

SEMI-MARKOV PROCESSES ON A GENERAL STATE SPACE: α -THEORY AND QUASI-STATIONARITY

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Abstract

By amalgamating the approaches of Tweedie (1974) and Nummelin (1977), an α -theory is developed for general semi-Markov processes. It is shown that α -transient, α -recurrent and α -positive recurrent processes can be defined, with properties analogous to those for transient, recurrent and positive recurrent processes. Limit theorems for α -positive recurrent processes follow by transforming to the probabilistic case, as in the above references: these then give results on the existence and form of quasi-stationary distributions, extending those of Tweedie (1975) and Nummelin (1976).

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0. Introduction

The analytic properties of the transition probabilities of Markov chains and semi-Markov processes, and the application of these properties to the investigation of quasi-stationary behaviour of such chains and processes, have been explored in a variety of contexts in recent years. For Markov chains in discrete time, and with a countable state space, the analytic behaviour ('*R*-theory') of the transition probabilities is investigated by Vere-Jones (1962) and (1967), and the application to quasi-stationary problems is given by Seneta and Vere-Jones (1966), following the initial work by Darroch and Seneta (1965) on quasi-stationarity for finite state space chains. The *R*-theory and quasi-stationarity for general state space is developed by Tweedie (1974a) and (1975), and the basic results of Vere-Jones are shown to hold without more than technical changes when the state space is generalized.

For Markov processes with continuous time parameter and countable state space the analytic properties (in this context called ' α -theory') of the transition pro-

babilities are studied by Kingman (1963), and some quasi-stationary results are described by Vere-Jones (1969) and Tweedie (1974b).

Finally, the α -theory and quasi-stationary behaviour of semi-Markov processes with a countable state space are considered by Cheong (1968) and (1970) and by Flashpohler and Holmes (1972); and further results in those areas are presented by Nummelin (1976) and (1977).

This paper is intended to complete this set of results by presenting the important aspects of an α -theory for semi-Markov processes whose state space is general rather than countable: this can then be seen as extending both the R -theory of Tweedie (1974a) and the countable space results of Nummelin (1976) and (1977), whose approach we follow quite closely here. From this α -theory we then deduce some quasi-stationarity results which again complement those previously discovered under restrictions on either the space or the time behaviour of the process.

1. Preliminaries

Suppose (E, \mathcal{E}) is an arbitrary measurable space, and write $(\mathbf{R}_+, \mathcal{A}_+)$ for the set $[0, \infty)$ and its Borel σ -field; we shall write \mathcal{F} for the product- σ -field $\mathcal{E} \otimes \mathcal{A}_+$. The basic object we shall study is a semi-Markov kernel $Q(x, B)$, $x \in E$, $B \in \mathcal{F}$; this is assumed to be such that $K(i)$: for every $B \in \mathcal{F}$, $Q(\cdot, B)$ is an \mathcal{E} -measurable function from E to \mathbf{R}_+ ; $K(ii)$: for every $x \in E$, $Q(x, \cdot)$ is a measure on \mathcal{F} , and $K(iii)$: for every $x \in E$, $Q(x, E \times \mathbf{R}_+) \leq 1$.

We shall use Δ to denote a point not in E , \bar{E} to denote $E \cup \{\Delta\}$, and $\bar{\mathbf{R}}_+$ to denote $\mathbf{R}_+ \cup \{\infty\}$; $\bar{\mathcal{E}}$ denotes the extension of \mathcal{E} to $E \cup \Delta$, $\bar{\mathcal{A}}_+$ the extension of \mathcal{A}_+ to $\bar{\mathcal{A}}_+$, and $\bar{\mathcal{F}}$ the extension of \mathcal{F} to $\bar{E} \times \bar{\mathbf{R}}_+$ in the obvious way. Let $Q_\Delta(x, \Gamma)$, $x \in E$, $\Gamma \in \bar{\mathcal{A}}_+$, be a transition kernel from E into $\bar{\mathcal{A}}_+$, called the absorption law. We assume that $Q(\cdot, E \times \mathbf{R}_+) + Q_\Delta(\cdot, \bar{\mathbf{R}}_+) \equiv 1$. Any measure μ on \mathcal{A}_+ is extended to $(-\infty, \infty)$ by setting $\mu(-\infty, 0) = 0$. Let $\{(X_n, T_n); n \geq 0\}$ be a Markov chain on $(\bar{E} \times \bar{\mathbf{R}}_+, \bar{\mathcal{F}})$ with the transition probabilities

$$\begin{aligned}
 P((x, t), A \times \Gamma) &= Q(x, A \times (\Gamma - t)), \\
 P((x, t), \{\Delta\} \times \Gamma) &= Q_\Delta(x, \Gamma - t), \\
 P((\Delta, t), \{\Delta\} \times \{\infty\}) &= 1, \quad x \in E, \quad A \in \mathcal{E}, \quad t \in \bar{\mathbf{R}}_+, \quad \Gamma \in \bar{\mathcal{A}}_+.
 \end{aligned}$$

