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RESEARCH ARTICLE

Null hypersurfaces in 4-manifolds endowed with a product structure

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Abstract
In a 4-manifold, the composition of a Riemannian Einstein metric with an almost paracomplex structure that is isometric and parallel defines a neutral metric that is conformally flat and scalar flat. In this paper, we study hypersurfaces that are null with respect to this neutral metric, and in particular we study their geometric properties with respect to the Einstein metric. Firstly, we show that all totally geodesic null hypersurfaces are scalar flat and their existence implies that the Einstein metric in the ambient manifold must be Ricci-flat. Then, we find a necessary condition for the existence of null hypersurface with equal nontrivial principal curvatures, and finally, we give a necessary condition on the ambient scalar curvature, for the existence of null (non-minimal) hypersurfaces that are of constant mean curvature.

1. Introduction

Einstein Riemannian 4-manifolds (M, g) with a parallel, isometric, almost paracomplex structure P exhibit many interesting properties through the metric g' defined by $g' = g(P., .)$. In particular, the metric g' is of neutral signature, locally conformally flat, and scalar flat and shares the same Levi-Civita connection and Ricci tensor with g [5].

Recently, Urbano in [11] and later Gao et al. in [4] have studied hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$, respectively, endowed with the Einstein product metric. In particular, they used two complex structures J_1, J_2 on those manifolds to study isoparametric and homogeneous hypersurfaces by considering the product $P = J_1 J_2$, which is an (almost) paracomplex structure that is parallel and isometric with respect to the product metric.

The space $\mathbb{L}(M^3)$ of oriented geodesics in the three-dimensional non-flat real space form M^3 is a four-dimensional manifold admiting an Einstein metric and a paracomplex structure P that is isometric and parallel. Therefore, there exists a neutral, locally conformally flat and scalar flat metric sharing the same Levi-Civita connection and Ricci tensor with the Einstein metric (see [1] and [2] for more details). The paracomplex structure P has been explicitly described by Anciaux in [2] in a similar manner as in the product of surfaces. More precisely, Anciaux constructed two (para) complex structures J_1 and J_2 so that $J_1 J_2 = J_2 J_1$ and then considered the product $P = J_1 J_2$. This paracomplex structure was used in [6], to study a class of hypersurfaces in $\mathbb{L}(M^3)$, called *tangential congruences*, that are sets of all tangent-oriented geodesics in a given surface in M. Particularly, it was shown that tangential congruences are null with respect to the neutral metric and if, additionally, they are tangent to a convex surface then they admit a contact structure. The space $\mathbb{L}(\mathbb{R}^3)$ of oriented lines in $\mathbb{R}^3$ is also a four-dimensional manifold admiting a neutral metric G that is locally conformally flat and scalar flat and is invariant under the Euclidean motions [1, 8]. M. Salvai showed that G is the only metric that is invariant of the group action of the Eucliean 3-space. The null hypersurfaces in $\mathbb{L}(\mathbb{R}^3)$ play an important role in the study of the ultrahyperbolic equation:

$$u_{x_1x_1} + u_{x_2x_2} - u_{x_3x_3} - u_{x_4x_4} = 0, \tag{1.1}$$

where $u = u(x_1, x_2, x_3, x_4)$ is a real function in $\mathbb{R}^4$ (see [3]). Specifically, let $\mathbb{R}^{2,2} = (\mathbb{R}^4, g_0 := dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2)$, and $f : \mathbb{L}(\mathbb{R}^3) \to \mathbb{R}^{2,2}$ be the conformal map defined according to $G = \omega^2 f^* g_0$, where ω is a strictly positive function. A function v is harmonic with respect to G, that is, $\Delta_G u = 0$, if and only if $\omega \cdot v \circ f$ is a solution of the ultrahyperbolic equation (1.1) [3]. This implies solving the ultrahyperbolic equation is equivalent to solving the Laplace equation with respect to the neutral metric G. Consider now the problem:

