Can. J. Math., Vol. XXXIII, No. 3, 1981, pp. 701–733

ON A PROBLEM OF TURÁN ABOUT POLYNOMIALS II

R. PIERRE AND Q. I. RAHMAN

1. It was proved by A. A. Markov [**3**] that if $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree at most n and $|p_n(x)| \leqq 1$ in the interval $-1 \leqq x \leqq 1$, then in the same interval

$$(1) \qquad |p_n'(x)| \leqq n^2.$$

The problem was proposed by the chemist Mendeleev who knew the answer for polynomials of degree 2. For a historical background of the problem see [**1**].

A. A. Markov's younger brother W. A. Markov considered the problem of determining exact bounds for the j-th derivative of $p_n(x)$ at a given point x_0 in $[-1, 1]$. His results appeared in a Russian journal in the year 1892; a German version of his remarkable paper was later published in [**4**]. Amongst other things he proved the following two theorems.

THEOREM A. *If $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree at most n such that $|p_n(x)| \leqq 1$ for $-1 \leqq x \leqq 1$, then*

$$(2) \qquad \max_{-1\leqq x\leqq 1} |p_n^{(j)}(x)| \leqq \frac{n^2(n^2-1^2)(n^2-2^2)\ldots(n^2-(j-1)^2)}{1\cdot 3\cdot 5\cdots(2j-1)}, \qquad j = 1, 2, \ldots, n,$$

where equality holds for the n-th Chebyshev polynomial of the first kind

$$(3) \qquad T_n(x) = \cos(n \arccos x) = 2^{n-1} \prod_{\nu=1}^{n} \{x - \cos((\nu - \tfrac{1}{2})\pi/n)\}.$$

THEOREM B. *Let $\sum_{\mu=0}^{m} t_{m,\mu} x^\mu =: T_m(x)$ denote the m-th Chebyshev polynomial of the first kind. If $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ satisfies the condition of Theorem A, then*

$$(4) \qquad |a_j| \leqq \begin{cases} |t_{n,j}| & \text{if } n-j \text{ is even} \\ |t_{n-1,j}| & \text{if } n-j \text{ is odd.} \end{cases}$$

W. A. Markov started out by taking a very general point of view: If $\alpha_0, \alpha_1, \ldots, \alpha_n$ are given constants and $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ satisfies the condition of Theorem A, what is the precise upper bound for the linear form $\sum_{\nu=0}^{n} a_\nu \alpha_\nu$? By suitably choosing the constants the linear form can be made equal to any derivative of $p_n(x)$ at any preassigned point.

Received February 4, 1980.

https://doi.org/10.4153/CJM-1981-055-8 Published online by Cambridge University Press

For polynomials with real coefficients the hypothesis $|p_n(x)| \leqq 1$ for $-1 \leqq x \leqq 1$ means that the graph of $p_n(x)$ on $[-1, 1]$ lies in the square

$$\{(x, y) \in \mathbf{R}^2: -1 \leqq x \leqq 1, -1 \leqq y \leqq 1\}.$$

At a conference held in Varna, Bulgaria in the year 1970 the late Professor Paul Turán asked the following question: "Given a polynomial $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ with real coefficients whose graph on $[-1, 1]$ lies in the unit disk, how large can its derivative be on the same interval? More generally, for an arbitrary non-negative function $\varphi(x)$ on $[-1, 1]$ let $\mathscr{P}(\varphi, n)$ denote the class of all polynomials p_n of degree at most n such that $|p_n(x)| \leqq \varphi(x)$ for $-1 \leqq x \leqq 1$. Then, how large can $|p_n^{(j)}(x_0)|$ be at a given point x_0 in $[-1, 1]$ as p_n varies in $\mathscr{P}(\varphi, n)$?" Problems of this type first occurred in approximation theory, notably, in the work of Dzyadyk [**2**] on converse type theorems concerning approximation by polynomials in $[-1, 1]$.

The next theorem briefly summarises what is now known in this connection ([**5**], [**6**], [**7**]).

THEOREM C. *Let*

$$P_m(x) = (1 - x^2)U_{m-2}(x) = \sum_{\mu=0}^{m} u_{m,\mu}x^\mu,$$

where $U_{m-2}(x) = (1 - x^2)^{-1/2} \sin\{(m - 1) \arc\cos x\}$ *is the Chebyshev polynomial of the second kind of degree* $m - 2$. *If* $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ *is a polynomial of degree at most* n *such that* $|p_n(x)| \leqq (1 - x^2)^{1/2}$ *for* $-1 \leqq x \leqq 1$ (*in case the coefficients are real this means that the graph of* $p_n(x)$ *on* $[-1, 1]$ *lies in the unit disk*), *then*

$$\max_{-1 \leqq x \leqq 1} |p_n^{(j)}(x)| \leqq |P_n^{(j)}(\pm 1)|, \quad j = 1, 2; \tag{5}$$

$$|a_j| \leqq \begin{cases} |u_{n,j}| & \text{if } n - j \text{ is even} \\ |u_{n-1,j}| & \text{if } n - j \text{ is odd.} \end{cases} \tag{6}$$

Inequality (5) leaves us wondering what happens when $j \geqq 3$, and, in fact, going back to the original question of Turán, what can we say about $|p_n^{(j)}(x_0)|$ if x_0 is a given point in $[-1, 1]$? These questions are answered in the present paper. Our approach to the problem is analogous to that of W. A. Markov [**4**] and gives rather complete results in the case of majorants of the form

$$\varphi(x) = (1 - x)^{\lambda/2}(1 + x)^{\mu/2}$$

where λ, μ are non-negative integers. Consideration of such majorants leads to the following strengthening of inequality (4) of W. A. Markov.

THEOREM 1. *If* $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ *is a polynomial of degree at most* n *with real coefficients such that* $|p_n(x)| \leqq 1$ *for* $-1 \leqq x \leqq 1$, *and*

$\sum_{\nu=0}^{n} t_{n,\nu}x^\nu$ is the n-th Chebyshev polynomial of the first kind, then

(7) $\quad |a_j| + |a_{j-1}| \leqq |t_{n,j}|$ *if* $n - j$ *is even.*

Here a_{-1} is supposed to be zero.

As a matter of fact, our method gives a more precise conclusion. For example, it shows that a similar strengthening of (4), in case $n - j$ is odd, is not possible. Indeed, if $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ satisfies the conditions of Theorem B and $n - j$ is odd, then $|a_j| + \epsilon|a_{j-1}|$ may not be less than $|t_{n-1,j}|$ if ϵ is positive. Besides, it shows that the left-hand side of (7) cannot in general be replaced by

$$|a_j| + \theta|a_{j-1}|$$

for any $\theta > 1$.

In fact, we shall prove the following more general

THEOREM 1′. *Let*

$$P_n(x) = \sum_{\nu=0}^{n} \gamma_{n,\nu}x^\nu = \begin{cases} (1 - x^2)^{\lambda/2}T_{n-\lambda}(x) & \text{if } \lambda \text{ is even} \\ (1 - x^2)^{(\lambda+1)/2}U_{n-\lambda-1}(x) & \text{if } \lambda \text{ is odd.} \end{cases}$$

If $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree at most n with real coefficients such that $|p_n(x)| \leqq (1 - x^2)^{\lambda/2}$ for $-1 \leqq x \leqq 1$, then in case $n - j$ is even, the inequality

(7′) $\quad |a_j| + \theta|a_{j-1}| \leqq |\gamma_{n,j}|$

holds provided $\theta \leqq 1$. At least when λ is even, this inequality does not hold for any $\theta > 1$. If $n - j$ happens to be odd, then

(7″) $\quad |a_j| \leqq |\gamma_{n-1,j}|,$

but $|a_j| + \epsilon|a_{j-1}|$ may not be less than $|\gamma_{n-1,j}|$ for any $\epsilon > 0$.

In addition to the Chebyshev polynomials of the first and second kinds, namely $T_m(x)$, $U_m(x)$ mentioned above we need to recall the Jacobi polynomials [**8**, p. 60]

$$P_m^{(-1/2,+1/2)}(x) = \frac{1\cdot 3\cdots(2m-1)}{2\cdot 4\cdots 2m}\,\frac{\cos\{\frac{1}{2}(2m+1)\arc\cos x\}}{\cos(\frac{1}{2}\arc\cos x)},$$

$$P_m^{(+1/2,-1/2)}(x) = \frac{1\cdot 3\cdots(2m-1)}{2\cdot 4\cdots 2m}\,\frac{\sin\{\frac{1}{2}(2m+1)\arc\cos x\}}{\sin(\frac{1}{2}\arc\cos x)}.$$

Throughout the paper, $Q_m(x)$, $R_m(x)$ will stand for the polynomials

$$\frac{1}{\sqrt{2}}\,\frac{2\cdot 4\cdots 2m}{1\cdot 3\cdots(2m-1)}\,P_m^{(-1/2,+1/2)}(x),$$

$$\frac{1}{\sqrt{2}}\,\frac{2\cdot 4\cdots 2m}{1\cdot 3\cdots(2m-1)}\,P_m^{(+1/2,-1/2)}(x)$$

respectively. Besides, for given non-negative integers λ, μ we will write $\nu(n)$ to abbreviate $n - ([(\lambda+1)/2] + [(\mu+1)/2]) + 1$. With these notations our principal result can be stated as follows.

THEOREM 2. *Let λ, μ be non-negative integers, and*

$$(8) \qquad P_m(x) = \begin{cases} (1-x)^{\lambda/2}(1+x)^{\mu/2}T_{\nu(m)-1}(x) & \text{if } \lambda, \mu \text{ are both even} \\ (1-x)^{(\lambda+1)/2}(1+x)^{(\mu+1)/2}U_{\nu(m)-1}(x) & \\ & \text{if } \lambda, \mu \text{ are both odd} \\ (1-x)^{\lambda/2}(1+x)^{(\mu+1)/2}Q_{\nu(m)-1}(x) & \text{if } \lambda \text{ is even, } \mu \text{ is odd} \\ (1-x)^{(\lambda+1)/2}(1+x)^{\mu/2}R_{\nu(m)-1}(x) & \text{if } \lambda \text{ is odd, } \mu \text{ is even.} \end{cases}$$

If $p_n(x)$ is a polynomial of degree at most n (where $n \geqq 3(\lambda+\mu)/2$ if $\lambda \neq \mu$) such that $|p_n(x)| \leqq (1-x)^{\lambda/2}(1+x)^{\mu/2}$ for $-1 \leqq x \leqq 1$, then

$$(9) \qquad \max_{-1\leqq x\leqq 1} |p_n^{(j)}(x)| \leqq \max\left\{\max_{-1\leqq x\leqq 1} |P_n^{(j)}(x)|, \max_{-1\leqq x\leqq 1} |P_{n-1}^{(j)}(x)|\right\},$$

$$\frac{\lambda+\mu}{2} \leqq j \leqq n.$$

From this we will deduce, in particular, that (5) holds for $j \geqq 3$ as well.

2. Consider the real linear space $\mathscr{P}_n$ of all polynomials

$$P(x) = a_0 + a_1x + \ldots + a_nx^n$$

of degree at most n with real coefficients having a zero of multiplicity at least $[(\lambda+1)/2]$ at 1 and a zero of multiplicity at least $[(\mu+1)/2]$ at -1. If for each $P \in \mathscr{P}_n$ we define

$$(10) \qquad ||P|| = \max_{-1\leqq x\leqq 1} |(1-x)^{-\lambda/2}(1+x)^{-\mu/2}P(x)|,$$

$\mathscr{P}_n$ becomes a normed linear space. Consider a general linear functional ω on $\mathscr{P}_n$. There exist real numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that

$$\omega(P) = a_0\alpha_0 + a_1\alpha_1 + \ldots + a_n\alpha_n \quad \left(P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu\right).$$

We want to determine its norm:

$$\|\omega\| = \sup_{\|P\|\leqq 1} |\omega(P)|.$$

For this let α be a real number different from zero and denote by $\mathscr{P}_{n,\alpha}$ the class of all polynomials $P \in \mathscr{P}_n$ for which $\omega(P) = \alpha$. Then P^* is a polynomial of smallest norm amongst all polynomials belonging to $\mathscr{P}_{n,\alpha}$ if and only if

$$\|\omega\| = |\omega(P^*/\|P^*\|)|.$$

It is therefore of fundamental importance for us to be able to recognize polynomials $P^* \in \mathscr{P}_{n,\alpha}$ whose norm is the smallest. Such polynomials (which always exist) will hereafter be referred to as *minimal*.

