direct limits in F and preserves fibrations of spectra. It is the analogue, in the homotopy category of spectra, of the more familiar hypercohomology constructions in algebraic abelian categories [**18**].

Let $K/l^\nu(A)$ denote the spectrum of the algebraic K-theory (mod l^ν) of A in Part 2.1 (a). By [**27**, **28**],

$$\pi_*(K/l^\nu(S)) = K_*(S; Z/l^\nu)$$

is the graded polynomial ring on one generator, β, in dimension two. $K/l^\nu(A)$ is a module spectrum over $K/l^\nu(A)$ so (following [**21**, IV, p. 134]) we may invert β to form the spectrum $K/l^\nu(A)[1/\beta]$; "Bott periodic" algebraic K-theory (see [**23**]). Similarly, let $K/l^\nu[1/\beta]$ denote the presheaf of fibrant spectra given by mod l^ν "Bott periodic" algebraic K-theory on the étale site of $X = \mathrm{Spec}(A)$.

Thomason's main result is the following:

2.3. THEOREM [**29**, Part 4.1]. *Let A be an S-algebra as in Parts* 2.1 (a)/(b). *Then* (*assuming $\nu \geqq 2$ if $l = 2$*),

$$\eta : K/l^\nu(A)[1/\beta] \to \mathbf{H}^{\bullet}_{et}(\mathrm{Spec}(A);\ K/l^\nu[1/\beta])$$

is a weak equivalence. Also, the spectral sequence of (2.3) *is strongly convergent and takes the form*

$$E_2^{s,t} = \begin{Bmatrix} H^s_{et}(\mathrm{Spec}(A);\ \mu_{l^\nu}(i)) & , t = 2i \\ 0 & , t \text{ odd} \end{Bmatrix} \Rightarrow \pi_{t-s}(K/l^\nu(A)[1/\beta]).$$

Note that, for some integer N, $E_2^{s,t} = 0$ when $s \geqq N$. Here $\mu_{l^\nu}(i)$ is the i-th Tate twist of the sheaf of l^ν-th roots of unity.

2.4. From [**27**, **28**] we have an isomorphism

$$K_*(S;\ Z/l^\nu) \cong \pi_*(BU;\ Z/l^\nu) \cong Z/l^\nu[\beta].$$

As a corollary, one has its Hermitian and self-conjugate analogues [**10**, **11**, **13**, **14**] which state that, in positive dimensions,

$${}_1L_*(S;\ Z/l^\nu) \cong \pi_*(BO;\ Z/l^\nu),$$

$${}_{-1}L_*(S;\ Z/l^\nu) \cong \pi_*(BSp;\ Z/l^\nu) \cong \pi_{*-4}(BO;\ Z/l^\nu),$$

and

$$KSC_*(S;\ Z/l^\nu) \cong \pi_*(BSC;\ Z/l^\nu)$$

where BO and BSC are the usual classifying spaces for real and self-conjugate K-theory [**1**, **2**, **3**, **9**]. In fact, the (co-)fibrations of spectra [**12**]

$${}_1V/l^\nu(S) \to {}_1L/l^\nu(S) \xrightarrow{F} K/l^\nu(S)$$

and

$${}_{-1}U/l^\nu(S) \to K/l^\nu(S) \xrightarrow{H} {}_{-1}L/l^\nu(S)$$

may be identified with the (co-)fibrations obtained by smashing

$$\Omega^{-1}BO \to BO \xrightarrow{C} BU \quad \text{and}$$

$$\Omega^{-2}BO \xrightarrow{\Omega^{-2}c} \Omega^{-2}BU \to \Omega^{-4}BO$$

respectively, with the mod l^ν Moore spectrum. Consequently, there exists generators

$$B \in {}_1L_8(S;\ Z/l^\nu) \quad \text{and} \quad b \in KSC_4(S;\ Z/l^\nu)$$

whose images under the forgetful map to $K_*(S;\ Z/l^\nu)$ are β^4 and β^2 respectively.

In a manner similar to Part 2.2, we may form the spectra

(2.5) $\quad {}_\epsilon L/l^\nu(A)[1/B] \quad \text{and} \quad KSC/l^\nu(A)[1/b].$

These spectra are the localizations of the spectra representing respectively the ϵ-Hermitian and the self-conjugate algebraic K-theories (mod l^ν) of A [**12**, **13**].