$$\Delta_G v = 0,$$

where the function v on $\mathbb{L}(\mathbb{R}^3)$ is given on the null hypersurface $H = \{\gamma \in \mathbb{L}(\mathbb{R}^3) |\ \gamma \parallel P_0\}$, with P_0 is a fixed plane in $\mathbb{R}^3$. In [7], Guilfoyle presented an inversion formula describing v on $\mathbb{L}(\mathbb{R}^3)$, using Fritz John's inversion formula (cf. [9]). It is then natural to ask whether an arbitrary real function defined on a null hypersurface can be uniquely extended to a harmonic function on $\mathbb{L}(M^3)$ with respect to the neutral metric, for any three-dimensional real space form M^3.

In this article, we study null hypersurfaces with respect to the neutral metric g_- of an Einstein four-dimensional manifold (M, g_+) endowed with an almost paracomplex structure P that is parallel and isometric so that $g_- = g(P_+., .)$.

Our first result deals with totally geodesic null hypersurfaces. In particular, we have the following:

Theorem 1. *Every totally geodesic null hypersurface is scalar flat. If M admits a totally geodesic null hypersurface then (M, g_+) is Ricci-flat.*

Let N be the unit normal vector field, with respect to the Riemannian Einstein metric g_+ along a null hypersurface. The principal curvature corresponding to the principal direction PN is zero. The other two principal curvatures are called *nontrivial*. The next result provides a necessary condition for the existence of null hypersurfaces with equal nontrivial principal curvatures.

Theorem 2. *Suppose (M, g) has nonnegative scalar curvature and Σ is a null hypersurface with equal nontrivial principal curvatures. Then, g is Ricci-flat and Σ is totally geodesic.*

Finally, we study (non-minimal) null hypersurfaces having constant mean curvature (CMC). In particular, we prove the following:

Theorem 3. *Let Σ be a CMC, non-minimal null hypersurface in (M, g). Then, all principal curvatures and the scalar curvature of Σ are constant. Furthermore, the scalar curvature of g is given by:*

$$\bar{R} = -8\lambda_1\lambda_2,$$

where λ_1, λ_2, denote the nontrivial principal curvatures of Σ.

2. Preliminaries

Let (M, g) be an Einstein 4-manifold endowed with a product structure P (specifically a type (1,1) tensor field with $P^2 = \mathrm{Id}$) such that:

1. The eigenbundles corresponding to the eigenvalues $+1$ and -1 have equal rank.
2. P is an isometry, that is,

$$g(P., P.) = g(., .).$$

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3. P is parallel, that is,

$$\overline{\nabla} P = 0,$$

where $\overline{\nabla}$ is the Levi-Civita connection of g.

In other words, P is an almost paracomplex structure that is parallel and isometric.

Define the metric g_- by:

$$g_- = g(P.,.),$$

and denote g by g_+. Then, g_- is of neutral signature, locally conformally flat and scalar flat [5]. Also, both metrics g_+ and g_- share the same Levi-Civita connection $\overline{\nabla}$ (see [2] for further details).

Let Σ^3 be an oriented hypersurface of M and consider the normal bundles:

$$\mathcal{N}_\pm(\Sigma) = \{\xi \in TM \mid g_\pm(X, \xi) = 0,\ \forall X \in T\Sigma\}.$$

Let $N_\pm$ be the normal vector of Σ with respect to $g_\pm$ so that

$$g_\pm(N_\pm, N_\pm) = \epsilon_\pm \in \{-1, 0, 1\},$$

(note that $\epsilon_+ = 1$) and define the functions $C_\pm$ on Σ according to

$$C_+ = g_+(PN_+, N_+) = g_-(N_+, N_+),$$

and

$$C_- = g_-(PN_-, N_-) = g_+(N_-, N_-).$$

Consider the tangential vector field along Σ:

$$X = PN_+ - C_+N_+.$$

Let ∇ be the Levi-Civita connection of g_+ induced on Σ. For a tangential vector field Y along Σ, we have

$$\begin{aligned} g_+(\nabla C_+, Y) &= \nabla_Y C_+ \\ &= 2\, g_+(\overline{\nabla}_Y N_+, X) \\ &= g_+(Y, -2A_+X), \end{aligned}$$

showing that

$$\nabla C_+ = -2A_+X, \tag{2.1}$$

where $A_\pm$ denotes the shape operator of Σ immersed in $(M, g_\pm)$.