Note that if $P \in \mathscr{P}_n$ then there exists a polynomial $\hat{P}$ of degree at most $n - ([(\lambda + 1)/2] + [(\mu + 1)/2])$ such that

$$P(x) = (1 - x)^{[(\lambda+1)/2]}(1 + x)^{[(\mu+1)/2]}\hat{P}(x).$$

Let

$$Z_P(x) = (1 - x)^{-\lambda}(1 + x)^{-\mu}P^2(x)$$

so that $\|P\|^2 = \max_{-1 \leqq x \leqq 1} Z_P(x)$, and denote the distinct roots of the equation

$$(11) \qquad \|P^2\| - Z_P(x) = 0$$

in $[-1, 1]$ by $x_1 < x_2 < \ldots < x_L$. Then clearly

$$(12) \qquad L \leqq \nu(n),$$

where $\nu(n)$ has been defined earlier.

We shall now state three results which are obtained by suitably modifying the proofs of (i) the lemma on p. 215 of [**4**], (ii) Theorem 1 on pp. 216–217 of [**4**], and (iii) Theorem 2 on pp. 219–220 of [**4**], respectively.

LEMMA 1. *Let $P \in \mathscr{P}_{n,\alpha}$ and let $x_1, x_2, \ldots, x_L$ be defined as above. Then P is minimal if and only if there does not exist a polynomial $g \in \mathscr{P}_n$ such that*

(i) $\omega(g) = 0$,
(ii) $\hat{g}(x_l)\hat{P}(x_l) < 0$, $l = 1, 2, \ldots, L$.

LEMMA 2. *Again for a given P in $\mathscr{P}_{n,\alpha}$, let $x_1 < x_2 < \ldots < x_L$ be the roots of* (11) *in* $[-1, 1]$. *Put*

$$(13) \qquad F(x) = (1 - x)^{[(\lambda+1)/2]}(1 - x)^{[(\mu+1)/2]} \prod_{l=1}^{L} (x - x_l)$$

and

$$(14) \qquad F_l(x) = F(x)/(x - x_l), \quad l = 1, 2, \ldots, L.$$

Then P is minimal if and only if (i) *the numbers*

$$(15) \qquad \omega(F_1)(-1)\hat{P}(x_1), \omega(F_2)(-1)^2\hat{P}(x_2), \ldots, \omega(F_L)(-1)^L\hat{P}(x_L)$$

are all of the same sign and (ii) *in case $L < \nu(n)$, we have*

$$(16) \qquad \omega(F\psi) = 0$$

for all polynomials ψ of degree at most $\nu(n) - L - 1$.

LEMMA 3. *If the minimal polynomial is not unique, then there always* *exists one for which the corresponding equation* (11) *has at most* $[(\nu(n) - 1)/2]$ *roots in* $[-1, 1]$ *if* λ, μ *are both odd, at most* $[(\nu(n) + 1)/2]$ *roots in* $[-1, 1]$ *if* λ, μ *are both even, and at most* $[(\nu(n))/2]$ *roots in* $[-1, 1]$ *if* λ, μ *are not of the same parity.*

Now for a given $t \in \mathbf{R}$ we consider the functional

$$\omega(P) = P^{(j)}(t) = \frac{d^j}{dx^j}[P(x)]\Big|_{x=t}, \quad P \in \mathscr{P}_n,$$

where (i) $0 < j < n$ if $t \in \mathbf{R}\backslash\{-1, 1\}$, (ii) if $t = 1$, then $[(\lambda + 1)/2] < j < n$ or $[(\lambda + 1)/2] \leqq j < n$ according as $\lambda = 0$ or $\lambda > 0$ respectively, (iii) if $t = -1$, then $[(\mu + 1)/2] < j < n$ or $[(\mu + 1)/2] \leqq j < n$ according as $\mu = 0$ or $\mu > 0$ respectively, (iv) $j = 0$ if $t \notin [-1, 1]$, (v) $j = n$.

In the first three cases $L \geqq \nu(n) - 1$. If not, (16), which takes the form

$$(17) \qquad F^{(j)}(t)\psi(t) + \binom{j}{1}F^{(j-1)}(t)\psi'(t) + \ldots + F(t)\psi^{(j)}(t) = 0$$

would be valid for all polynomials $\psi(x)$ of degree $\nu(n) - L - 1$ or less. Since the numbers $\psi(t), \psi'(t), \ldots, \psi^{(\nu(n)-L-1)}(t)$ may be chosen arbitrarily, it would follow that

$$F^{(j)}(t) = F^{(j-1)}(t) = \ldots = F(t) = 0 \quad \text{if} \quad j \leqq \nu(n) - L - 1,$$

whereas

$$F^{(j)}(t) = F^{(j-1)}(t) = \ldots = F^{(j-\nu(n)+L+1)}(t) = 0 \qquad \text{if} \quad j > \nu(n) - L - 1.$$

Thus if L were less than $\nu(n) - 1$ then $F^{(j)}(t)$, $F^{(j-1)}(t)$ would both vanish which is clearly impossible. Hence $L \geqq \nu(n) - 1$ and by Lemma 3, the minimal polynomial is unique.

It is similarly seen that in the cases (iv), (v), L is equal to $\nu(n)$ and the minimal polynomial is unique.

For the determination of the minimal polynomial we will need some further lemmas, namely Lemmas 5, 6, 7. They concern polynomials whose zeros are all real.

We mention that between two consecutive zeros of a polynomial $f(x)$ having only real zeros there is one and only one zero of $f'(x)$.

The proof of Lemma 5 depends on the following simple fact which we state without proof.

LEMMA 4. *If the roots of the equation*

$$(18) \qquad g(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_1x + c_0 = 0$$

are all real, then for real t

$$\{g^{(j)}(t)\}^2 - g^{(j-1)}(t)g^{(j+1)}(t) \geqq 0,$$

where the sign of equality holds if and only if t is a root of (18) *of multiplicity greater than j.*

LEMMA 5. *Let $x_1, x_2, \ldots, x_n$ be real and let*

$$g(x) = \prod_{\nu=1}^{n} (x - x_\nu), \quad g_l(x) = g(x)/(x - x_l) \quad (l = 1, 2, \ldots, n).$$

If t is a root of the equation $g^{(j)}(x) = 0$ where $j < n$, then

$$(19) \qquad g_1^{(j)}(t), g_2^{(j)}(t), \ldots, g_n^{(j)}(t), g^{(j+1)}(t)$$

are all of the same sign. Further, if any one of the numbers (19) *is zero, then t must be a zero of $g(x)$ of multiplicity greater than j.*

Proof. Clearly

$$(20) \qquad g^{(m)}(x) = (x - x_l)g_l^{(m)}(x) + mg_l^{(m-1)}(x)$$

so that

$$g_l^{(j-1)}(t) = -\frac{(t - x_l)g_l^{(j)}(t)}{j},$$

$$g_l^{(j+1)}(t) = \frac{-(j + 1)g_l^{(j)}(t) + g^{(j+1)}(t)}{t - x_l} \text{ provided } t \neq x_l.$$

Hence by the preceding lemma we obtain for $t \neq x_l$

$$\{g_l^{(j)}(t)\}^2 - \left\{-\frac{(t - x_l)g_l^{(j)}(t)}{j}\right\}\left\{\frac{-(j + 1)g_l^{(j)}(t) + g^{(j+1)}(t)}{t - x_l}\right\} \geqq 0,$$

i.e.,

$$(21) \qquad g_l^{(j)}(t)g^{(j+1)}(t) \geqq \{g_l^{(j)}(t)\}^2 \geqq 0.$$

Further, equality holds if and only if t is a zero of $g_l^{(j)}(x)$ and so of $g(x)$ of multiplicity greater than j.

If $t = x_l$, then from (20) we see that $g^{(j+1)}(t)$ is equal to $(j + 1)g_l^{(j)}(t)$ and so (21) is satisfied in this case as well. It is clear that equality can hold only if $g_l^{(j)}(t)$ or $g^{(j+1)}(t)$ is zero. But $g^{(j)}(t)$ is zero and as is seen from (20), $g_l^{(j-1)}(t)$ is zero too. Hence, again equality holds in (21) if and only if t is a zero of $g(x)$ of multiplicity greater than j.

LEMMA 6. *Let A, B be two positive numbers, and $a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_s$ real numbers such that*

$$(22) \qquad b_1 \leqq a_1 \leqq b_2 \leqq a_2 \leqq \ldots \leqq b_s \leqq a_s.$$

If

$$g(x) = A \prod_{l=1}^{s} (x - a_l), \quad h(x) = B \prod_{l=1}^{s} (x - b_l)$$

https://doi.org/10.4153/CJM-1981-055-8 Published online by Cambridge University Press

and x_0 is a root of the equation $g^{(j)}(x) = 0$, then $h^{(j)}(x_0)$, $g^{(j+1)}(x_0)$ are of the same sign.

Proof. Case (i): $g^{(j+1)}(x_0) = 0$. In this case x_0 must be a zero of $g(x)$ of multiplicity at least $j + 2$, and so, in view of (22), it must be a zero of $h(x)$ of multiplicity at least $j + 1$, i.e., $h^{(j)}(x_0) = 0$.

Case (ii): $g^{(j+1)}(x_0) \neq 0$. In this case we shall show that

$$\{h^{(j)}(x_0)\}/\{g^{(j+1)}(x_0)\} \geqq 0.$$

If not,

$$\{h^{(j)}(x_0)\}/\{g^{(j+1)}(x_0)\} = -\delta < 0.$$

It is easily seen that the a_l's and b_l's can be slightly modified so that $g(x)$ becomes

$$G(x) = A \prod_{l=1}^{s} (x - a_l^*),$$

$h(x)$ becomes

$$H(x) = B \prod_{l=1}^{s} (x - b_l^*)$$

with $b_1^* < a_1^* < b_2^* < a_2^* < \ldots < b_s^* < a_s^*$ while $G^{(j)}(x_0)$ is still zero and

$$(23) \qquad H^{(j)}(x_0)/G^{(j+1)}(x_0) \leqq -\delta/2 < 0.$$

If $G_l(x) = G(x)/(x - x_l)$, then by the Lagrange interpolation formula

$$H(x) = \sum_{l=1}^{s} \frac{H(a_l^*)}{G'(a_l^*)} G_l(x) + cG(x),$$

where c is a constant. Consequently,

$$H^{(j)}(x_0) = \sum_{l=1}^{s} \frac{H(a_l^*)}{G'(a_l^*)} G_l^{(j)}(x_0).$$

But the sign of $H(a_l^*)$ is that of $(-1)^{s-l}$ and also the sign of $G'(a_l^*)$ is that of $(-1)^{s-l}$, i.e., $H(a_1^*)/G'(a_1^*)$, $H(a_2^*)/G'(a_2^*), \ldots, H(a_s^*)/G'(a_s^*)$ are all positive. Since, by Lemma 5, $G_1^{(j)}(x_0)$, $G_2^{(j)}(x_0), \ldots, G_s^{(j)}(x_0)$ are all of the same sign as $G^{(j+1)}(x_0)$, it follows that $H^{(j)}(x_0)$ and $G^{(j+1)}(x_0)$ are of the same sign. This contradicts (23) and hence $h^{(j)}(x_0)/g^{(j+1)}(x_0)$ must be $\geqq 0$.

We can similarly prove:

LEMMA 6′. *If $g(x)$, $h(x)$ are as in Lemma 6, but*

$$(22') \qquad a_1 \leqq b_1 \leqq a_2 \leqq b_2 \leqq \ldots \leqq a_s \leqq b_s$$

then $h^{(j)}(x)g^{(j+1)}(x) \leqq 0$ at all the roots of the equation $g^{(j)}(x) = 0$.