2.6. THEOREM. *Let l be a prime and ν an integer ($\nu \geqq 2$ if $l = 2$). Let A be a commutative S-algebra (where $2l$ is invertible in S) satisfying the conditions of Parts* 2.1 (a)/(b). *Then*

a) $\quad \eta: KSC/l^\nu(A)[1/b] \rightarrow \mathbf{H}^{\bullet}_{et}(\text{Spec } A;\ KSC/l^\nu[1/b])$

is a weak equivalence;

b) *if* $\epsilon = \pm 1$,

$$\eta_\epsilon: {}_\epsilon L/l^\nu(A)[1/B] \rightarrow \mathbf{H}^{\bullet}_{et}(\text{Spec } A;\ {}_\epsilon L/l^\nu[1/B])$$

is a weak equivalence.

c) *consequently there are strongly convergent spectral sequences* ($x = 1$, 2, 3),

$$E_2^{s,t} = H^s_{et}(A;\ \mathbf{F}(x)_t) \Rightarrow \pi_{t-s}(F(x)),$$

where

$$F(1) = KSC/l^\nu(A)[1/b], \quad F(2) = {}_1L/l^\nu(A)[1/B]$$

and

$$F(3) = {}_{-1}L/l^\nu(A)[1/B].$$

In these cases, the sheaf $\mathbf{F}(x)_t$ *is given by the following table.*

TABLE 1

t mod 8	$\mathbf{F}(1)_t$, l *odd*	$\mathbf{F}(1)_t$, $\nu \geqq 2$, $l = 2$	$\mathbf{F}(x)_t$, $x = 2, 3$; l *odd*	$\mathbf{F}(2)_t$, $\nu \geqq 2$, $l = 2$	$\mathbf{F}(3)_t$, $\nu \geqq 2$, $l = 2$
$8k$	$\mu_{l^\nu}(4k)$	$\mu_{2^\nu}(4k)$	$\mu_{l^\nu}(4k)$	$\mu_{2^\nu}(4k)$	$\mu_{2^\nu}(4k)$
$8k + 1$	0	$\mu_{2^\nu}(4k + 1) \otimes Z/2$	0	$\mu_{2^\nu}(4k) \otimes Z/2$	0
$8k + 2$	0	$\mu_2(4k + 2)$	0	$(\mu_{2^\nu}(4k) \otimes Z/2)^2$	0
$8k + 3$	$\mu_{l^\nu}(4k + 2)$	$\mu_{2^\nu}(4k + 2)$	0	$\mu_{2^\nu}(4k) \otimes Z/2$	0
$8k + 4$	$\mu_{l^\nu}(4k + 2)$	$\mu_{2^\nu}(4k + 2)$	$\mu_{l^\nu}(4k + 2)$	$\mu_{2^\nu}(4k + 2)$	$\mu_{2^\nu}(4k + 2)$
$8k + 5$	0	$\mu_{2^\nu}(4k + 3) \otimes Z/2$	0	0	$\mu_{2^\nu}(4k + 2) \otimes Z/2$
$8k + 6$	0	$\mu_2(4k + 3)$	0	0	$(\mu_{2^\nu}(4k + 2) \otimes Z/2)^2$
$8k + 7$	$\mu_{l^\nu}(4k + 4)$	$\mu_{2^\nu}(4k + 4)$	0	0	$\mu_{2^\nu}(4k + 2) \otimes Z/2$

2.7. *Remark.* Aside from Tate twists, the coefficients in Table 1 are given by

$$\mathbf{F}(1)_t \cong \pi_t(BSC;\ Z/l^\nu),\quad \mathbf{F}(2)_t \cong \pi_t(BO;\ Z/l^\nu)$$

and

$$\mathbf{F}(3)_t \cong \pi_t(BSp;\ Z/l^\nu).$$

These coefficients have the usual

$${}_1L_*(S;\ Z/l^\nu) \cong \pi_*(BO;\ Z/l^\nu)$$

module structure; that is, the one from topological K-theory. For example, when $l = 2$, $\nu \geqq 2$, let η generate ${}_1L_1(S;\ Z/2^\nu)$ and let $\{\eta^2, g\}$ generate