We denote by \mathbf{P}_x the probability on the space $((\bar{E} \times \bar{\mathbf{R}}_+)^{\infty}, \bar{\mathcal{F}}^{\infty})$ associated with the transition probabilities $P(\cdot, \cdot)$ and the initial conditions $X_0 = x$, $T_0 = 0$; the corresponding expectation is denoted by \mathbf{E}_x . If $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n; T_0, \dots, T_n)$, then clearly the collection

$$(X, T) = \{(\bar{E} \times \bar{\mathbf{R}}_+)^{\infty}, \bar{\mathcal{F}}^{\infty}, \mathcal{F}_n, X_n, T_n, (\mathbf{P}_x)_{x \in E}\}$$

is a Markov renewal process in the sense of Cinlar (1974). If we let $M(t) = \sup \{n \geq 0: T_n \leq t\}$ and $Z = \sup \{T_n; n \geq 0\}$ then the semi-Markov process corresponding to this Markov renewal process can be defined by

$$X(t) = X_{M(t)} \text{ for } t < Z \text{ and } X(t) = \Delta \text{ for } t \geq Z.$$

We shall let, for any set $A \in \mathcal{E}$, $\tau_A = \inf \{n > 0: X_n \in A\}$, and work with the following quantities:

$$R(x, A \times \Gamma) = \mathbf{E}_x \left[\sum_{n=0}^{\infty} 1_{A \times \Gamma}(X_n, T_n) \right],$$

$${}_B R(x, A \times \Gamma) = \mathbf{E}_x \left[\sum_{n=1}^{\tau_B} 1_{A \times \Gamma}(X_n, T_n) \right].$$

Both $R(\cdot, \cdot)$ and ${}_B R(\cdot, \cdot)$ satisfy $K(i)$ and $K(ii)$ above. If $K: E \times \mathcal{F} \rightarrow \bar{\mathbf{R}}_+$ satisfies $K(i)$ and $K(ii)$ we call K a *kernel*.

Let K and L be arbitrary kernels, μ a measure on \mathcal{F} , g a measurable function $E \times \mathbf{R}_+ \rightarrow \mathbf{R}_+$, λ a real number and f a measurable function $E \rightarrow \mathbf{R}_+$. We shall use the following notation throughout this paper:

$$K * L(x, A \times \Gamma) = \int_E \int_{\mathbf{R}_+} K(x, dy \times dt) L(y, A \times (\Gamma - t)),$$

$$\mu * K(A \times \Gamma) = \int_E \int_{\mathbf{R}_+} \mu(dx \times dt) K(x, A \times (\Gamma - t)),$$

$$K * g(x, t) = \int_E \int_{\mathbf{R}_+} K(x, dy \times du) g(y, t - u),$$

$$I_f g(x, t) = f(x)g(x, t), \quad \mu I_f(dy \times \Gamma) = \mu(dy \times \Gamma)f(y),$$

$$\mu^\lambda(A \times dt) = e^{\lambda t} \mu(A \times dt), \quad g^\lambda(x, t) = e^{\lambda t} g(x, t),$$

$$\tilde{\mu}(A) = \mu(A \times \mathbf{R}_+), \quad \tilde{g}(x) = \int_{\mathbf{R}_+} g(x, t) dt.$$

(We write $\tilde{\mu}^\lambda$ instead of $(\mu^\lambda)^\sim$, and similarly for g .)

For any *transition kernel* $N(x, A)$ on (E, \mathcal{E}) we denote the n -step iterates by $N^{(n)}(x, A)$; otherwise we use the standard operator-theoretic notation of Revuz (1975).