LEMMA 7. *Let*

$$-1 \leqq x_0 < x_1 < \ldots < x_s \leqq 1,$$

and consider the polynomial

$$H(x) = (x-1)^p(x+1)^q \prod_{r=0}^{s} (x - x_r).$$

If

$$H_r(x) = H(x)/(x - x_r), \quad r = 0, 1, 2, \ldots, s,$$

then $H_0^{(j)}(t), H_1^{(j)}(t), \ldots, H_s^{(j)}(t)$ *are all of the same sign if and only if*

$$H_0^{(j)}(t)H_s^{(j)}(t) \geqq 0.$$

Proof. Set $n = p + q + s + 1$. Denote by $\xi_1' \leqq \xi_2' \leqq \ldots \leqq \xi_{n-1-j}'$ the roots of the equation

$$(24) \qquad H_s^{(j)}(x) = 0,$$

and by $\eta_1' \leqq \eta_2' \leqq \ldots \leqq \eta_{n-1-j}'$ the roots of

$$(25) \qquad H_0^{(j)}(x) = 0.$$

Further, let

$$H_{r,s}(x) = \frac{H_r(x)}{x - x_s} = \frac{H_s(x)}{x - x_r} = \frac{H(x)}{(x - x_r)(x - x_s)},$$

$$(r = 0, 1, 2, \ldots, s - 1)$$

and denote by $\zeta_1, \zeta_2, \ldots, \zeta_{n-2-j}$ the roots of

$$(26) \qquad H_{0,j}^{(j)}(x) = 0.$$

Using the trivial fact $H_s^{(j)}(\zeta_l) = jH_{0,s}^{(j-1)}(\zeta_l)$ it can be easily shown that the roots of (24), (26) separate each other, i.e.,

$$(27) \qquad \xi_1' \leqq \zeta_1 \leqq \ldots \leqq \xi_l' \leqq \zeta_l \leqq \ldots \leqq \zeta_{n-2-j} \leqq \xi_{n-1-j}'.$$

When applied to $H(-x)$ this implies that also

$$(28) \qquad \eta_1' \leqq \zeta_1 \leqq \ldots \leqq \eta_l' \leqq \zeta_l \leqq \ldots \leqq \zeta_{n-2-j} \leqq \eta_{n-1-j}'.$$

Keeping these facts in mind and using

$$H_s^{(j)}(\eta_l') = (x_s - x_0)H_{0,s}^{(j)}(\eta_l'),$$

$$H_0^{(j)}(\xi_l') = (x_0 - x_s)H_{0,s}^{(j)}(\xi_l')$$

in conjunction with Lemma 5 we can show that $\xi_1' \leqq \eta_1' \leqq \ldots \leqq \xi_l' \leqq$

$\eta_l' \leqq \ldots \leqq \xi'_{n-1-j} \leqq \eta'_{n-1-j}$, i.e., we have

$$(29) \quad \xi_1' \leqq \eta_1' \leqq \zeta_1 \leqq \ldots \leqq \xi_l' \leqq \eta_l' \leqq \zeta_l \leqq \ldots \leqq \zeta_{n-2-j} \leqq \xi'_{n-1-j} \leqq \eta'_{n-1-j}.$$

It is clear that $H_0^{(j)}(t)H_s^{(j)}(t) \geqq 0$ if and only if t lies in one of the intervals

$$(30) \quad (-\infty, \xi_1'], [\eta_1', \xi_2'], \ldots, [\eta'_{n-2-j}, \xi'_{n-1-j}], [\eta'_{n-1-j}, \infty).$$

No doubt, some of these intervals may just be points $-1, +1$. The lemma will be proved if we show that $H_1^{(j)}(t), \ldots, H_s^{(j)}(t)$ are all of the same sign if t lies in any one of the intervals (30). We have

$$H_r^{(j)}(\xi_l') = \frac{d^j}{dx^j}\{(x - x_s)H_{r,s}(x)\}\Big|_{x=\xi_l'} = (x_r - x_s)H_{r,s}^{(j)}(\xi_l').$$

Hence if $-1 < \xi_l' < \eta_l' < 1$, then by Lemma 5, $H_{r,s}^{(j)}(\xi_l')$ and $H_s^{(j+1)}(\xi_l')$ have the same sign, i.e., that of $(-1)^{n-1-j-l}$. Thus $H_r^{(j)}(\xi_l')$ is of the same sign as $(-1)^{n-j-l}$. By a similar reasoning we see that $H_r^{(j)}(\eta_l')$ is of the same sign as $(-1)^{n-j-l+1}$. Consequently,

$$(31) \quad H_r^{(j)}(x) = 0 \quad (1 \leqq r \leqq s - 1)$$

must have a root in the interval (ξ_l', η_l'). If -1 is a root of multiplicity m_1 of (24) then (31) as well has a root of multiplicity m_1 at -1. Besides, if 1 is a root of multiplicity m_2 of (24), then (31) has a root of multiplicity $m_2 + 1$ or m_2 at 1 according as $x_s = 1$ or $x_s < 1$. Further, note that if $x_s < 1$, then

$$\xi'_{n-1-j-m_2} < \eta'_{n-1-j-m_2} < 1,$$

and so (31) has a root in $(\xi'_{n-1-j-m_2}, \eta'_{n-1-j-m_2})$. Thus all the roots of (31) lie in $[\xi_1', \eta'_{n-1-j}]$ and precisely one lies in (ξ_l', η_l') if $-1 < \xi_l' < \eta_l' < 1$. Now it is readily seen that $H_1^{(j)}(t), \ldots, H_s^{(j)}(t)$ are all of the same sign if t lies in any one of the intervals (30).

3. Now let us return to the study of the functional $\omega(P) = P^{(j)}(t)$ for the values of t and j specified earlier. We know that in all the five cases $L \geqq \nu(n) - 1$. Hence there are only two possibilities to be considered, namely, $L = \nu(n)$ or $L = \nu(n) - 1$.

Consider first $L = \nu(n)$. If and when this happens, the minimal polynomial must satisfy the differential equation

$$(32) \quad ||P||^2 - Z_P(x) = \sigma_n^{-2}(Z_P'(x))^2 \frac{1 - x^2}{Z_P(x)},$$

where

$$(33) \quad \sigma_n = \begin{cases} 2(\nu(n) - 1) & \text{if } \lambda, \mu \text{ are both even} \\ 2\nu(n) & \text{if } \lambda, \mu \text{ are both odd} \\ 2\nu(n) - 1 & \text{if } \lambda \not\equiv \mu \bmod 2. \end{cases}$$

Solving the differential equation by separation of variables, we obtain

$$Z_P(x) = \frac{\|P\|^2}{2}\{1 - \cos(\sigma_n \arccos x + D)\},$$

where D is a constant. If λ is even, then L can be equal to $\nu(n)$ only if $+1$ is a root of (11), i.e., $Z_P(1) = \|P\|^2$ and so $D = (2k+1)\pi$ where k is an integer. On the other hand if λ is odd, then $D = 2k\pi$ since in that case $Z_P(1) = 0$. Thus, the minimal polynomial P^* must be a constant multiple of

$$(34)\quad P_n(x) = \begin{cases} (1-x)^{\lambda/2}(1+x)^{\mu/2}T_{\nu(n)-1}(x) & \text{if } \lambda, \mu \text{ are both even} \\ (1-x)^{(\lambda+1)/2}(1+x)^{(\mu+1)/2}U_{\nu(n)-1}(x) & \text{if } \lambda, \mu \text{ are both odd} \\ (1-x)^{\lambda/2}(1+x)^{(\mu+1)/2}Q_{\nu(n)-1}(x) & \text{if } \lambda \text{ is even, } \mu \text{ is odd} \\ (1-x)^{(\lambda+1)/2}(1+x)^{\mu/2}R_{\nu(n)-1}(x) & \text{if } \lambda \text{ is odd, } \mu \text{ is even.} \end{cases}$$

Now, let us recall that

$$\omega(P^*) = \frac{d^j}{dx^j}P^*(x)\bigg|_{x=t} = \alpha,$$

and so the minimal polynomial is

$$(35)\quad P^*(x) = \frac{\alpha}{P_n^{(j)}(t)}P_n(x)$$

where $P_n(x)$ is given in (34). According to Lemma 2, the polynomial (35) will be minimal for a given t if and only if the corresponding numbers

$$(-1)\hat{P}^*(x_1)\omega(F_1),\ (-1)^2\hat{P}^*(x_2)\omega(F_2), \dots, (-1)^{\nu(n)}\hat{P}^*(x_{\nu(n)})\omega(F_{\nu(n)})$$

are of the same sign. But clearly

$$(-1)\hat{P}^*(x_1),\ (-1)^2\hat{P}^*(x_2), \dots, (-1)^{\nu(n)}\hat{P}^*(x_{\nu(n)})$$

are of the same sign. Hence we only have to look at the signs of $\omega(F_1)$, $\omega(F_2), \dots, \omega(F_{\nu(n)})$. It is easily seen that if $0 \leqq j < n$, then according as (a) λ, μ are both even, (b) λ, μ are both odd, (c) λ is even, μ is odd, (d) λ is odd, μ is even, $\omega(F_l)$ is

$$\frac{d^j}{dx^j}\left\{\frac{(1-x)^{\lambda/2}(1+x)^{\mu/2}}{2^{\nu(n)-2}}\frac{(x^2-1)U_{\nu(n)-2}(x)}{x-x_l}\right\}\bigg|_{x=t},$$

$$\frac{d^j}{dx^j}\left\{\frac{(1-x)^{(\lambda+1)/2}(1+x)^{(\mu+1)/2}}{2^{\nu(n)-1}}\frac{T_{\nu(n)}(x)}{x-x_l}\right\}\bigg|_{x=t},$$

$$\frac{d^j}{dx^j}\left\{\frac{(1-x)^{\lambda/2}(1+x)^{(\mu+1)/2}}{2^{\nu(n)-1/2}}\frac{(x-1)R_{\nu(n)-1}(x)}{x-x_l}\right\}\bigg|_{x=t},$$

$$\frac{d^j}{dx^j}\left\{\frac{(1-x)^{(\lambda+1)/2}(1+x)^{\mu/2}}{2^{\nu(n)-1/2}}\frac{(x+1)Q_{\nu(n)-1}(x)}{x-x_l}\right\}\bigg|_{x=t},$$

respectively, where $(d^0/dx^0)f(x)|_{x=t}$ means $f(t)$. Finally, if $j = n$, then

$$\omega(F_l) = (-1)^{[(\lambda+1)/2]}n!.$$

Thus, condition (i) of Lemma 2 is satisfied for $j = 0$, $t \notin [-1, 1]$ as well as for $j = n$, and so (35) must be minimal, since in these two cases $L = \nu(n)$. Further, it follows from Lemma 7 that if $0 < j < n$ then (35) is minimal for a given t if and only if

$$(36) \qquad F_1^{(j)}(t)F_{\nu(n)}^{(j)}(t) \geqq 0.$$

So if $\xi_1 \leqq \xi_2 \leqq \ldots \leqq \xi_{n-j}$, $\eta_1 \leqq \eta_2 \leqq \ldots \leqq \eta_{n-j}$ are the roots of

$$(37) \qquad F_{\nu(n)}^{(j)}(x) = 0,$$

$$(38) \qquad F_1^{(j)}(x) = 0$$

respectively, then (35) is minimal if and only if t lies in one of the intervals

$$(30^*) \qquad (-\infty, \xi_1], [\eta_1, \xi_2], \ldots, [\eta_{n-j-1}, \xi_{n-j}], [\eta_{n-j}, \infty).$$

Suppose now that $L = \nu(n) - 1$. Again using Lemma 2 we see that a polynomial $P(x)$ will be minimal if and only if

$$(39) \qquad F^{(j)}(t) = \frac{d^j}{dx^j}\left\{(1-x)^{[(\lambda+1)/2]}(1+x)^{[(\mu+1)/2]}\prod_{l=1}^{\nu(n)-1}(x-x_l)\right\}\Bigg|_{x=t} = 0$$

and

$$(-1)\hat{P}(x_1)F_1^{(j)}(t),\ (-1)^2\hat{P}(x_2)F_2^{(j)}(t), \ldots,$$
$$(-1)^{\nu(n)-1}\hat{P}(x_{\nu(n)-1})F_{\nu(n)-1}^{(j)}(t)$$

are of the same sign. But by Lemma 5, the numbers $F_1^{(j)}(t)$, $F_2^{(j)}(t), \ldots$, $F_{\nu(n)-1}^{(j)}(t)$ are all of the same sign as $F^{(j+1)}(t)$ and so it is enough to check the signs of

$$(40) \qquad (-1)\hat{P}(x_1),\ (-1)^2\hat{P}(x_2), \ldots, (-1)^{\nu(n)-1}\hat{P}(x_{\nu(n)-1}).$$

Now we distinguish four different cases:

Case I: The degree of $P(x)$ is $n - 1$. If this happens the polynomial $P(x)$ must satisfy the differential equation (32) with σ_n replaced by σ_{n-1}. The minimal polynomial is then given by (35) with $n - 1$ in place of n. Since (40) is obviously satisfied the values of t for which such a polynomial will be minimal are precisely the roots

$$(41) \qquad \rho_1, \rho_2, \ldots, \rho_{n-j}$$

of the corresponding equation (39).