$${}_1L_2(S;\ Z/2^\nu) \cong Z/2 \oplus Z/2$$

(as $\nu \geqq 2$). If ι_k generates $\mathbf{F}(2)_{8k}$, then

$$\mathbf{F}(2)_{8k+1} = \langle \eta\iota_k \rangle$$

$$\mathbf{F}(2)_{8k+2} = \langle \eta^2\iota_k,\ g\iota_k \rangle,$$

$$\mathbf{F}(2)_{8k+3} = \langle \eta g\iota_k \rangle$$

and $\eta^3\iota_k = 0$.

2.8. *Proof of theorem* 2.6. Firstly, we appeal to the rigidity theorems of [**27**, **6**, **7**, **11**] which show that the sheaves $\mathbf{F}(x)_*$ are constant on the étale site and isomorphic in positive dimensions to

$$KSC_*(S;\ Z/l^\nu) \cong \pi_*(BSC;\ Zl^\nu),$$

$${}_1L_*(S;\ Z/l^\nu) \cong \pi_*(BO;\ Z/l^\nu)$$

or

$${}_{-1}L_*(S;\ Z/l^\nu) \cong \pi_*(BSp;\ Z/l^\nu).$$

These yield the groups, with the appropriate Tate twists, which are recorded in Table I. This proves (c), as explained in Part 2.2.

Part (a) is an immediate corollary of Thomason's theorem (Part 2.3), the fibration sequence of

$$KSC/l^\nu(-)[1/b] \to K/l^\nu(-)[1/\beta] \xrightarrow{1-T} K/l^\nu(-)[1/\beta]$$

and the five lemma. Here we have used the fact, mentioned in Part 2.2, that $\mathbf{H}^{\bullet}_{ee}(\mathrm{Spec}\ A;\ -)$ preserves fibrations of presheaves of fibrant spectra.

When $\epsilon = \pm 1$, consider the commutative diagram of fibrations of spectra [**13**, Part 4.10].

$$
(2.9)_\epsilon \qquad
\begin{tikzcd}
\Omega^2{}_\epsilon L/l^\nu(A) \arrow[r] \arrow[d, "{}_\epsilon\gamma"] & * \arrow[r] \arrow[d] & \Omega_\epsilon L/l^\nu(A) \arrow[d] \\
{}_{-\epsilon}L/l^\nu(A) \arrow[r] \arrow[d] & K/l^\nu(A) \arrow[r] \arrow[d, "1"] & {}_\epsilon U/l^\nu(A) \arrow[d] \\
KSC/l^\nu(A) \arrow[r] & K/l^\nu(A) \arrow[r, "1 - T"] & K/l^\nu(A)
\end{tikzcd}
$$

Firstly, we claim that ${}_\epsilon\gamma$ is induced by multiplication by an element, also denoted by ${}_\epsilon\gamma$, in ${}_{-1}L_{-2}(Z[1/2])$. For all the maps in $(2.9)_\epsilon$ are "L_*-module" maps; in the sense of [**12**, **13**]. Generalising [**12**, **13**] marginally one can construct categorical models realising the sum of $(2.9)_{-1}$ and $(2.9)_1$ in such a way that the resulting map

$$
\Omega^2({}_{-1}L/l^\nu(A) \vee {}_1L/l^\nu(A)) \xrightarrow{\begin{pmatrix} 0 & {}_{-1}\gamma \\ {}_1\gamma & 0 \end{pmatrix}} ({}_{-1}L/l^\nu(A) \vee {}_1L/l^\nu(A))
$$

is seen to be an "$({}_{-1}L_* \oplus {}_1L_*)$-module" map. Hence, it must be multiplication by a matrix of elements

$$
\begin{pmatrix} 0 & {}_{-1}\gamma \\ {}_1\gamma & 0 \end{pmatrix}
$$

as required.