We now concentrate on the embedded chain X_n , whose one-step transition kernel is given by $\tilde{Q}(x, A)$, defined to be $Q(x, A \times \mathbf{R}_+)$. Our *first basic assumption throughout this paper* will be that, for some σ -finite measure φ on E , $\{X_n\}$ is φ -irreducible: that is $\varphi(A) > 0$ implies $\sum_{n=1}^{\infty} \tilde{Q}^{(n)}(x, A) > 0$ for all $x \in E$. From Tweedie (1974a) and (1976)

there is then at least one *maximal* irreducibility measure M satisfying

I(i): $M(A) > 0 \Rightarrow \sum_1^\infty \tilde{Q}^{(n)}(x, A) > 0$ for all x ,

I(ii): $M(A) = 0 \Rightarrow M\left\{x: \sum_1^\infty \tilde{Q}^{(n)}(x, A) > 0\right\} = 0$,

I(iii): if $\{X_n\}$ is φ -irreducible, then $\varphi << M$.

We shall use M to denote a fixed finite measure satisfying I(i)–I(iii), and put $\mathcal{E}^+ = \{A \in \mathcal{E}: M(A) > 0\}$.

The semi-Markov kernel Q will be called M -irreducible, M -recurrent, and so on, according as \tilde{Q} has these properties. For details of the classifications of general state-space chains, the reader can consult, for example, Jain and Jamison (1967), Tuominen (1976) or Tweedie (1976).

Our second basic assumption throughout this paper will be that the semi-Markov kernel is *nondegenerate*:

$$\int_E M(dx) Q(x, E \times (0, \infty)) > 0.$$

2. Solidarity properties and α -recurrent processes

Our first step in describing the α -properties of a semi-Markov process is a solidarity theorem (see Theorem 1 of Tweedie (1974a), and Theorem 1 of Nummelin (1977)).

THEOREM 1. *Suppose Q is M -irreducible. For any fixed $\lambda \geq 0$, either*

- (i) $\tilde{R}^\lambda(x, A) = \infty$ for all $x \in E$ and $A \in \mathcal{E}^+$; or
- (ii) *there exists a countable partition $\{A(j)\}$ for E and an M -null set N_λ such that $\tilde{R}^\lambda(x, A(j)) < \infty$ for all $x \notin N_\lambda$ and all j .*

PROOF. Suppose (i) does not hold, and let $B \in \mathcal{E}^+$ and $z \in E$ be such that $\tilde{R}^\lambda(z, B) < \infty$. Write, for $n, j \geq 1$,

$$\bar{B}(n, j) = \{y \in E: (\tilde{Q}^\lambda)^m(y, B) = 0, m < n; (\tilde{Q}^\lambda)^n(y, B) \in [(j+1)^{-1}, j^{-1}]\},$$

$$N_\lambda(B) = \{y \in E: \tilde{R}^\lambda(y, B) = \infty\}.$$

As in Theorem 1 of Tweedie (1974a), it can be shown that $\{\bar{B}(n, j)\}$ is the required partition and $N_\lambda(B)$ the required M -null set for (ii) to hold.

From Theorem 1 it is now possible to define α , the *convergence parameter* of Q , as

(2.1) $\alpha = \sup \{\lambda \geq 0: \tilde{R}^\lambda(x, A) < \infty, \text{ some } x \in E, A \in \mathcal{E}^+\}.$

The nondegeneracy assumption guarantees that α is finite. We shall call Q λ -transient if $\tilde{R}^\lambda(x, A) < \infty$ for some x and some $A \in \mathcal{E}^+$, and call Q α -recurrent if it is not α -transient. If Q is 0-transient (0-recurrent) we shall merely call Q transient (recurrent).

In our next theorem we give the connection between the α -properties of Q and the properties of \tilde{Q}^λ considered as a transition kernel on (E, \mathcal{E}) . For the latter we use the nomenclature of Theorem 1 of Tweedie (1974a). We omit the proof, which is fairly straightforward.