It can be shown that

$$(42) \qquad \xi_l \leqq \rho_l \leqq \eta_l \quad (l = 1, 2, \ldots, n - j).$$

Let us verify it in the case when λ, μ are both even. Setting

$$g(x) = \frac{(1-x)^{\lambda/2}(1+x)^{\mu/2}}{2^{\nu(n)-2}} \frac{(x^2-1)U_{\nu(n)-2}(x)}{x-1},$$

and

$$h(x) = \frac{(1-x)^{\lambda/2}(1+x)^{\mu/2}}{2^{\nu(n)-3}} (x^2-1)U_{\nu(n)-3}(x),$$

we see that the conditions of Lemma 6′ are satisfied. Hence at the points ξ_l which are the roots of $g^{(j)}(x) = 0$ we have

$$h^{(j)}(x)g^{(j+1)}(x) \leqq 0$$

and therefore

$$h^{(j)}(\xi_l)h^{(j)}(\xi_{l+1}) \leqq 0.$$

This implies that between two consecutive ξ_l's there is a root of $h^{(j)}(x) = 0$ and so in fact

$$\xi_1 \leqq \rho_1 \leqq \xi_2 \leqq \rho_2 \leqq \ldots \leqq \xi_{n-j} \leqq \rho_{n-j}.$$

Since the minimal polynomial is of precise degree n when t lies in one of the intervals (30*) it follows that

$$-\infty < \xi_1 \leqq \rho_1 \leqq \eta_1 \leqq \xi_2 \leqq \rho_2 \leqq \eta_2 \leqq \xi_3 \leqq \rho_3 \leqq \ldots$$
$$\leqq \xi_{n-j} \leqq \rho_{n-j} \leqq \eta_{n-j} < \infty.$$

Case II: The degree of $P(x)$ is n, λ is even but 1 is not a root of (11).

Now according as μ is odd or even the equation (11) has either $\nu(n) - 1$ double roots or $\nu(n) - 2$ double roots and one simple root at -1. This implies that the minimal polynomial must satisfy the differential equation

$$||P||^2 - Z_P(x) = \tau_n^{-2}(Z_P'(x))^2 \frac{(1+x)(c-x)}{Z_P(x)}$$

where

$$\tau_n = \begin{cases} 2(\nu(n)-1) & \text{if } \mu \text{ is even} \\ 2\nu(n)-1 & \text{if } \mu \text{ is odd.} \end{cases}$$

In view of this if

$$u = \begin{cases} 1 + 2\tan^2 \dfrac{\pi}{2(\nu(n)-1)} & \text{if } \mu \text{ is even} \\[2ex] 1 + 2\tan^2 \dfrac{\pi}{2\nu(n)-1} & \text{if } \mu \text{ is odd,} \end{cases}$$

and for $c \in (1, u)$,

$$(43) \quad P_{n,c}(x) = \begin{cases} (1-x)^{\lambda/2}(1+x)^{\mu/2}T_{\nu(n)-1}\left(\dfrac{2x+1-c}{c+1}\right) & \text{if } \mu \text{ is even} \\ (1-x)^{\lambda/2}(1+x)^{(\mu+1)/2}Q_{\nu(n)-1}\left(\dfrac{2x+1-c}{c+1}\right) & \text{if } \mu \text{ is odd,} \end{cases}$$

then, the minimal polynomial must be of the form $(\alpha/P_{n,c}{}^{(j)}(t))P_{n,c}(x)$. For a given c the values of t for which $(\alpha/P_{n,c}{}^{(j)}(t))P_{n,c}(x)$ is minimal are the roots $\epsilon_1(c), \epsilon_2(c), \ldots, \epsilon_{n-j}(c)$ of the equation (39) where

$$x_l = x_l(c) = \begin{cases} \dfrac{c+1}{2}\cos\left(\dfrac{(\nu(n)-l)\pi}{\nu(n)-1}\right) + \dfrac{c-1}{2} & \text{if } \mu \text{ is even} \\ \dfrac{c+1}{2}\cos\left(\dfrac{2(\nu(n)-l)\pi}{2\nu(n)-1}\right) + \dfrac{c-1}{2} & \text{if } \mu \text{ is odd} \end{cases}$$

for $l = 1, 2, \ldots, \nu(n) - 1$. Now we wish to show that for each k, $\epsilon_k(c)$ is an increasing function of c in the range $(1, u)$. For this we observe that ϵ_k is defined implicitly by (39). Hence if $\epsilon_k \neq \pm 1$, then

$$\frac{d\epsilon_k}{dc} F^{(j+1)}(\epsilon_k) + \left.\frac{\partial F^{(j)}}{\partial c}\right|_{x=\epsilon_k} = 0.$$

But

$$\left.\frac{\partial F^{(j)}}{\partial c}\right|_{x=\epsilon_k} = -\sum_{l=1}^{\nu(n)-1} \frac{\partial x_l}{\partial c} F_l{}^{(j)}(\epsilon_k)$$

and so

$$\frac{d\epsilon_k}{dc} = \sum_{l=1}^{\nu(n)-1} \frac{\partial x_l}{\partial c} \frac{F_l{}^{(j)}(\epsilon_k)}{F^{(j+1)}(\epsilon_k)}.$$

Now note that $\partial x_l/\partial c > 0$ except when $l = 1$ and μ is even, in which case it is equal to zero. Thus using Lemma 5 we conclude that $d\epsilon_k/dc > 0$, i.e., ϵ_k is an increasing function of c in $(1, u)$. Moreover, calculating $\lim_{c\to 1+} x_l(c)$ we easily see that

$$\lim_{c\to 1+} \epsilon_l(c) = \xi_l$$

where ξ_l is a root of (37). Now, if $\lambda_l = \lim_{c\to u-} \epsilon_l(c)$, then from the above discussion it follows that for each t in (ξ_l, λ_l) there is one and only one c in $(1, u)$ for which $(\alpha/P_{n,c}{}^{(j)}(t))P_{n,c}(x)$ is minimal.

For $t = \lambda_l$, the minimal polynomial is $(\alpha/P_{n,u}{}^{(j)}(t))P_{n,u}(x)$, where

$$P_{n,u}(x) = \begin{cases} (1-x)^{\lambda/2}(1+x)^{\mu/2}T_{\nu(n)-1}\left(\dfrac{2x+1-u}{u+1}\right), & \text{if } \mu \text{ is even} \\ (1-x)^{\lambda/2}(1+x)^{(\mu+1)/2}Q_{\nu(n)-1}\left(\dfrac{2x+1-u}{u+1}\right), & \text{if } \mu \text{ is odd.} \end{cases}$$

Finally, we remark that $\lambda_l < \rho_l$ or else we would have two minimal polynomials for $t = \rho_l$.

Case III: The degree of $P(x)$ is n, μ is even but -1 is not a root of (11). Set

$$v = \begin{cases} 1 + 2\tan^2 \dfrac{\pi}{2(\nu(n) - 1)} & \text{if } \lambda \text{ is even} \\ 1 + 2\tan^2 \dfrac{\pi}{2\nu(n) - 1} & \text{if } \lambda \text{ is odd} \end{cases}$$

and for $-v \leqq d < -1$ let

$$P_{n,d}(x) = \begin{cases} (1-x)^{\lambda/2}(1+x)^{\mu/2}T_{\nu(n)-1}\left(\dfrac{2x-1-d}{1-d}\right) & \text{if } \lambda \text{ is even} \\ (1-x)^{(\lambda+1)/2}(1+x)^{\mu/2}R_{\nu(n)-1}\left(\dfrac{2x-1-d}{1-d}\right) & \text{if } \lambda \text{ is odd.} \end{cases} \tag{44}$$

By a reasoning similar to the one used in Case II we can show that if $\mu_1, \mu_2, \ldots, \mu_{n-j}$ are the roots of the equation

$$\frac{d^j}{dx^j}\left\{(1-x)^{[(\lambda+1)/2]}(1+x)^{\mu/2}\prod_{l=2}^{\nu(n)}(x - x_l)\right\} = 0$$

where

$$x_l = \begin{cases} \dfrac{v+1}{2}\cos\dfrac{(\nu(n)-l)\pi}{\nu(n)-1} - \dfrac{v-1}{2} & \text{if } \lambda \text{ is even} \\ \dfrac{v+1}{2}\cos\dfrac{(2(\nu(n)-l)+1)\pi}{2(\nu(n)-1)} - \dfrac{v-1}{2} & \text{if } \lambda \text{ is odd,} \end{cases}$$

$$(l = 2, 3, \ldots, \nu(n)),$$

then for each t in $[\mu_l, \eta_l)$ there is one and only one d in the interval $[-v, -1)$ for which $(\alpha/P_{n,d}^{(j)}(t))P_{n,d}(x)$ is minimal. Here μ_l must be greater than ρ_l.

Summarizing, we may say that we now know the minimal polynomial when t lies in the shaded intervals indicated in Figures 1, 2, 3 and 4.

To simplify the subsequent discussion we set $\lambda_l = \xi_l$ if λ is odd and $\mu_l = \eta_l$ if μ is odd.

Case IV: Here we consider the possibilities not covered previously. For this we wish to investigate how the minimal polynomial varies as t grows in the interval (λ_l, μ_l), $l = 1, 2, \ldots, n - j$. In order to facilitate the study let us exhibit the parameter t explicitly; for example write the minimal polynomial corresponding to a given value t as

$$P^*(x, t) = \sum_{\nu=0}^{n} a_\nu^*(t)x^\nu.$$

Further, we use the notation $f_{j,k}(x, t)$ for $(\partial^{j+k}/\partial x^j \partial t^k)f(x, t)$.

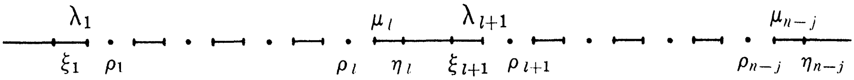

FIG. 1. (λ, μ both even)

FIG. 2. (λ, μ both odd)

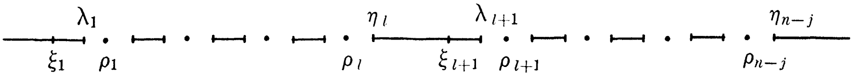

FIG. 3. (λ even, μ odd)

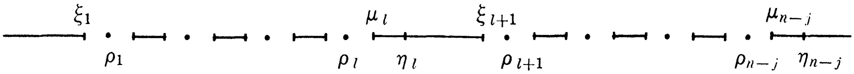

FIG. 4. (λ odd, μ even)

Now let $x_1(t), x_2(t), \ldots, x_{\nu(n)-1}(t)$ be the roots of the corresponding equation (11). The fact that the quantities (40) have to be of the same sign implies that there exists one and only one root $y_l(t)$ of $Z_{P*}(x, t) = 0$ in $(x_l(t), x_{l+1}(t))$, $l = 1, 2, \ldots, \nu(n) - 2$. Hence if

$$W(x, t) = ||P^*||^{-2} Z_{P*}(x, t) = \sum_{l=0}^{N} c_l(t) x^{N-l}$$

where N is the degree of W, then the form of Z_{P*} shows that $W(x, t)$ must have one further double root $\delta(t)$ which must necessarily be outside the interval $[x_1, x_{\nu(n)-1}]$. Consequently, $W_{1,0}(x, t)$ vanishes at all the double roots of (11), as well as at the points $y_1(t), y_2(t), \ldots, y_{\nu(n)-2}(t)$, $\delta(t)$. In addition it must have one more real root $\beta(t)$. Depending on the parity of λ, μ we will have four different possibilities. Let us examine them in the case $\beta(t) > x_{\nu(n)-1}$; the case $\beta(t) < x_1$ being symmetrical.

a) Suppose that λ, μ are both even:

Under this hypothesis equation (11) has $\nu(n) - 3$ double roots in $(-1, 1)$ along with a simple root at each of the points $-1, +1$. This situation is illustrated in Fig. 5.