Set

$$
{}_\epsilon L' = {}_\epsilon L/l^\nu(A)[1/B], \quad K' = KSC/l^\nu(A)[1/b]
$$

with associated presheaves of spectra ${}_\epsilon\mathbf{L}'$, $\mathbf{K}'$. For $\epsilon = \pm 1$, we have a diagram of exact sequences of the following form:

$$
(2.10)_\epsilon \qquad
\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \pi_{m+1}(K') \arrow[r] \arrow[d, "\eta"', "\cong"] & \pi_{m+2}({}_{-\epsilon}L') \arrow[r] \arrow[d, "\eta_{-\epsilon}"'] & \pi_m({}_\epsilon L') \arrow[r, "H"] \arrow[d, "\eta_\epsilon"'] & \pi_m(K') \arrow[r] \arrow[d, "\eta"', "\cong"] & \cdots \\
\cdots \arrow[r] & \pi_{m+1}(\mathbf{H}^{\cdot}_{et}(A;\, \mathbf{K}')) \arrow[r, "\pi"] & \pi_{m+2}(\mathbf{H}^{\cdot}_{et}(A;\, {}_{-\epsilon}\mathbf{L}')) \arrow[r] & \pi_m(\mathbf{H}^{\cdot}_{et}(A;\, {}_\epsilon\mathbf{L}') \arrow[r, "H"] & \pi_m(\mathbf{H}^{\cdot}_{et}(A;\, \mathbf{K}')) \arrow[r] & \cdots
\end{tikzcd}
$$

The isomorphism comes from part (a).

Next we observe that ${}_{-1}\gamma_1\gamma$ is zero because ${}_\epsilon\gamma$ multiplies through its image in

$$
{}_{-1}L_{-2}(S;\, Z/l^\nu)[1/B] \cong \pi_6(BSp;\, Z/l^\nu)
$$
$$
\cong \begin{cases} Z/2 \oplus Z/2 & \text{if } l = 2,\ \nu \geqq 2, \\ 0 & \text{if } l \text{ is odd.} \end{cases}
$$

Furthermore, ${}_{-1}\gamma$ and ${}_1\gamma$ are integral classes so that their product in

$$ {}_1L_{-4}(S;\ Z/2^{\nu})[1/B] \cong \pi_{12}(BO;\ Z/2^{\nu}) $$

is zero, from the known ring product structure in topological K-theory.

Thus, if $x \in \pi_m({}_\epsilon L')$ and $\eta_\epsilon(x) = 0$, then

$$ H(x) = 0 \quad \text{and} \quad x = {}_{-\epsilon}\gamma y, \quad (y \in \pi_{m+2}({}_{-\epsilon}L')). $$

Therefore,

$$ \eta_{-\epsilon}(y) = \pi\eta(z) \quad (z \in \pi_{m+1}(K')), $$

so that

$$ \eta_{-\epsilon}(y - \pi(z)) = 0 \quad \text{and} $$

$$ y - \pi(Z) = {}_\epsilon\gamma w \quad (w \in \pi_{m+4}({}_\epsilon L')) $$

which implies

$$ \begin{aligned} x &= {}_{-\epsilon}\gamma y = {}_{-\epsilon}\gamma(y - \pi(z)) \\ &= {}_{-\epsilon}\gamma_\epsilon \gamma w = 0, \end{aligned} $$

so that η_ϵ is one-one.

We conclude by showing that η_ϵ is onto. If

$$ x \in \pi_m(\mathbf{H}^{\bullet}_{et}(A,\ {}_\epsilon\mathbf{L}')), $$

then $H(x) = \eta(y)$ for $y \in \pi_m(K')$ and, because $\eta_{-\epsilon}$ is one-one,

$$ y = H(z) \quad (z \in \pi_m({}_\epsilon L')). $$

Therefore,

$$ x - \eta_\epsilon(z) = {}_{-\epsilon}\gamma w \quad (w \in \pi_{m+2}(\mathbf{H}^{\bullet}_{et}(A,\ {}_{-\epsilon}\mathbf{L}'))). $$

Similarly

$$ \begin{aligned} w &= \eta_{-\epsilon}(p) + {}_{-\epsilon}\gamma_\epsilon \gamma q \\ &= \eta_{-\epsilon}(p) \end{aligned} $$

so that $x = \eta_\epsilon(z) + \eta_\epsilon({}_{-\epsilon}\gamma p)$ as required.