THEOREM 2. *The semi-Markov kernel Q is λ -transient if and only if the transition kernel \tilde{Q}^λ is 1-transient; and Q is α -recurrent if and only if \tilde{Q}^α is 1-recurrent.*

The main use of Theorem 2 is in providing an easy route to the existence of α -invariant measures and functions: the results of Section 3 of Tweedie (1974a), applied to the kernels \tilde{Q}^λ , allow us to assert immediately

THEOREM 3. (i) *If Q is α -recurrent, there exists a σ -finite measure π on \mathcal{E} satisfying (in the notation of Revuz (1975))*

$$(2.2) \quad \pi \geq \pi \tilde{Q}^\alpha.$$

The measure π is unique up to constant multiples, satisfies (2.2) with equality, and is equivalent to M . For any set $A \in \mathcal{E}^+$, π is connected to the taboo renewal measures ${}_A R$ by

$$(2.3) \quad \pi = \pi I_A {}_A \tilde{R}^\alpha.$$

(ii) *If Q is α -recurrent, there exists a strictly positive finite measurable function h on E satisfying*

$$(2.4) \quad h(x) \geq \tilde{Q}^\alpha h(x), \quad M\text{-almost every } x \in E.$$

The function h is unique up to constant multiples and definition on null sets, and satisfies (2.4) with equality M -almost everywhere. For any set $A \in \mathcal{E}^+$, h satisfies

$$(2.5) \quad h = {}_A \tilde{R}^\alpha I_A h, \quad M\text{-almost everywhere.}$$

In the discrete state space case, it is often convenient to use the representation (Theorem 2 of Nummelin (1977)) of π and h in terms of last exit and first entrance distributions to some fixed reference state. Note that when some fixed state $z \in E$ is such that $\pi(z) > 0$, then (2.3) and (2.5) with $A = \{z\}$ have this useful form. When no such state exists then one can use the splitting technique of Nummelin (1978) to derive a similar relationship.

We will use π to denote the invariant measure of Theorem 3(i), and h to denote one version of the invariant function; if we denote the M -null set on which this version of the invariant function fails to satisfy (2.4) with equality by N , and write $N' = \{y: \tilde{R}(y, N) > 0\}$, then we will use N_h to denote $N \cup N'$. From I(ii), $M(N_h) = 0$.

We now use h to define a transform \hat{Q} of Q ; we write (when Q is α -recurrent)

$$\hat{Q}(x, \cdot) = I_{1/h} Q^\alpha I_h(x, \cdot), \quad x \notin N_h$$

and

$$\hat{Q}(x, \cdot) = \hat{Q}(x_0, \cdot), \quad x \in N_h,$$

where x_0 is an arbitrary but fixed point not in N_h .

In a straightforward manner we get

THEOREM 4. *Suppose Q is α -recurrent. Then \hat{Q} is also an M -irreducible semi-Markov kernel, and \hat{Q} is 0-recurrent.*

The kernel \hat{Q} is more useful than \tilde{Q}^α precisely because it is semi-Markov as well as 0-recurrent; this enables us to exploit known results for 0-recurrent processes when deriving α -limit properties for Q .

Finally in this section we shall delineate the relationship between our definition of α in terms of \tilde{R}^α , and a convergence norm for the transition probabilities

$$P_t(x, A) = \mathbf{P}_x\{X(t) \in A\}, \quad A \in \mathcal{E},$$

where $\{X(t)\}$ is the semi-Markov process associated with Q .

Let us call P_t λ -transient if there is a partition $\{C(j)\}$ of E such that

$$\tilde{P}^\lambda(x, C(j)) = \int_{\mathbf{R}^+} P_t(x, C(j)) e^{\lambda t} dt$$

is finite for M -almost all x ; call P_t α -recurrent if P_t is λ -transient for $\lambda < \alpha$ and if, for all $x \in E$ and $A \in \mathcal{E}^+$, $\tilde{P}^\alpha(x, A)$ diverges. Writing

$$Q(x, \bar{E} \times \cdot) = Q(x, E \times \cdot) + Q_\Delta(x, \cdot), \quad f(x, t) = 1 - Q(x, \bar{E} \times [0, t])$$

we have

$$(2.6) \quad P_t(x, A) = \int_A \int_0^t R(x, dy \times ds) f(y, t-s) = R * I_A f(x, t),$$

hence

$$(2.7) \quad \tilde{P}^\lambda(x, A) = \tilde{R}^\lambda I_A \tilde{f}^\lambda(x).$$

From the definitions and (2.7) we get easily by using Fubini's theorem

THEOREM 5. *Suppose that $Q(x, \bar{E} \times (0, \infty)) > 0$ for each $x \in E$ (that is, there are no instantaneous states).*

(i) If P_t is λ -transient, then so is Q , and

$$(2.8) \quad \text{for } M\text{-almost all } x: \tilde{Q}^\lambda(x, \bar{E}) < \infty \text{ (if } \lambda > 0) \text{ and} \\ \int_{\mathbf{R}} tQ(x, \bar{E} \times dt) < \infty \text{ (if } \lambda = 0).$$

Conversely, if (2.8) holds and Q is λ -transient, so is P_t .