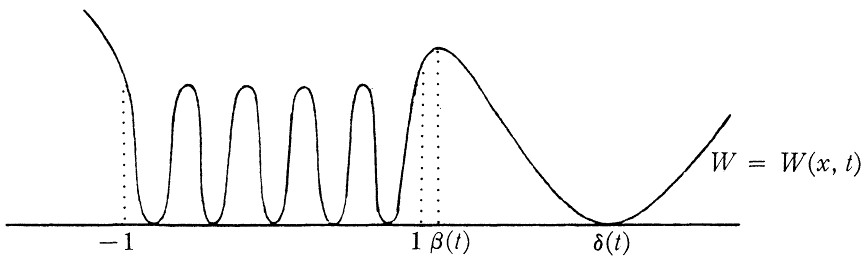

FIG. 5

b) Suppose that λ, μ are both odd:

Under this hypothesis equation (11) has $\nu(n) - 1$ double roots in $(-1, 1)$. The following three diagrams illustrate how $W(x, t)$ may possibly look:

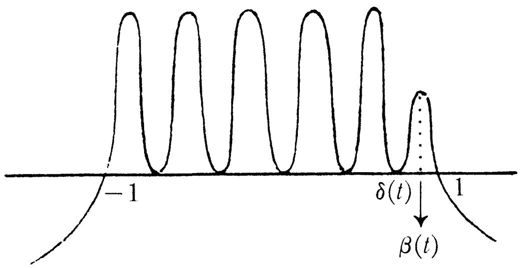

FIG. 6.1

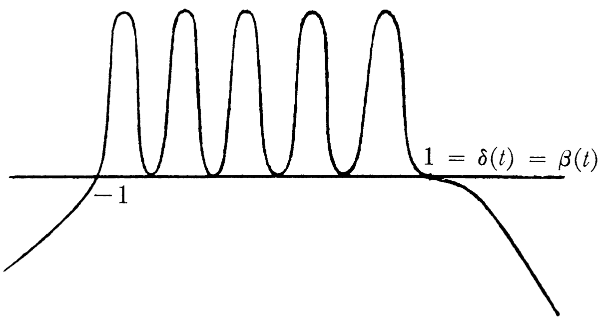

FIG. 6.2

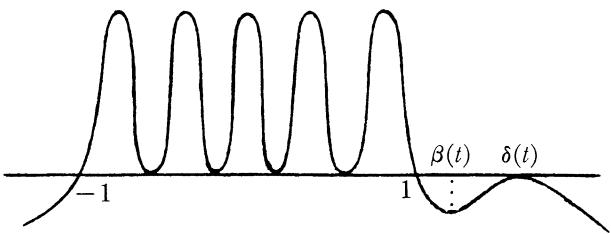

FIG. 6.3

c) Assume now that λ is even, μ odd:

Under this hypothesis equation (11) has $\nu(n) - 2$ double roots in $(-1, +1)$ along with a simple root at $+1$ (see Fig. 7).

d) Finally, suppose that λ is odd, μ even:

Under this hypothesis equation (11) has $\nu(n) - 2$ double roots in $(-1, +1)$ and a simple root at -1 (see Figures 8.1, 8.2, 8.3).

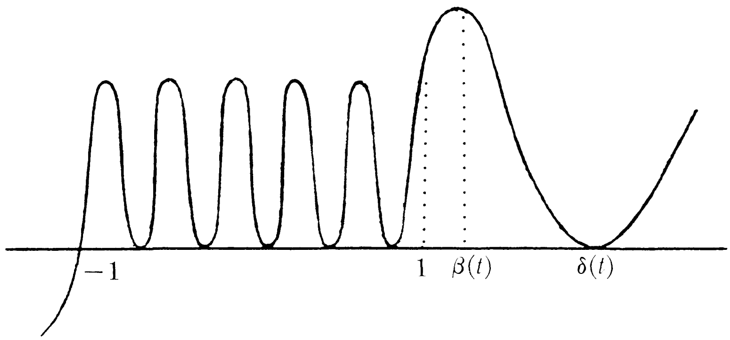

Fig. 7

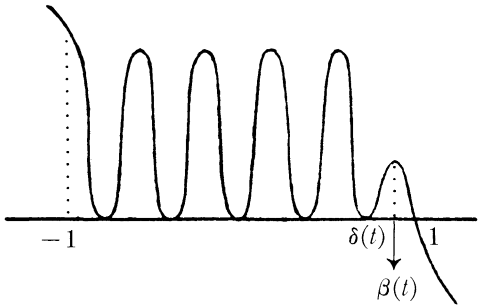

Fig. 8.1

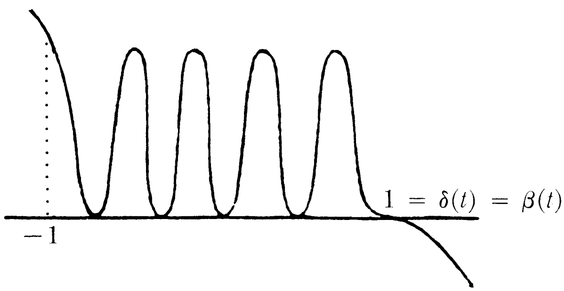

Fig. 8.2

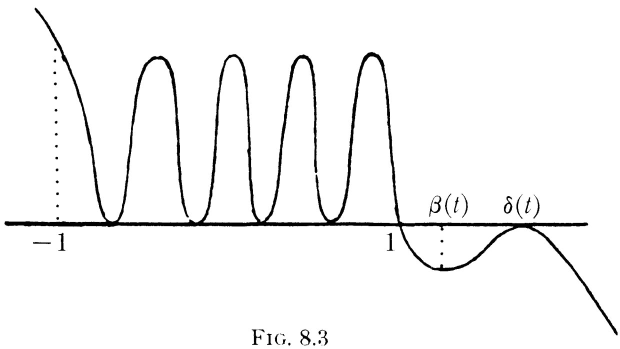

Fig. 8.3

https://doi.org/10.4153/CJM-1981-055-8 Published online by Cambridge University Press

We will now study possibility d) in detail. The other three can be dealt with similarly. The possibility d) can be analytically described as follows:

The degree N of $W(x, t)$ is $2\nu(n) - 1$, and the $4\nu(n) - 2$ quantities $c_0(t), c_1(t), \ldots, c_{2\nu(n)-1}(t)$, $x_2(t), x_3(t), \ldots, x_{\nu(n)-1}(t)$, $y_1(t), y_2(t), \ldots, y_{\nu(n)-2}(t)$, $\delta(t)$, $\beta(t)$ are implicitly defined by the system of $4\nu(n) - 2$ equations

$$(45)\quad \begin{cases} W(1, t) = 0, & W(-1, t) = 1 \\ W(x_l(t), t) = 1, & W_{1,0}(x_l(t), t) = 0, \ (l = 2, 3, \ldots, \nu(n) - 1) \\ W(y_l(t), t) = 0, & W_{1,0}(y_l(t), t) = 0, \ (l = 1, 2, \ldots, \nu(n) - 2) \\ W(\delta(t), t) = 0, & W_{1,0}(\delta(t), t) = 0 \\ & W_{1,0}(\beta(t), t) = 0 \\ F_{j,0}(t, t) = 0. & \end{cases}$$

Observe that $x_l(t)$ cannot be equal to $y_l(t)$. Also, $\lambda_l = \xi_l$ so that the system (45) is satisfied for $t = \lambda_l$ except that the last equation is to be replaced by

$$\left.\left\{\frac{d^j}{dx^j} F_{\nu(n)}(x, \lambda_l)\right\}\right|_{x=\lambda_l} = 0.$$

Further

$$(46)\quad \begin{aligned} -1 = x_1(\lambda_l) < y_1(\lambda_l) < \ldots < x_{\nu(n)-1}(\lambda_l) \\ < y_{\nu(n)-1}(\lambda_l) = \delta(\lambda_l) < \beta(\lambda_l) = x_{\nu(n)}(\lambda_l) < 1. \end{aligned}$$

So if we show that $x_l(t)$, $2 \leqq l \leqq \nu(n) - 1$ and $y_l(t)$, $1 \leqq l \leqq \nu(n) - 2$ are increasing functions of t in (λ_l, ρ_l), then we would have

$$(47)\quad -1 = x_1(t) < y_1(t) < x_2(t) < \ldots < x_{\nu(n)-1}(t) < +1$$

for all t in (λ_l, ρ_l). Differentiating the system (45) with respect to t we obtain

$$(48)\quad \begin{cases} W_{0,1}(-1, t) = W_{0,1}(1, t) = 0 \\ W_{0,1}(x_l(t), t) + x_l'(t) W_{1,0}(x_l(t), t) = 0, \ (l = 2, 3, \ldots, \nu(n) - 1) \\ W_{0,1}(y_l(t), t) + y_l'(t) W_{1,0}(y_l(t), t) = 0, \ (l = 1, 2, \ldots, \nu(n) - 2) \\ W_{0,1}(\delta(t), t) + \delta'(t) W_{1,0}(\delta(t), t) = 0, \end{cases}$$

$$(49)\quad \begin{cases} W_{1,1}(x_l(t), t) + x_l'(t) W_{2,0}(x_l(t), t) = 0, \ (l = 2, 3, \ldots, \nu(n) - 1) \\ W_{1,1}(y_l(t), t) + y_l'(t) W_{2,0}(y_l(t), t) = 0, \ (l = 1, 2, \ldots, \nu(n) - 2) \\ W_{1,1}(\delta(t), t) + \delta'(t) W_{2,0}(\delta(t), t) = 0 \\ W_{1,1}(\beta(t), t) + \beta'(t) W_{2,0}(\beta(t), t) = 0, \end{cases}$$

https://doi.org/10.4153/CJM-1981-055-8 Published online by Cambridge University Press

and

$$(50)\qquad F_{j+1,0}(t,t) - \sum_{k=2}^{\nu(n)-1} x_k'(t)\left\{\frac{\partial^j}{\partial x^j} F_k(x,t)\right\}\Big|_{x=t} = 0.$$

Using (45) we deduce from (48) that

$$W_{0,1}(-1,t) = W_{0,1}(1,t) = W_{0,1}(x_l(t),t) = W_{0,1}(y_m(t),t) \\ = W_{0,1}(\delta(t),t) = 0$$

for $l = 2, 3, \ldots, \nu(n) - 1$ and $m = 1, 2, \ldots, \nu(n) - 2$. Therefore

$$(51)\qquad W_{0,1}(x,t) = c_0'(t)\cdot(x-\delta(t))(x^2-1) \\ \times \prod_{l=2}^{\nu(n)-1}(x-x_l(t))\prod_{m=1}^{\nu(n)-2}(x-y_m(t)).$$

Now set

$$f(x,t) = \prod_{l=2}^{\nu(n)-1}(x-x_l(t)),\quad q(x,t) = (x-\delta(t))\prod_{m=1}^{\nu(n)-2}(x-y_m(t)).$$

Then clearly

$$W_{1,1}(x,t) = c_0'(t)\cdot(x^2-1)\{f(x,t)q_{1,0}(x,t) + f_{1,0}(x,t)q(x,t)\} \\ + 2c_0'(t)xf(x,t)q(x,t).$$

Now substituting for $W_{1,1}$ in (49) we get

$$(52)\qquad \begin{cases} x_l'(t) = -c_0'(t)(x_l^2(t)-1)f_{1,0}(x_l(t),t)q(x_l(t),t)/W_{2,0}(x_l(t),t), \\ \qquad\qquad (l = 2, 3, \ldots, \nu(n)-1) \\ y_l'(t) = -c_0'(t)(y_l^2(t)-1)f(y_l(t),t)q_{1,0}(y_l(t),t)/W_{2,0}(y_l(t),t), \\ \qquad\qquad (l = 1, 2, \ldots, \nu(n)-2) \\ \delta'(t) = -c_0'(t)(\delta^2(t)-1)f(\delta(t),t)q_{1,0}(\delta(t),t)/W_{2,0}(\delta(t),t) \\ \beta'(t) = -c_0'(t)\dfrac{\partial}{\partial x}\{f(x,t)q(x,t)(x^2-1)\}|_{x=\beta(t)}\{1/W_{2,0}(\beta(t),t)\}. \end{cases}$$

But

$$W_{1,0}(x,t) = Nc_0(t)f(x,t)q(x,t)(x-\beta(t)),$$

and so

$$W_{2,0}(x,t) = Nc_0(t)\{f_{1,0}(x,t)q(x,t)(x-\beta(t)) \\ + f(x,t)q_{1,0}(x,t)(x-\beta(t)) + f(x,t)q(x,t)\}.$$

https://doi.org/10.4153/CJM-1981-055-8 Published online by Cambridge University Press

Hence

$$(53)\quad \begin{cases} x_l'(t) = -\dfrac{c_0'(t)}{Nc_0(t)}\dfrac{x_l^2(t)-1}{x_l(t)-\beta(t)}, & (l = 2, 3, \ldots, \nu(n)-1) \\[2ex] y_l'(t) = -\dfrac{c_0'(t)}{Nc_0(t)}\dfrac{y_l^2(t)-1}{y_l(t)-\beta(t)}, & (l = 1, 2, \ldots, \nu(n)-2) \\[2ex] \delta'(t) = -\dfrac{c_0'(t)}{Nc_0(t)}\dfrac{\delta^2(t)-1}{\delta(t)-\beta(t)}. \end{cases}$$