2.12. *Remark*. In the proof of Theorem 2.6 (b), we used the fact that

$$ {}_{-1}\gamma_1\gamma \in {}_1L_{-4}(S;\ Z/l^{\nu})[1/B] $$

is zero. However, ${}_{-1}\gamma_1\gamma$ is not generally nilpotent. For example, if we were to replace the strictly Hensel local ring, S, by $Z[1/2]$, from [**12**], the fundamental theorem of [**13**], the fact that

$$ K_m(Z[1/2]) = 0 \quad \text{for } m < 0 $$

and the fact that

$$ {}_{-1}L_0(Z[1/2]) \cong Z \oplus Z/2, $$

one finds

$$_{-1}L_{-2}(Z[1/2]) \cong Z \oplus Z/2 \oplus Z/2.$$

In this group, $_{\pm 1}\gamma$ is not a torsion element. Furthermore, no power, $(_{-1}\gamma_1\gamma)^m$, is zero for in large negative dimensions it yields an isomorphism

$$_{-1}L_n(Z[1/2]) \xrightarrow{\cong} {}_{-1}L_{n-4m}(Z[1/2]).$$

3. The analogue of Wood's theorem.

3.1. By [**27**, **28**, **14**, **15**, **10**, **11**] we have isomorphisms

$$\begin{aligned}[\Sigma^{-2}\mathbf{C}P^2, K/l^\nu(S)] &= \left[S^0 \underset{\eta}{\cup} e^2, K/l^\nu(S)\right] \\ &\cong KU^0\left(S^0 \underset{\eta}{\cup} e^2; Z/l^\nu\right)\end{aligned}$$

and

$$\left[S^0 \underset{\eta^2}{\cup} e^3, KSC/l^\nu(S)\right] \cong KSC^0\left(S^0 \underset{\eta^2}{\cup} e^3; Z/l^\nu\right).$$

Consequently, we have maps

$$h:S^0 \underset{\eta}{\cup} e^2 \to K/l^\nu(S) \quad \text{and} \quad h_1:S^0 \underset{\eta^2}{\cup} e^2 \to KSC/l^\nu(S),$$

corresponding to the topological maps of Part 1.2.

We may form maps, analogous to those of Section 1,

$$\begin{aligned}\Phi:({}_\epsilon L/l^\nu(A)) \wedge \left(S^0 \underset{\eta}{\cup} e^2\right) &\xrightarrow{F \wedge h} K/l^\nu(A) \wedge K/l^\nu(S) \\ &\xrightarrow{\text{mult}} K/l^\nu(A)\end{aligned}$$

and

$$\begin{aligned}\Psi:({}_\epsilon L/l^\nu(A)) \wedge \left(S^0 \underset{\eta^2}{\cup} e^3\right) &\xrightarrow{1 \wedge h_1} {}_\epsilon L/l^\nu(A) \wedge KSC/l^\nu(S) \\ &\xrightarrow{\text{mult}} KSC/l^\nu(A)\end{aligned}$$

for $\epsilon = \pm 1$. Here F is the forgetful map. These maps are "L_*-module" maps in the sense of [**12**, **13**] so that they induce maps

(3.2) $$\Phi:{}_\epsilon L/l^\nu(A)[1/B] \wedge \left(S^0 \underset{\eta}{\cup} e^2\right) \to K/l^\nu(A)[1/\beta]$$

and

$$\Psi : {}_{\epsilon}L/l^{\nu}(A)[1/B] \wedge \left(S^0 \underset{\eta^2}{\cup} e^3\right) \to KSC/l^{\nu}(A)[1/b].$$

3.3. THEOREM. *Let l be a prime and let ν be an integer* (*if* $l = 2$, $\nu \geqq 2$). *Let S be a separately closed field or a strictly Hensel local ring in which $2l$ is invertible. Let A be a commutative Hermitian ring which is an S-algebra satisfying the conditions of Parts* 2.1 (a)/(b). *Then Φ and Ψ are weak equivalences in* (3.2) *when* $\epsilon = \pm 1$.

Proof. We will prove that Φ is a weak equivalence, the case of Ψ being entirely similar.