(ii) If P_t is α -recurrent and (2.8) holds with $\lambda = \alpha$, then Q is α -recurrent; conversely, if (2.8) holds for all $\lambda < \alpha$, and Q is α -recurrent then so is P_t .

REMARKS. In the Markov case with countable state space (2.8) always holds, since $\lambda \leq \alpha < q_i$ for all i , where $Q(i, \bar{E} \times \cdot)$ is exponential with parameter q_i (Kingman (1963)). For $\lambda > 0$ (2.8) is in any case automatically satisfied if Q is λ -transient and admits the decomposition

$$(2.9) \quad Q(x, A \times \Gamma) = Q(x, \bar{E} \times \Gamma) \tilde{Q}(x, A);$$

that is, the semi-Markov process has holding times in state x with distribution function $Q(x, \bar{E} \times \cdot)$ independent of the next state to be visited. This follows easily from the inequality

$$\tilde{R}^\lambda(x, A) \geq \tilde{Q}^\lambda(x, \bar{E}) \tilde{Q} \tilde{R}^\lambda(x, A).$$

Convergence norms for $\{P_t(x, A)\}$ are studied by Kingman (1963) in the countable space Markov case, and by Cheong (1968) in the countable space semi-Markov case: it is clear that our definition of α corresponds with that of Kingman, and when (2.9) holds with that of Cheong. However, Cheong's (1968) work contains an error: equation (1), on which much of his analysis is based, is valid only if (2.9) holds. It is not obvious in general, however, that a convergence norm exists for $\{P_t(x, A)\}$, even on a countable space with $Q(x, E \times \mathbf{R}_+) \equiv 1$, and certainly our results show that if $Q(x, E \times \mathbf{R}_-) < 1$ then the transition probabilities $\{P_t(x, A)\}$ need not have the same type of solidarity properties exhibited by the kernels \tilde{R}^λ .

3. Limit theorems for α -positive recurrent processes

In this section our basic aim is to find limit theorems for terms of the form

$$(3.1) \quad e^{\alpha t} \mu * R * f(t),$$

where α is the convergence parameter of Q , f is a function from $E \times \mathbf{R}_+$ into \mathbf{R}_+ and μ is a measure on \mathcal{F} . As in Nummelin (1976) this will enable us to deduce, in the next section, quasi-stationary results for $P_t(x, A)$ conditional on $X(t)$ not being absorbed in Δ until some time later than t .

As in Tweedie (1974a) and Nummelin (1976), our method involves using the limit theorems for the transformed kernel \hat{Q} , shown in Theorem 4 to be 0-recurrent. Unfortunately, in the context of Markov renewal processes with a general state space, a wholly satisfactory version of the key renewal theorem does not yet exist, although many partial results are known; the reader is referred to Jacod (1974), Kesten (1974), Arjas *et al.* (1978), McDonald (1978) or Nummelin (1978) for some of these.

Since such key renewal theorems are the $\alpha = 0$ case of our α -limit theorems, it is clear that the α -theorems cannot be developed more completely than the key renewal theorems. We wish to highlight the fact that both the α -theory and the probabilistic theory can be developed essentially to exactly the same stage, and so we take the perhaps unusual step of assuming the existence of a 'global theorem': the above references all give examples of this global theorem for different sets of conditions Ξ and correspondingly different classes of measures $\mathcal{M}(\Xi)$ and functions $\mathcal{D}(\Xi)$.

KEY RENEWAL THEOREM FOR MARKOV RENEWAL PROCESSES. *Let M denote the irreducibility measure of Section 1, and assume that Q is M -recurrent (or equivalently, that the embedded Markov chain $\{X_n\}$ is recurrent in the sense of Harris; the reader should not confuse this with α -recurrence). Let π be the invariant measure of \hat{Q} . If Q satisfies a certain set Ξ of conditions, then there exists a class of finite measures $\mathcal{M}(\Xi)$ and a class of functions $\mathcal{D}(\Xi)$ such that for $\mu \in \mathcal{M}(\Xi)$ and $f \in \mathcal{D}(\Xi)$,*

$$(3.2) \quad \lim_{t \rightarrow \infty} \mu * R * f(t) = [\pi(m)]^{-1} \mu(E \times \mathbb{R}_+) \pi(\tilde{f}),$$

where

$$m(x) = \int_{\mathbb{R}_+} tQ(x, E \times dt)$$

denotes the mean sojourn time in state $x \in E$.