In order to determine the sign of $c_0'(t)/c_0(t)$ we substitute the value of $x_k'(t)$ in (50) to obtain

$$F_{j+1,0}(t, t) = -\frac{c_0'(t)}{Nc_0(t)}\sum_{k=2}^{\nu(n)-1}\left\{\left(\frac{x_k^2(t)-1}{x_k(t)-\beta(t)}\right)\left(\frac{\partial^j}{\partial x^j}F_k(x, t)\right)\bigg|_{x=t}\right\}.$$

Keeping in mind the fact that $F_{j,0}(t, t) = 0$, we may apply Lemma 5 to conclude that

$$F_{j+1,0}(t, t) \quad \text{and} \quad \left(\frac{\partial^j}{\partial x^j}F_k(x, t)\right)\bigg|_{x=t}, \quad (k = 2, 3, \ldots, \nu(n)-1)$$

are all of the same sign. Hence $-c_0'(t)/c_0(t) \geqq 0$. From this it follows that $x_l'(t)$, $(l = 2, 3, \ldots, \nu(n)-1)$, $y_l'(t)$, $(l = 1, 2, \ldots, \nu(n)-2)$, and $\delta'(t)$ are all positive. Now we wish to show that the same is true about $\beta'(t)$. In fact

$$\beta'(t) = -\frac{c_0'(t)}{Nc_0(t)}\left\{\frac{\partial}{\partial x}\left(f(x, t)q(x, t)(x^2-1)\right)\right\}\bigg|_{x=\beta(t)}\frac{Nc_0(t)}{W_{2,0}(\beta(t), t)},$$

and so it is enough to apply Lemma 6 to the polynomials

$$g(x) := W(x, t)/Nc_0(t),\ h(x) := f(x, t)q(x, t)(x^2-1)$$

at the point $x = \beta(t)$ to obtain $\beta'(t) \geqq 0$.

If $\beta(t)$ were bounded in $[\lambda_l, \rho_l)$, then at the point $t = \rho_l$ we would have two minimal polynomials, one of degree $n-1$ and another of degree n. So for each l where $l = 1, 2, \ldots, n-j$ there exists a point τ_l in $(\lambda_l, \rho_l]$ such that if $t \in [\lambda_l, \tau_l)$ then the minimal polynomial is a solution of the system (45) for which (47) holds while

$$\lim_{t\to\tau_l-}\beta(t) = +\infty,$$

and, a fortiori,

$$\lim_{t\to\tau_l-}\delta(t) = +\infty.$$

On the other hand, when $t \to \tau_l-$, (47) remains true except that the last inequality may not be strict. This means that the coefficients of the polynomial

$$h(x, t) = \frac{W_{1,0}(x, t)}{Nc_0(t)(x - \beta(t))(x - \delta(t))} = \prod_{l=2}^{\nu(n)-1} (x - x_l(t)) \prod_{l=1}^{\nu(n)-2} (x - y_l(t))$$

stay bounded as $t \to \tau_l -$. But

$$W(x, t) = Nc_0(t) \int_{-1}^{x} (z - \beta(t))(z - \delta(t))h(z, t)dz + W(-1, t); \tag{54}$$

moreover since $W(y_1, t) - W(-1, t) = -1$ and $h(z, t)$ does not change sign on $[-1, y_1)$ we get

$$Nc_0(t) = -\left\{ \int_{-1}^{y_1(t)} z^2 h(z, t)dz - (\beta(t) + \delta(t)) \int_{-1}^{y_1(t)} zh(z, t)dz + \beta(t)\delta(t) \int_{-1}^{y_1(t)} h(z, t)dz \right\}^{-1},$$

$$Nc_0(t)(\beta(t) + \delta(t)) = -\left\{ \frac{1}{\beta(t) + \delta(t)} \int_{-1}^{y_1(t)} z^2 h(z, t)dz - \int_{-1}^{y_1(t)} zh(z, t)dz + \left(\frac{1}{\beta(t)} + \frac{1}{\delta(t)} \right)^{-1} \int_{-1}^{y_1(t)} h(z, t)dz \right\}^{-1}$$

$$Nc_0(t)\beta(t)\delta(t) = -\left\{ \frac{1}{\beta(t)\delta(t)} \int_{-1}^{y_1(t)} z^2 h(z, t)dz - \left(\frac{1}{\beta(t)} + \frac{1}{\delta(t)} \right) \int_{-1}^{y_1(t)} zh(z, t)dz + \int_{-1}^{y_1(t)} h(z, t)dz \right\}^{-1}.$$

Thus we simultaneously have

$$\begin{aligned} &\lim_{t \to \tau_l -} c_0(t) = 0, \quad \lim_{t \to \tau_l -} c_0(t)(\beta(t) + \delta(t)) = 0, \\ &\lim_{t \to \tau_l -} c_0(t)\beta(t)\delta(t) = L < \infty. \end{aligned} \tag{55}$$

Now taking the limit in (54) as $t \to \tau_l -$ we see that $W(x, t)$ converges uniformly on $[-1, 1]$ to a polynomial of degree $N - 2$, which implies that the polynomial $P^*(x, t)$ converges to a polynomial of degree $n - 1$ for which $L = \nu(n) - 1$. But then this polynomial must be the minimal polynomial arising in Case I, and τ_l must be equal to ρ_l.

By symmetry, the minimal polynomial for a value t lying in (ρ_l, μ_l) corresponds to a solution of the system (45) for which $\beta(t) < x_1(t)$.

Section 4.

Proof of Theorem 1'. Let δ be a small positive number. For a fixed c such that $1 - \delta \leqq c \leqq 1$ consider the normed linear space $\pi_{n+1,c}$ of all polynomials P of degree at most $n + 1$ vanishing at $-c$ and having a zero of multiplicity at least $[(\lambda + 1)/2]$ at 1 and a zero of multiplicity at least $[(\lambda + 1)/2]$ or $[(\lambda + 3)/2]$ at -1 according as $c \neq 1$ or $c = 1$, and where

$$\|P\| = \max_{-1 \leqq x \leqq 1} |(x + c)^{-1}(1 - x^2)^{-\lambda/2}P(x)|.$$

We wish to determine the norm of the functional $\omega(P) = P^{(j)}(0)/j!$ defined on $\pi_{n+1,c}$. It is easily checked that the reasoning of Section 2 leading to the characterisation of the extremal polynomials remains valid, and in particular if

$$(34') \quad P_n(x) = \begin{cases} (1 - x^2)^{\lambda/2}T_{n-\lambda}(x) & \text{if } \lambda \text{ is even} \\ (1 - x^2)^{(\lambda+1)/2}U_{n-\lambda-1}(x) & \text{if } \lambda \text{ is odd} \end{cases}$$

and

$$F(x) = \begin{cases} (1 - x^2)^{\lambda/2}(x + c)(x^2 - 1)U_{n-\lambda-1}(x) & \text{if } \lambda \text{ is even} \\ (1 - x^2)^{(\lambda+1)/2}(x + c)T_{n-\lambda}(x) & \text{if } \lambda \text{ is odd} \end{cases}$$

then $(x + c)P_n(x)$ is extremal if and only if

$$F_1{}^{(j)}(0)F^{(j)}_{\nu(n+1)}(0) \geqq 0.$$

Let us write $F(x) = (x^2 - 1)(x + c)G(x)$, where $G(x)$ is

$$(1 - x^2)^{\lambda/2}U_{n-\lambda-1}(x) \quad \text{if } \lambda \text{ is even} \quad \text{and}$$
$$-(1 - x^2)^{(\lambda-1)/2}T_{n-\lambda}(x) \quad \text{if } \lambda \text{ is odd.}$$

If we define

$$F_0(x) = F(x)/(x + 1) = (x - 1)(x + c)G(x) \quad \text{and}$$
$$F_{\nu(n+1)+1}(x) = F(x)/(x - 1) = (x + 1)(x + c)G(x)$$

then by Lemma 7,

$$F_1{}^{(j)}(0)F^{(j)}_{\nu(n+1)}(0) \geqq 0$$

if the same is true for $F_0{}^{(j)}(0)F^{(j)}_{\nu(n+1)+1}(0)$.

Clearly

$$\begin{aligned} &F_0{}^{(j)}(0)F^{(j)}_{\nu(n+1)+1}(0) \\ &\quad = \{-cG^{(j)}(0) - (1 - c)jG^{(j-1)}(0) + j(j - 1)G^{(j-2)}(0)\} \\ &\quad \times \{cG^{(j)}(0) + (1 + c)jG^{(j-1)}(0) + j(j - 1)G^{(j-2)}(0)\}. \end{aligned}$$

Note that for any n the polynomial $G(x)$ (which is of degree $n - 1$) is either even or odd. If $n - j$ is even then $n - 1 - j$ is odd and so

$$G^{(j)}(0) = G^{(j-2)}(0) = 0.$$

Hence

$$F_0{}^{(j)}(0)F^{(j)}_{\nu(n+1)+1}(0) = -(1 - c^2)j^2(G^{(j-1)}(0))^2,$$

and consequently for the polynomial $(x + c)P_n(x)$ to be extremal it is sufficient that c be equal to 1. The condition $c = 1$ is also necessary if λ is even since in that case

$$F_0(x) = F_1(x), \quad F_{\nu(n+1)+1}(x) = F_{\nu(n+1)}(x).$$

Now we note that if $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree at most n with real coefficients such that $|p_n(x)| \leqq (1 - x^2)^{\lambda/2}$ for $-1 \leqq x \leqq 1$, then

$$|(x+1)p_n(x)| = \left| a_0 + \sum_{\nu=1}^{n} (a_\nu + a_{\nu-1})x^\nu + a_n x^{n+1} \right|$$
$$\leqq (x+1)(1-x^2)^{\lambda/2} \quad \text{on } [-1, 1]$$

and so provided $n - j$ is even

$$|a_j + a_{j-1}| \leqq |\gamma_{n,j} + \gamma_{n,j-1}| = |\gamma_{n,j}|.$$

Similarly

$$|a_j - a_{j-1}| \leqq |\gamma_{n,j} - \gamma_{n,j-1}| = |\gamma_{n,j}|.$$

Consequently (7′) holds if $\theta \leqq 1$. Besides, it follows from above that if $\theta > 1$ then at least when λ is even there exists a polynomial $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ of degree at most n with real coefficients such that $|p_n(x)| \leqq (1 - x^2)^{\lambda/2}$ for $-1 \leqq x \leqq 1$ but for which

$$|a_j| + \theta|a_{j-1}| > |\gamma_{n,j}|.$$

Now we wish to show that if $n - j$ is odd, then

$$|a_j| \leqq |\gamma_{n-1,j}|.$$

We know that $P_{n-1}(x)$ is extremal for the functional $\omega(P) = P^{(j)}(0)/j!$ defined on the normed linear space $\mathscr{P}_n$ if and only if $F^{(j)}(0) = 0$ where

$$F(x) = \begin{cases} (1-x^2)^{\lambda/2}(x^2-1)U_{n-\lambda-1}(x) & \text{if } \lambda \text{ is even} \\ (1-x^2)^{(\lambda+1)/2}T_{n-\lambda-1}(x) & \text{if } \lambda \text{ is odd.} \end{cases}$$

Note that $F(x)$ is even or odd according as n is even or odd respectively. Hence $n - j$ being odd, $F^{(j)}(0) = 0$.

In order to verify the last statement of Theorem 1′ we consider the normed linear space $\pi_{n+1,c}$ of all polynomials P of degree at most $n + 1$ vanishing at $-c$ (c real and $|c| > 1$) and having zeros of multiplicity $[(\lambda + 1)/2]$ at $+1$, -1, and where

$$\|P\| = \max_{-1 \leqq x \leqq 1} |(x + c)^{-1}(1 - x^2)^{-\lambda/2}P(x)|.$$

We claim that again the reasoning of Section 2 leading to the characterisation of the extremal polynomials corresponding to the functional $\omega(P) = P^{(j)}(t)$ remains valid. Thus, in order to show that $(x + c)P_{n-1}(x)$ is never extremal we simply need to check that $F^{(j)}(0) \neq 0$ where

$$F(x) = \begin{cases} (1-x^2)^{\lambda/2}(x+c)(x^2-1)U_{n-\lambda-1}(x) & \text{if } \lambda \text{ is even} \\ (1-x^2)^{(\lambda+1)/2}(x+c)T_{n-\lambda-1}(x) & \text{if } \lambda \text{ is odd.} \end{cases}$$

https://doi.org/10.4153/CJM-1981-055-8 Published online by Cambridge University Press

Again we write $F(x) = (x^2 - 1)(x + c)G(x)$ where, for every n, $G(x)$ is either even or odd. Calculating $F^{(j)}(0)$ we get

$$F^{(j)}(0) = -cG^{(j)}(0) - jG^{(j-1)}(0) + cj(j-1)G^{(j-2)}(0) + j(j-1)(j-2)G^{(j-3)}(0).$$

Since $n - j$ is odd $G^{(j)}(0) = G^{(j-2)}(0) = 0$. Also, on examining $G(x)$ more closely we notice that the non-zero coefficients alternate in sign, i.e., $G^{(j-1)}(0)$ and $G^{(j-3)}(0)$ are of opposite sign and hence $F^{(j)}(0) \neq 0$.