The map, Φ, is natural in A. Therefore it induces a map of spectral sequences (2.3) and Part 2.6 for

$$F = {}_{\epsilon}L/l^{\nu}[1/B] \wedge (\Sigma^{-2}\mathbf{C}P^2)$$

and for

$$F = K/l^{\nu}[1/\beta].$$

The assumptions of Part 2.1 (b) ensure that these spectral sequences converge strongly to the homotopy of the range and domain of Φ. Hence, we have only to verify the result on stalks; that is, when A itself is a strictly Hensel local ring. However, this case, by [**27**, **28**, **14**, **15**, **10**, **11**], may be identified with the topological result (Part 1.2) with mod l^{ν} coefficients, which completes the proof.

4. An application to Stiefel-Whitney classes of symmetric bilinear forms.

4.1. Let K be a field of characteristic not equal to two. Suppose that $\phi : V \times V \to K$ is a non-singular, symmetric bilinear form, of rank m, over K. To (V, ϕ) one may attach Stiefel-Whitney classes [**5**]

(4.2) $\quad w_i(V, \phi) \in H^i(K; Z/2)$

where $H^i(K; Z/2)$ is the i-th Galois cohomology group of K (see [**18**]). To define $w_i(V, \phi)$, one diagonalises (V, ϕ) to the form

$$\begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_m \end{bmatrix}.$$

Set $w_i(V, \phi)$ equal to the i-th elementary symmetric function in $l(\alpha_1), \ldots, l(\alpha_m)$

$$w_i(V, \phi) = \sum_{j_1 < j_2 < \ldots < j_i} l(\alpha_{j_1}) \ldots l(\alpha_{j_i})$$

where $l(\alpha) \in H^1(K; Z/2)$ is represented by the function $\mathrm{Gal}(\bar{K}/K) \to Z/2$ given by

$$l(\alpha)(g) = g(\sqrt{\alpha})/\sqrt{\alpha} \in \{\pm 1\} \cong Z/2.$$

4.2. Suppose that L/K is a finite, separable field extension and that $\psi: X \times X \to L$ is a symmetric, non-singular bilinear form. The *Scharlau transfer* of (X, ψ) is the composition

$$X \times X \xrightarrow{\psi} L \xrightarrow{\text{trace}} K$$

considered as a non-singular bilinear form over K, $\mathrm{Tr}^S_{L/K}(X, \psi)$.

The question arises: What is

(4.3) $\quad w_i(\mathrm{Tr}^S_{L/K}(X, \psi)) \in H^i(K; Z/2)$?

This question was first studied by Serre [**20**], who found an elegant formula for the discriminant and Hasse-Witt invariants (w_1 and w_2). See also [**24**]. The problem was taken up by B. Kahn [**16**]. Below, I will sketch an approach to defining $w_i(V, \phi)$ which uses algebraic K- and L-theory and with it I will answer (4.3) for fields K which have a separably closed subfield. In a future paper [**25**], I will develop this outline to give the solution to (4.3) (at least, whenever an eighth-root of unity lies in L).

Recently I have learnt that B. Kahn has obtained a complete solution to (4.3) by methods which do not involve algebraic K-theory [**17**]. Nevertheless, I believe that the theory developed in [**25**] will be of independent interest.

4.4. *Definition*. If (X, ψ) is as in Part 4.2 and has rank t, let

$$[X, \psi]: \mathrm{Gal}(\bar{K}/L) \to (Z/2)^t$$

denote $(l(\beta_1), \dots, l(\beta_t))$ where

$$\begin{bmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_t \end{bmatrix}$$

is any diagonalisation of (X, ψ).

If $n = [L:K]$, we may form the induced representation

(4.5) $\quad \mathrm{Tr}^V_{L/K}[X, \psi]: \mathrm{Gal}(\bar{K}/K) \to \Sigma_n \int (Z/2)^t \subset O_{nt}(K),$

where Tr^V denotes for "vector bundle transfer" and $H \int G$ denotes a wreath product. The topological Stiefel-Whitney classes restrict to classes

$$w_i^{\text{top}} \in H^i\left(\Sigma_n \int (Z/2)^t; Z/2\right)$$

(by [**27**, **28**, **14**, **15**, **11**], w_i^{top} actually originates in the cohomology of $O_{nt}(K)$). Thus we may form