Assuming the existence of the preceding type of key renewal theorem we now turn to applying it to derive an α -limit theorem for our semi-Markov processes.

From now on we shall assume that the σ -field \mathcal{E} is countably generated, that is that there is a sequence $\{E_n\}$ such that $\mathcal{E} = \sigma(E_n; n = 0, 1, \dots)$.

We first note that if Q is α -recurrent, then \hat{Q} is recurrent, from Theorem 4; and hence (as is well known; see, for example, Jain and Jamison (1967)) by the preceding assumption on \mathcal{E} there is an M -null subset N of E such that $E \setminus N$ is stochastically closed and \hat{Q} restricted to $E \setminus N$ is M -recurrent.

THEOREM 6. *Suppose that Q is α -recurrent; let π and h be the left and right invariant vectors of \tilde{Q}^α , respectively, and denote by C the transition kernel*

$$C(x, A) = \int_{\mathbf{R}_+} t e^{\alpha t} Q(x, A \times dt).$$

Suppose further that \tilde{Q} satisfies the set of conditions Ξ in the key renewal theorem above. Let μ be such that $\mu(N \times \mathbf{R}_+) = 0$ and $\hat{\mu} = \mu^\alpha I_h$ is in $\mathcal{M}(\Xi)$; and f be such that $\hat{f} = I_{1/h} f^\alpha$ is in $\mathcal{D}(\Xi)$. Then

$$\lim_{t \rightarrow \infty} e^{\alpha t} \mu * R * f(t) = (\pi C h)^{-1} \tilde{\mu}^\alpha(h) \pi(\tilde{f}^\alpha).$$

PROOF. By simple calculations and by the key renewal theorem

$$e^{\alpha t} \mu * R * f(t) = \hat{\mu} * \hat{R} * \hat{f}(t) \rightarrow [\hat{\pi}(\hat{m})]^{-1} \hat{\mu}(E \times \mathbf{R}_+) \hat{\pi}((\hat{f})^\sim), \text{ as } t \rightarrow \infty,$$

where $\hat{\pi} = \pi I_h$ is the unique invariant measure of $(\tilde{Q})^\sim$. Now

$$(3.3) \quad \hat{m}(x) = \int_{\mathbf{R}_+} t \tilde{Q}(x, E \times dt) = \int_{\mathbf{R}_+} t e^{\alpha t} I_{1/h} Q I_h(x, E \times dt) = I_{1/h} C h(x),$$

$$(3.4) \quad \hat{\mu}(E \times \mathbf{R}_+) = \mu^\alpha I_h(E \times \mathbf{R}_+) = \tilde{\mu}^\alpha(h)$$

and

$$(3.5) \quad (\hat{f})^\sim = I_{1/h} \tilde{f}^\alpha.$$

Hence

$$\begin{aligned} & [\hat{\pi}(\hat{m})]^{-1} \hat{\mu}(E \times \mathbf{R}_+) \hat{\pi}((\hat{f})^\sim) \\ &= [\pi I_h I_{1/h} C h]^{-1} \tilde{\mu}^\alpha(h) \pi I_n(I_{1/h} \tilde{f}^\alpha) \\ &= (\pi C h)^{-1} \tilde{\mu}^\alpha(h) \pi(\tilde{f}^\alpha). \end{aligned}$$

This result covers Theorems 6 and 7 of Nummelin (1977) as special cases, and the proof is similar. Note that in the special case of $\mu = \varepsilon_{(x,0)}$, the Dirac measure at $(x, 0)$, the term $\tilde{\mu}^\alpha(h)$ becomes merely $h(x)$.