Proof of Theorem 2. Let

$$M_n = \sup \{\max_{[-1,1]} |P^{(j)}(x)| \ |P \in \mathscr{P}_n, \|P\| \leqq 1\}.$$

It is obvious that if $Q \in \mathscr{P}_n$ is such that $\|Q\| \leqq 1$,

$$|Q^{(j)}(t_0)| = \max_{-1 \leqq x \leqq 1} |Q^{(j)}(x)| = M_n$$

then $Q(x) = p(x, t_0)$ where $p(x, t_0)$ is an extremal polynomial for the functional $\omega(P) = P^{(j)}(t_0)$ on $\mathscr{P}_n$ and so

$$M_n = \max_{-1 \leqq t \leqq 1} |p_{j,0}(t, t)|.$$

Clearly, $p(x, t)$ is of norm 1. If for a given α, $P^*(x, t)$ is the minimal polynomial in $\mathscr{P}_{n,\alpha}$ then

$$P^*(x, t) = \frac{\alpha}{\omega(p)} p(x, t).$$

Using the same notation as before we get

$$W(x, t) = Z_{P^*}(x, t)/\|P^*\|^2 = (P^*(x, t))^2/\|P^*\|^2(1 - x)^\lambda(1 + x)^\mu = p^2(x, t)/(1 - x)^\lambda(1 + x)^\mu.$$

Thus if $p(x, t) = \sum_{\nu=0}^n d_\nu(t)\, x^{n-\nu}$ then

$$c_0(t) = (-1)^\lambda d_0{}^2(t).$$

We know that for

$$t \in (-\infty, \xi_1] \cup \left\{ \bigcup_{l=1}^{n-j-1} [\eta_l, \xi_{l+1}] \right\} \cup [\eta_{n-j}, \infty),$$

$$p(x, t) = P_n(x)$$

whereas for $t = \rho_l$ $(l = 1, 2, \ldots, n - j)$,

$$p(x, t) = P_{n-1}(x).$$

Hence the desired result will follow if we show that the function $|p_j(t, t)|$ cannot have a local maximum on the intervals (ξ_l, ρ_l), (ρ_l, η_l) $(l = 1, 2, \ldots, n - j)$.

Let us put $A(t) = p_{j,0}(t, t)$. In order to show that a local extremum of $|A(t)|$ is necessarily a minimum it is enough to show that, if t is a point belonging to any of the intervals (ξ_l, λ_l), (λ_l, ρ_l), (ρ_l, μ_l), (μ_l, η_l) such that $A'(t) = 0$, then $A(t)A''(t) \geqq 0$ whereas if $A'(t_0) = 0$ for $t_0 = \lambda_l$ or $t_0 = \mu_l$ then

$$\lim_{t\to t_0-} A(t)A''(t) \geqq 0 \quad \text{and} \quad \lim_{t\to t_0+} A(t)A''(t) \geqq 0.$$

This consideration is made necessary by the fact that at the points λ_l, μ_l the function $A''(t)$ is not continuous (see [**9**, p. 193, Theorem 6]). Here we shall restrict ourselves to an interval of the form (ξ_l, ρ_l), the corresponding result for intervals of the form (ρ_l, η_l) being obtained by symmetry.

Using (51) and the fact that

$$F(x, t) = (1 - x)^{[(\lambda+1)/2]}(1 + x)^{[(\mu+1)/2]} \prod_{l=1}^{\nu(n)-1} (x - x_l(t)),$$

$$p(x, t) = (-1)^{[(\lambda+1)/2]}d_0(t)(1 - x)^{[(\lambda+1)/2]}(1 + x)^{[(\mu+1)/2]} \times \prod_{l=1}^{\nu(n)-2} (x - y_l(t))(x - \delta(t))$$

(where $\delta(t) = y_{\nu(n)-1}(t)$ if $t \in (\xi_l, \lambda_l])$ we easily obtain

(56) $$p_{0,1}(x, t) = (-1)^{[(\lambda+1)/2]}d_0'(t)F(x, t).$$

Since on the interval under consideration $F_{j,0}(t, t) = 0$ this implies that

$$\begin{aligned} A'(t) &= p_{j+1,0}(t, t) + p_{j,1}(t, t) \\ &= p_{j+1,0}(t, t) + (-1)^{[(\lambda+1)/2]}d_0'(t)F_{j,0}(t, t) \\ &= p_{j+1,0}(t, t). \end{aligned}$$

Further

$$A''(t) = p_{j+2,0}(t, t) + (-1)^{[(\lambda+1)/2]}d_0'(t)F_{j+1,0}(t, t).$$

Let us now suppose that there exists a point $t_0 \in (\xi_l, \rho_l)$ such that

$$A'(t_0) = p_{j+1,0}(t_0, t_0) = 0$$

and consider the product $A(t)A''(t)$ for t in a neighbourhood of t_0. For this we recall that if

$$\phi(x, t) = \begin{cases} \displaystyle\sum_{k=1}^{\nu(n)-1} (x_k(t) + 1)F_k(x, t) & \text{for } t \in (\xi_l, \lambda_l) \\ \displaystyle\sum_{k=1}^{\nu(n)-1} \frac{x_k^2(t) - 1}{x_k(t) - \beta(t)} F_k(x, t) & \text{for } t \in (\lambda_l, \rho_l) \end{cases}$$

then

(57) $$F_{j+1,0}(t, t)/\phi_{j,0}(t, t) = -c_0'(t)/Nc_0(t) \geqq 0.$$

It is a matter of simple calculation that

$$\phi(x, t) = -nF(x, t) + (x + \beta(t))F_{1,0}(x, t) + \psi(x, t)$$

(here and in the sequel $\beta(t)$ is taken to be 1 for $t \in (\xi_l, \lambda_l)$) where

$$(58) \quad (x - 1)\psi(x, t) = -2\left[\frac{\lambda + 1}{2}\right]F(x, t) \quad \text{if } t \in (\xi_l, \lambda_l)$$

whereas

$$\frac{\psi(x, t)}{\beta^2(t) - 1} = \left[\frac{\lambda + 1}{2}\right] \frac{F(x, t)}{(x - 1)(1 - \beta(t))}$$

$$+ \left[\frac{\mu + 1}{2}\right] \frac{F(x, t)}{(x + 1)(-1 - \beta(t))} + \sum_{l=1}^{\nu(n)-1} \frac{F_l(x, t)}{x_l(t) - \beta(t)} \quad \text{if } t \in (\lambda_l, \rho_l).$$

As well, it can be easily checked that in case $t \in (\lambda_l, \rho_l)$

$$(59) \quad (x - \beta(t)) \frac{\psi(x, t)}{\beta^2(t) - 1} = F_{1,0}(x, t) - \frac{F_{1,0}(\beta(t), t)}{F(\beta(t), t)} F(x, t).$$

Differentiating the two sides of (58) and of (59) j times with respect to x and then putting $x = t$ we obtain for $t \in (\xi_l, \lambda_l)$

$$(60) \quad (t - 1)\psi_{j,0}(t, t) + j\psi_{j-1,0}(t, t) = -2\left[\frac{\lambda + 1}{2}\right]F_{j,0}(t, t)$$

and for $t \in (\lambda_l, \rho_l)$

$$(t - \beta(t))\psi_{j,0}(t, t) + j\psi_{j-1,0}(t, t)$$

$$= (\beta^2(t) - 1)\left\{F_{j+1,0}(t, t) - \frac{F_{1,0}(\beta(t), t)}{F(\beta(t), t)} F_{j,0}(t, t)\right\}$$

which, in view of the fact that $F_{j,0}(t, t) = 0$, gives us

$$\psi_{j,0}(t, t) = \frac{(\beta^2(t) - 1)F_{j+1,0}(t, t) - j\psi_{j-1,0}(t, t)}{t - \beta(t)}.$$

Similarly

$$\phi_{j,0}(t, t) = (t + \beta(t))F_{j+1,0}(t, t) + \psi_{j,0}(t, t).$$

Combining the two preceding relations we obtain

$$\phi_{j,0}(t, t) = \frac{t^2 - 1}{t - \beta(t)} F_{j+1,0}(t, t) - \frac{j}{t - \beta(t)} \psi_{j-1,0}(t, t).$$

Examining the system (45), we see that

$$W_{1,0}(x, t) = Nc_0(t) \prod_{l \in \mathscr{L}} (x - x_l(t)) \prod_{m=1}^{\nu(n)-2} (x - y_m(t))(x - \delta(t)) \times (x - \beta(t))$$

https://doi.org/10.4153/CJM-1981-055-8 Published online by Cambridge University Press

where $\mathscr{L}$ is the set of indices of the double roots of (11). This can be rewritten as

$$\frac{d}{dx}\left\{\frac{p^2(x,t)}{(1-x)^\lambda(1+x)^\mu}\right\} = (-1)^{[(\lambda+1)/2]}\frac{Nd_0(t)F(x,t)p(x,t)(x-\beta(t))}{(1-x)^\lambda(1+x)^\mu(x^2-1)},$$

from which we deduce

$$(61)\quad p_{1,0}(x,t)(x^2-1)+p(x,t)\left\{\left(\frac{\mu-\lambda}{2}\right)-\left(\frac{\lambda+\mu}{2}\right)x\right\} = (-1)^{[(\lambda+1)/2]}\frac{Nd_0(t)}{2}F(x,t)(x-\beta(t)).$$

Differentiating the two sides $j+1$ times with respect to x and putting $x=t$ we obtain

$$(62)\quad p_{j+2,0}(t,t)(t^2-1)+p_{j+1,0}(t,t)\left\{\left(2(j+1)-\left(\frac{\lambda+\mu}{2}\right)\right)t + \frac{\mu-\lambda}{2}\right\} + p_{j,0}(t,t)\left(j-\frac{\lambda+\mu}{2}\right)(j+1) = (-1)^{[(\lambda+1)/2]}\frac{Nd_0(t)}{2}F_{j+1,0}(t,t)(t-\beta(t)).$$

Hence

$$\begin{aligned}
A''(t) &= p_{j+2,0}(t,t)+(-1)^{[(\lambda+1)/2]-1}\frac{Nd_0(t)}{2}\frac{F_{j+1,0}(t,t)}{\phi_{j,0}(t,t)}F_{j+1,0}(t,t)\\
&= p_{j+2,0}(t,t)-\frac{F_{j+1,0}(t,t)}{\phi_{j,0}(t,t)}\\
&\quad\times\frac{p_{j+2,0}(t,t)(t^2-1)+(j+1)\left(j-\frac{\lambda+\mu}{2}\right)p_{j,0}(t,t)}{t-\beta(t)}\\
&\quad-\frac{F_{j+1,0}(t,t)}{\phi_{j,0}(t,t)}\frac{p_{j+1,0}(t,t)}{t-\beta(t)}\left\{\left(2(j+1)-\frac{\lambda+\mu}{2}\right)t+\frac{\mu-\lambda}{2}\right\}\\
&= \frac{A(t)}{\beta(t)-t}\left\{j\frac{\psi_{j-1,0}(t,t)}{F_{j+1,0}(t,t)}\frac{p_{j+2,0}(t,t)}{p_{j,0}(t,t)}+(j+1)\left(j-\frac{\lambda+\mu}{2}\right)\right\}\\
&\quad\times\frac{F_{j+1,0}(t,t)}{\phi_{j,0}(t,t)}+\frac{A'(t)}{\beta(t)-t}\frac{F_{j+1,0}(t,t)}{\phi_{j,0}(t,t)}\\
&\quad\times\left\{\left(2(j+1)-\frac{\lambda+\mu}{2}\right)t+\frac{\mu-\lambda}{2}\right\}.
\end{aligned}$$

Let us now assume that $A'(t_0)=p_{j+1,0}(t_0,t_0)=0$. Using Lemma 4 we may conclude that for all t in $[\xi_l, \rho_l]$ we have

$$\{p_{j+1,0}(t, t)\}^2 - p_{j+2,0}(t, t)p_{j,0}(t, t) < 0$$

where the inequality is strict since $t \neq \pm 1$. Hence by continuity there exists a constant M such that for all $t \in [\xi_l, \rho_l]$

$$\{p_{j+1,0}(t, t)\}^2 - p_{j+2,0}(t, t)p_{j,0}(t, t) < M < 0.$$

From this it follows that if $p_{j+1,0}(t_0, t_0) = 0$, then there exists an interval around t_0 wherein

$$p_{j+2,0}(t, t)p_{j,0}(t, t) < 0.$$

The quantity $(j + 1)(j - (\lambda + \mu)/2)$ appearing in the expression for $A''(t)$ is clearly non-negative and because of (57) so is the factor $F_{j+1,0}(t, t)/\phi_{j,0}(t, t)$ for $t \neq \lambda_l$ so that

$$\lim_{t\to\lambda_l -} F_{j+1,0}(t, t)/\phi_{j,0}(t, t) \geqq 0 \quad \text{and}$$
$$\lim_{t\to\lambda_l +} F_{j+1,0}(t, t)/\phi_{j,0}(t, t) \geqq 0$$

as well.