(4.6) $\quad w_i^{\mathrm{top}}(\mathrm{Tr}^V_{L/K}[X, \psi]) \in H^i(K; Z/2).$

4.7. THEOREM. *Let K be a field of characteristic different from two, containing a separably closed subfield* (*or more generally the quotient field of a strictly Hensel local ring*). *Then, with the notation of Parts* 4.1/4.6, *for all* $i \geqq 1$,

$$w_i\mathrm{Tr}^S_{L/K}(X, \psi) = w_i^{\mathrm{top}}\mathrm{Tr}^V_{L/K}[X, \psi] \in H^i(K; Z/2).$$

Sketch proof. Let $S \subset K$ be a separable closed field or a strictly Hensel local ring. In the following construction, any such K is the direct limit of fields satisfying the conditions of Parts 2.1 (a)/(b) (and hence of Part 3.1) and by taking limits we may assume that K satisfies Part 3.1.

By [**21**, **22**, **14**, **15**, **10**, **11**], there is an isomorphism ($X_+ = X_\sqcup$ (base point))

(4.8) $\quad [\Sigma^n\mathrm{Gal}(\bar{K}/K), B_1O(S)^+; Z/2^\nu] \cong KO^{-n}(B\,\mathrm{Gal}(\bar{K}/K); Z/2^\nu)$

(for all ν), where $\mathrm{Gal}(\bar{K}/K)$ has the profinite topology. Since Thomason's spectrum $\mathbf{H}(\mathrm{Spec}\,K;\ {}_1L/2^\nu)$ of Part 2.2 is the homotopy fixed points,

$$\mathrm{Map}_{\mathrm{Gal}(\bar{K}/K)}(E\,\mathrm{Gal}(\bar{K}/K)_+,\ {}_1L/2^\nu(\bar{K})),$$

the natural $\mathrm{Gal}(\bar{K}/K)$-equivariant map of spectra (with trivial action on the first one)

$$(\Sigma^\infty/2^\nu) \wedge B_1O(S)^+ \to {}_1L/2^\nu(\bar{K})$$

induces a map (using (4.8) and Theorem 2.6 (b))

(4.9) $\quad \gamma_\nu = KO^{-*}(B\,\mathrm{Gal}(\bar{K}/K); Z/2^\nu) \overset{\cong}{\to} {}_1L_*(K; Z/2^\nu)[1/B].$

However, (4.9) is an isomorphism. For by [**26**] the analogous map

$$\rho_\nu : KU^{-*}(B\,\mathrm{Gal}(\bar{K}/K); Z/2^\nu) \overset{\cong}{\to} K_*(K; Z/2^\nu)[1/\beta]$$

is an isomorphism. Hence (4.9) follows from the fact that Φ in (3.2) is an equivalence (with $\epsilon = 1$) and a diagram chase of the resulting diagram, namely (setting $X = B\,\mathrm{Gal}(\bar{K}/K)$)

(4.10)
$$\begin{array}{ccccccccc}
\cdots \to & KO^m(X; Z/2^\nu) & \xrightarrow{n} & KO^{m+1}(X; Z/2^\nu) & \to & KU^{m+1}(X; Z/2^\nu) & \to \cdots \\
& \downarrow \gamma_\nu & & \downarrow \gamma_\nu & & \cong \downarrow \rho_\nu & \\
\cdots \to & {}_1L_{-m}(K; Z/2^\nu)[1/B] & \to & {}_1L_{-m-1}(K; Z/2^\nu)[1/B] & \to & K_{-m-1}(K; Z/2^\nu)[1/\beta] & \to \cdots
\end{array}$$

From (4.9) we may define $w_i(V, \phi)$ as follows: (V, ϕ) defines a class in ${}_1L_0(K)$ and hence one in

$$\lim_{\substack{\leftarrow \\ \nu}} {}_1L_0(K;\, Z/2^\nu)[1/B] \cong \lim_{\substack{\leftarrow \\ \nu}} KO^0(B\ \mathrm{Gal}(\bar{K}/K);\, Z/2^\nu).$$

However, the topological Stiefel-Whitney classes are defined on the latter group. Furthermore, one can verify, using [**24**, Part 3.2], and the behaviour under transfer of the above isomorphisms, that the class of $w_i(\mathrm{Tr}^S_{L/K}(X, \psi))$ by this construction is $w_i^{\mathrm{top}}(\mathrm{Tr}^V_{L/K}[X, \psi])$.