4. Quasi-stationary distributions

We define the *epoch of absorption* by

$$(4.1) \quad \tau = \inf\{t \in \mathbf{R}_+ : X(t) = \Delta\}$$

(note that $\tau = Z = \sup\{T_n : n \geq 0\}$ on the set $\bigcap_{n=0}^\infty \{X_n \in E\}$ and $\tau = \inf\{T_n : X_n = \Delta\}$ on the set $\bigcup_{n=0}^\infty \{X_n = \Delta\}$). Denote for all $x \in E$ the *probability*

of ultimate absorption from state x by

$$(4.2) \quad p(x) = \mathbf{P}_x\{\tau < \infty\}.$$

It is easy to see that p satisfies

$$(4.3) \quad p(x) = \int_E \tilde{Q}(x, dy)p(y), \quad x \in E; \quad p(\Delta) = 1.$$

We are interested in the following two types of quasi-stationary distributions (provided they exist independently of the starting point $x \in E \setminus N$):

$$(4.4) \quad \pi^{(1)}(A) = \lim_{t \rightarrow \infty} \mathbf{P}_x\{X(t) \in A \mid t < \tau < \infty\}$$

(I-type quasi-stationary distribution),

and

$$(4.5) \quad \pi^{(2)}(A) = \lim_{t \rightarrow \infty} (\lim_{s \rightarrow \infty} \mathbf{P}_x\{X(t) \in A \mid t+s < \tau < \infty\})$$

(II-type quasi-stationary distribution).

We use the notation

$$(4.6) \quad P'_t(x, A) = \mathbf{P}_x\{X(t) \in A, \tau < \infty\},$$

$$f(x, t) = \int_E Q(x, dy \times (t, \infty))p(y), \quad x \in E, \quad A \in \mathcal{E}, \quad t \in \mathbf{R}_+.$$

THEOREM 7. *Suppose that Q is α -positive recurrent and that \hat{Q} satisfies Ξ . Suppose that for all $A \in \mathcal{E}$, \hat{f}_A defined by*

$$\hat{f}_A(x, t) = [h(x)]^{-1} 1_A(x) f(x, t) e^{\alpha t}$$

is in $\mathcal{L}(\Xi)$ and that $\pi(\tilde{f}^\alpha)$ is finite. Suppose further that for all $x \in E \setminus N$, $e_{(x,0)}$ belongs to $\mathcal{M}(\Xi)$. Then the I-type quasi-stationary distribution exists, and is given by

$$\pi^{(1)}(A) = (\pi \tilde{f}^\alpha)^{-1} \pi 1_A \tilde{f}^\alpha.$$

PROOF. We have $P'_t(x, A) = R 1_A * f(x, t)$. By our assumptions and by Theorem 6

$$\lim_{t \rightarrow \infty} e^{\alpha t} P'_t(x, A) = (\pi C h)^{-1} h(x) \pi 1_A \tilde{f}^\alpha$$

so that the assertion follows from

$$\mathbf{P}_x\{X(t) \in A \mid t < \tau < \infty\} = \frac{e^{\alpha t} P'_t(x, A)}{e^{\alpha t} P'_t(x, E)}.$$

Before stating the result about the II-type quasi-stationary distribution, we prove the following interesting result which gives a probabilistic characterization for the semi-Markov process $\{\hat{X}(t)\}$ associated with the transformed kernel \hat{Q} .

THEOREM 8. *Let $x \in E$, $N, t \in \mathbf{R}_+$ be fixed. Suppose that Q is x -positive recurrent and that \hat{Q} satisfies Ξ . Suppose further that $R(x, E \times [0, t])$ is finite; that f in (4.6) satisfies the regularity conditions f^2 bounded, $\pi(\tilde{f}^2)$ finite, and $\lim_{s \rightarrow x} f(y, s) = 0$ for all $y \in E$; and that $\hat{f} = I_{1,h} f^2$ belongs to $\mathcal{G}(\Xi)$ and for all $A \in \mathcal{E}$, $\hat{\mu}_A$ defined by*

$$\hat{\mu}_A(dy \times du) = e^{xu} h(y) \int_0^t \int_A R(x, dz \times dv) Q(z, dy \times d(u+t-v))$$

belongs to $\mathcal{M}(\Xi)$. Then for all A

$$\begin{aligned} \lim_{s \rightarrow \infty} \mathbf{P}_x\{X(t) \in A \mid t+s < \tau < \infty\} &= \hat{\mathbf{P}}_x\{\hat{X}(t) \in A\} \\ &= \hat{\mathbf{P}}_x(x, A), \end{aligned}$$

where $\{\hat{X}(t)\}$ is the semi-Markov process associated with \hat{Q} .