Let us now consider the factor $\beta(t) - t$. For this we rewrite (62) in the form

$$(62') \quad \frac{p_{j+2,0}(t, t)}{d_0(t)}(t^2 - 1) + \frac{p_{j+1,0}(t, t)}{d_0(t)} \times \left\{\left(2(j + 1) - \left(\frac{\lambda + \mu}{2}\right)\right)t + \left(\frac{\mu - \lambda}{2}\right)\right\} + \frac{p_{j,0}(t, t)}{d_0(t)}(j + 1)\left(j - \frac{\lambda + \mu}{2}\right) = (-1)^{[(\lambda+1)/2]}\frac{N}{2}F_{j+1,0}(t, t)(t - \beta(t)).$$

We have already noticed that there exists an interval around t_0 wherein

$$(t^2 - 1)p_{j+2,0}(t, t)p_{j,0}(t, t) > 0.$$

Further, since $F_{j,0}(t, t) = 0$ for each t, we can apply Lemma 6′ to the polynomials

$$g(x) = (-1)^{[(\lambda+1)/2]}F(x, t) \quad \text{and} \quad h(x) = \frac{p(x, t)}{d_0(t)}$$

to obtain

$$(-1)^{[(\lambda+1)/2]}F_{j+1,0}(t, t)\frac{p_{j,0}(t, t)}{d_0(t)} \geqq 0.$$

Hence if $p_{j+1,0}(t_0, t_0) = 0$, the two sides of (62′) can be of the same sign if and only if $\beta(t_0) - t_0 \geqq 0$.

It is clear that the proof will be complete if we show that

$$\psi_{j-1,0}(t,t)/F_{j+1,0}(t,t) \leqq 0$$

for all $t \neq \lambda_l$ so that

$$\lim_{t\to\lambda_l-} \psi_{j,0}(t,t)/F_{j+1,0}(t,t) \leqq 0 \quad \text{and}$$
$$\lim_{t\to\lambda_l+} \psi_{j,0}(t,t)/F_{j+1,0}(t,t) \leqq 0.$$

Let us first consider the case $t \in (\xi_l, \lambda_l)$. In view of (60) we have

$$\frac{\psi_{j-1,0}(t,t)}{F_{j+1,0}(t,t)} = -\frac{t-1}{j}\frac{\psi_{j,0}(t,t)}{F_{j+1,0}(t,t)}$$
$$= 2\left[\frac{\lambda+1}{2}\right]\frac{t-1}{j}\frac{\left.\frac{\partial^j}{\partial x^j}\left\{\frac{F(x,t)}{x-1}\right\}\right|_{x=t}}{F_{j+1,0}(t,t)},$$

and we can apply Lemma 5 to conclude that this latter quantity is negative. We are thus left with the case $t \in (\lambda_l, \rho_l)$.

Let us first show that for all $t \in (\lambda_l, \rho_l)$

$$(63) \qquad (\beta^2(t)-1)\, F_{1,0}(\beta(t),t)/F(\beta(t),t) \geqq 0.$$

If $\beta(t) \geqq 1$ this is a consequence of the fact that all the roots of $F(x,t)$ lie in $[-1, 1]$. If $\beta(t) \leqq 1$ we show that

$$F_{1,0}(\beta(t),t)/F(\beta(t),t) \leqq 0$$

for every n if $\lambda = \mu$ and for $n \geqq 3(\lambda+\mu)/2$ if $\lambda \neq \mu$. We recall that the situation $\beta(t) \leqq 1$ occurs only when λ is odd and that in this case we have the inequalities

$$-1 \leqq x_1(t) < y_1(t) < \ldots < x_{\nu(n)-1}(t) < \delta(t) < \beta(t) \leqq 1.$$

Writing

$$\frac{F_{1,0}(\beta(t),t)}{F(\beta(t),t)} = \frac{\left[\frac{\lambda+1}{2}\right]}{\beta(t)-1} + \frac{\left[\frac{\mu+1}{2}\right]}{\beta(t)+1} + \sum_{l=1}^{\nu(n)-1}\frac{1}{\beta(t)-x_l(t)},$$

$$\frac{p_{1,0}(\beta(t),t)}{p(\beta(t),t)} = \frac{\left[\frac{\lambda+1}{2}\right]}{\beta(t)-1} + \frac{\left[\frac{\mu+1}{2}\right]}{\beta(t)+1}$$
$$+ \sum_{m=1}^{\nu(n)-2}\frac{1}{\beta(t)-y_m(t)} + \frac{1}{\beta(t)-\delta(t)},$$

we see that

$$F_{1,0}(\beta(t),t)/F(\beta(t),t) \leqq p_{1,0}(\beta(t),t)/p(\beta(t),t).$$

From (61) we readily obtain

$$p_{1,0}(\beta(t), t)/p(\beta(t), t) = ((\mu - \lambda)/2) - ((\mu + \lambda)/2)\beta(t))/(1 - \beta^2(t))$$

and so (63) will be proved if we show that $\beta(t) \geqq (\mu - \lambda)/(\mu + \lambda)$. Using the fact that

$$\beta(t) \geqq x_{\nu(n)}(\xi_l) = \cos \frac{\pi}{2n - (\lambda + \mu)}$$

and $\lambda \geqq 1$ we can easily check that this inequality holds for $n \geqq 3(\lambda + \mu)/2$ if $\lambda \neq \mu$. If $\lambda = \mu$ then it obviously holds for all n.

Now we will prove that

$$\psi_{j-1,0}(t, t)/F_{j+1,0}(t, t) \leqq 0,$$

giving the details only in the case when λ is odd and μ is even. Since

$$F(x, t) = (-1)^{(\lambda+1)/2}(x - 1)^{(\lambda+1)/2}(x + 1)^{(\mu+2)/2} \prod_{l=2}^{\nu(n)-1} (x - x_l(t))$$

the functions $F_{1,0}(x, t)$, $\psi(x, t)$ must be of the form

$$F_{1,0}(x, t) = (-1)^{(\lambda+1)/2}n(x - 1)^{(\lambda-1)/2}(x + 1)^{\mu/2} \prod_{l=1}^{\nu(n)-1} (x - X_l(t))$$

$$\psi(x, t) = (-1)^{(\lambda-1)/2}(\beta^2(t) - 1) \frac{F_{1,0}(\beta(t), t)}{F(\beta(t), t)} (x - 1)^{(\lambda-1)/2} \times (x + 1)^{\mu/2} \prod_{m=1}^{\nu(n)-1} (x - Y_m(t))$$

where $X_l(t)$, $Y_m(t)$ belong to the interval $[-1, 1]$ and

$$\psi(X_l(t), t) = (\beta^2(t) - 1) \frac{F_{1,0}(\beta(t), t)}{F(\beta(t), t)} \frac{F(X_l(t), t)}{\beta(t) - X_l(t)}.$$

We observe that the inequality $\beta(t) \geqq X_{\nu(n)-1}(t)$ is always true (if $\beta(t) \geqq 1$ it is evident whereas if $\beta(t) \leqq 1$ it is a consequence of $F_{1,0}(\beta(t), t)/F(\beta(t), t) \leqq 0$).

Since

$$(\beta^2(t) - 1)\ F_{1,0}(\beta(t), t)/F(\beta(t), t) \geqq 0$$

the dominating coefficients of $\psi(x, t)$ and $F(x, t)$ are of opposite signs whereas at each of the points $X_l(t)$,

$$\operatorname{sign} \psi(X_l(t), t) = \operatorname{sign} F(X_l(t), t).$$

Consequently, for $x > 1$ we have

$$\operatorname{sign} \psi(x, t) = -\operatorname{sign} F(x, t)$$

while

$$\operatorname{sign} \psi(X_{\nu(n)-1}(t), t) = \operatorname{sign} F(X_{\nu(n)-1}(t), t).$$

But the number of sign changes at $+1$ is one less for $\psi(x, t)$ than for $F(x, t)$ so that $\psi(x, t)$ vanishes in $(X_{\nu(n)-1}(t), 1)$ an even number of times. On the other hand for $x < -1$,

$$\operatorname{sign} \psi(x, t) = \operatorname{sign} F(x, t)$$

while

$$\operatorname{sign} \psi(X_1(t), t) = \operatorname{sign} F(X_1(t), t);$$

so, repeating the previous argument, we can conclude that $\psi(x, t)$ vanishes in $(-1, X_1(t))$ an odd number of times.

Moreover, $\psi(x, t)$ vanishes at least once in each of the intervals $(X_l(t), X_{l+1}(t))$. These remarks lead to the inequalities

$$-1 \leqq Y_1(t) \leqq X_1(t) \leqq Y_2(t) \leqq \ldots \leqq Y_{\nu(n)-1}(t) \leqq X_{\nu(n)-1}(t) \leqq +1.$$

Using again the fact that $F_{j,0}(t, t) = 0$ we can apply Lemma 6 to obtain

$$\psi_{j-1,0}(t, t)/F_{j+1,0}(t, t) \leqq 0$$

which completes the proof.

In conclusion we verify that (5) is valid for $j \geqq 3$. For this we apply Theorem 2 with $P_n(x) = (1 - x^2)T'_{n-1}(x)/(n - 1)$. Using the well known differential equation for T_{n-1} we obtain

$$P_n^{(j)}(x) = -\frac{1}{n-1}\{xT_{n-1}^{(j)}(x) + T_{n-1}^{(j-1)}(x)((j-1) + (n-1)^2)\}.$$

But $\max_{[-1,+1]}|T_n^{(j)}(x)| = T_n^{(j)}(1)$ is an increasing function of n so that

$$\max_{[-1,+1]}|P_n^{(j)}(x)| = |P_n^{(j)}(1)|$$

is also an increasing function of n.

References

1. R. P. Boas, Jr., *Inequalities for the derivatives of polynomials*, Math. Mag. *42* (1969), 165–174.
2. V. K. Dzyadyk, *On a constructive characteristic of functions satisfying the Lipschitz condition* α $(0 < \alpha < 1)$, *on a finite segment of the real axis*, Izv. Akad. Nauk SSSR (seriya mat.) *20* (1956), 623–642.
3. A. A. Markov, *On a problem of D. I. Mendeleev*, Zap. Imp. Akad. Nauk *62* (1889), 1–24.
4. W. A. Markov, *Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen*, Math. Ann. *77* (1916), 218–258.
5. R. Pierre and Q. I. Rahman, *On a problem of Turán about polynomials*, Proc. Amer. Math. Soc. *56* (1976), 231–238.

6. Q. I. Rahman, *On a problem of Turán about polynomials with curved majorants*, Trans. Amer. Math. Soc. *163* (1972), 447–455.

7. ——— *Addendum to "On a problem of Turán about polynomials with curved majorants"*, Trans. Amer. Math. Soc. *168* (1972), 517–518.

8. G. Szegö, *Orthogonal polynomials*, American Mathematical Society Colloquium Publications *23* (American Mathematical Society, Providence, Rhode Island, Third edition, 1967).

9. E. V. Voronovskaja, *The functional method and its applications*, Translations of Mathematical Monographs *28* (American Mathematical Society, Providence, Rhode Island, 1970).

Université Laval,
Québec, Québec;
Université de Montréal,
Montréal, Québec