References

1. J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lecture Notes in Mathematics, (1974).

2. D. W. Anderson, *The real K-theory of classifying spaces*, Proc. Nat. Acad. Sci. *51* (1964), 634-636.

3. M. F. Atiyah, *K-theory and reality*, Quart. J. Math., Oxford (2), *17* (1966), 367-386.

4. W. Browder, *Algebraic K-theory with coefficients Z/p*, Springer Verlag Lecture Notes in Mathematics *654* (1978), 40-84.

5. M. A. Delzant, *Definition des classes de Stiefel-Whitney d'un module quadratique sur un corps de charactéristique differente de* 2, C.R. Acad. Sci., Paris *255* (1962), 1366-1368.

6. O. Gabber, Lecture at France – U.S.A. *K*-theory conference, Luminy (1983).

7. H. Gillet and R. W. Thomason, *The K-theory of strict Hensel local rings and a theorem of Suslin*, JPAA *34* (1984), 241-251.

8. D. Grayson (after D. G. Quillen), *Higher algebraic K-theory* II, Springer-Verlag Lecture Notes in Mathematics *551*, 217-240.

9. P. S. Green, *A cohomology theory based upon self-conjugacies of complex vector bundles*, Bull. A.M. Soc. (1964), 522-524.

10. J. F. Jardine, *Simplicial objects in a Grothendieck topos*, preprint (1983).

11. ——— *A rigidity theorem for L-theory*, preprint (1983).

12. M. Karoubi, *Theorie de Quillen et homologie du groupe orthogonal*, Ann. Math. *112* (1980), 206-257.

13. ——— *Le theoreme fondamental de la K-theorie hermitienne*, Ann. Math. *112* (1980), 259-282.

14. ——— *Homology of the infinite orthogonal and symplectic groups over algebraically closed fields*, Inventiones Math. *73* (1983), 247-250.

15. ——— *Relations between algebraic K-theory and Hermitian K-theory*, preprint (1984).

16. B. Kahn, *La deuxième classe de Stiefel-Whitney d'une représentation régulière, I & II*, C.R. Acad. Sci., Paris *297* (1983), 313-316 and 573-576.

17. ——— *Classes de Stiefel-Whitney de formes quadratiques et de représentations Galoisiennes reélles*, Inventiones Math. *78* (1984), 223-256.

18. J. S. Milne, *Étale cohomology*, Princeton Math. Series *33* (1980).

19. D. G. Quillen, *Higher algebraic K-theory I*, Springer-Verlag Lecture Notes in Mathematics *341* (1973), 85-147.

20. J-P. Serre, *Sur* 1-*invariant de Witt de la forme* $Tr(x^2)$, Comm. Math. Helv. *59* (1984), 651-676.

21. V. P. Snaith, *Algebraic cobordism and K-theory*, Mem. A. M. Soc. *221* (1979).

22. ——— *Localised stable homotopy and algebraic K-theory*, Mem. A. M. Soc. *280* (1983).

23. ——— *A brief survey of Bott periodic K-theory*, Can. Math. Soc. Conf. Proc. *2*, Part I (1982).

24. ——— *Stiefel-Whitney classes of symmetric bilinear forms – a formula of Serre*, Can. Bull. Math. *28* (1985), 218-222.

25. ——— *Algebraic K-theory and bilinear forms*, in preparation.
26. ——— *K-theory of the classifying spaces of Galois groups*, to appear Proc. Conf., St. John's, Newfoundland (1983) in A. M. Soc. Contemporary Math. series.
27. A. A. Suslin, *On the K-theory of algebraically closed fields*, Inventiones Math. *73* (1983), 241-245.
28. ——— *On the K-theory of local fields*, to appear J. Pure and Appl. Alg.
29. R. W. Thomason, *Algebraic K-theory and étale cohomology*, preprint.
30. J. B. Wagoner, *Delooping the classifying spaces of algebraic K-theory*, Topology *11* (1972), 349-370.

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