PROOF. Denote the residual holding time at t by

$$V^+(t) = T_{M(t)+1} - t$$

and let

$$a(A, s) = e^{2t+s} \mathbf{P}_x\{X(t) \in A, X(t+s) \in E, V^+(t) \leq s, \tau < \infty\}.$$

We have

$$\begin{aligned} (4.7) \quad e^{2t+s} \mathbf{P}_x\{X(t) \in A, V^+(t) > s, \tau < \infty\} \\ = e^{2t+s} \int_0^t \int_A R(x, dy \times dv) f(y, t-v+s) \leq e^{2t} \int_A R(x, dy \times [0, t]) f^2(y, s). \end{aligned}$$

The right-hand side of (4.7) converges by dominated convergence to zero as $s \rightarrow \infty$. Hence

$$\begin{aligned} (4.8) \quad \lim_{s \rightarrow \infty} e^{2t+s} \mathbf{P}_x\{X(t) \in A, X(t+s) \in E, \tau < \infty\} \\ = \lim_{s \rightarrow \infty} a(A, s) \\ = \lim_{s \rightarrow \infty} e^{2t+s} \mu_A * R * f(s) \\ = e^{2t} (\pi Ch)^{-1} \tilde{\mu}_A^2(h) \pi(\tilde{f}^2) \quad (\text{by Theorem 6}), \end{aligned}$$

where μ_A is defined by

$$\mu_A(dy \times du) = \int_0^t \int_A R(x, dz \times dv) Q(z, dy \times d(u+t-v)).$$

We have

$$\begin{aligned} \tilde{\mu}_A^x(h) &= \int_{v=0}^t \int_{u=0}^x \int_{z \in A} \int_{y \in E} R(x, dz \times dv) Q(z, dy \times d(u+t-v)) e^{xu} h(y) \\ &= e^{-xt} \int_0^t \int_A \int_E R^x(x, dz \times dv) Q^x(z, dy \times [t-v, x]) h(y) \\ &= e^{-xt} h(x) \int_0^t \hat{R}(x, dz \times dv) \hat{Q}(z, E \times [t-v, x]) \\ &= e^{-xt} h(x) \hat{P}_t(x, A), \end{aligned}$$

from which we get by (4.8)

$$\begin{aligned} \lim_{s \rightarrow x} \mathbf{P}_x \{ X(t) \in A \mid t+s < \tau < x \} &= \frac{\lim_{s \rightarrow x} e^{x(t+s)} \mathbf{P}_x \{ X(t) \in A, X(t+s) \in E, \tau < x \}}{\lim_{s \rightarrow x} e^{x(t+s)} \mathbf{P}_x \{ X(t+s) \in E, \tau < x \}} \\ &= \frac{\lim_{s \rightarrow x} a(A, s)}{\lim_{s \rightarrow x} a(E, s)} \\ &= \hat{P}_t(x, A). \end{aligned}$$

Now it is easy to get the following theorem which gives sufficient conditions for the II-type quasi-stationary distribution to exist. We abbreviate $\hat{B}(x, t) = \hat{Q}(x, E \times [0, t])$.

THEOREM 9. *Suppose that the assumption of Theorem 8 are satisfied for any $x \in E \setminus N$, $t \in \mathbf{R}_+$. Suppose further that for all $A \in \mathcal{E}$, $I_A(1 - \hat{B})$ belongs to $\mathcal{C}(\Xi)$. Then the II-type quasi-stationary distribution exists, and is given by*

$$\pi^{(2)}(A) = (\pi Ch)^{-1} \pi I_A Ch.$$

PROOF. Denote by $\hat{\pi} = \pi I_h$ the unique invariant measure of the recurrent transition kernel $(\hat{Q})^\sim$ and by $\hat{m}(x)$ the function given by (3.3). By the preceding

theorem and by (3.2) for any $x \in E \setminus N$

$$\begin{aligned}\pi^{(2)}(A) &= \lim_{t \rightarrow \infty} \hat{P}_t(x, A) \\ &= [\hat{\pi}(\hat{m})]^{-1} \hat{\pi}(I_A(1 - \hat{B})^\sim) \quad (\text{by assumption and by the} \\ &\quad \text{key renewal theorem),} \\ &= (\pi Ch)^{-1} \pi I_h(I_A(1 - \hat{B})^\sim) \quad (\text{by (3.3)}).\end{aligned}$$

The assertion follows after observing that

$$(1 - \hat{B})^\sim(x) = \hat{m}(x) = I_{1,h} Ch.$$

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