Also,

$$\nabla_Y X = -P^T A_+ Y + C_+ A_+ Y, \tag{2.2}$$

where P^T stands for the orthogonal projection of P on Σ. Let $R_\pm$, $H_\pm$, and $\sigma_\pm$ be, respectively, the scalar curvature, the mean curvature, and the second fundamental form of Σ immersed in $(M, g_\pm)$.

Proposition 1. *The Hessian of C_+ is*

$$\nabla^2 C_+(u, v) = -2(\nabla_u \sigma_+)(X, v) - 2C_+ g_+(A_+u, A_+v) + 2g_+(PA_+u, A_+v). \tag{2.3}$$

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Proof. In this proof, we omit the subscript $+$ unless it is necessary.
Using (2.1) on the tangential vector fields u, v, we have

$$\begin{aligned}\nabla^2 C(u,v) &= g(\nabla_u(-2AX), v)\\ &= -2g(\nabla_u AX, v)\\ &= -2\nabla_u(g(AX, v)) + 2g(AX, \nabla_u v)\\ &= -2\nabla_u(g(X, Av)) + 2g(AX, \nabla_u v)\\ &= -2g(\nabla_u X, Av) - 2g(X, \nabla_u Av) + 2g(AX, \nabla_u v)\\ &= -2g(\epsilon CAu - P^T Au, Av) - 2g(X, \nabla_u Av) + 2g(AX, \nabla_u v)\\ &= -2\epsilon Cg(Au, Av) + 2G(PAu, Av) - 2g(X, \nabla_u Av) + 2g(AX, \nabla_u v)\end{aligned}$$

Note that $\sigma(u, v) = g(Au, v)$ and for simplicity use $\nabla_u\sigma(X, v)$ to denote $(\nabla_u\sigma)(X, v)$. We now have

$$\begin{aligned}\nabla_u\sigma(X,v) &= u(\sigma(X,v)) - \sigma(\nabla_u X, v) - \sigma(X, \nabla_u v)\\ &= u(G(X, Av)) - g(\nabla_u X, Av) - g(AX, \nabla_u v)\\ &= g(\nabla_u X, Av) + g(X, \nabla_u Av) - g(\nabla_u X, Av) - g(AX, \nabla_u v)\\ &= g(X, \nabla_u Av) - g(AX, \nabla_u v),\end{aligned}$$

and therefore,

$$\nabla^2 C(u,v) = -2\epsilon Cg(Au, Av) + 2g(PAu, Av) - 2\nabla_u\sigma(X, v).$$

□

Proposition 2. *If Δ denotes the Laplacian of the metric g_+ induced on the hypersurface Σ, then*

$$\Delta C_+ = -6\, g_+(X_+, \nabla H_+) - 2C_+|\sigma_+|^2 + 2\, Tr(P^T A_+^2),$$

where H_+ denotes the mean curvature and A_+ is the shape operator.

Proof. In the proof, we omit the subscript $+$ unless it is necessary. The Codazzi–Mainardi equation for Σ is

$$g(R(u,v)z, N) = (\nabla_u\sigma)(v,z) - (\nabla_v\sigma)(u,z).$$

Consider the orthonormal frame (e_1, e_2, e_3) of Σ, where $Ae_i = \lambda_i e_i$. The fact that g is Einstein gives

$$\begin{aligned}\sum_{i=1}^{3}\big((\nabla_{e_i}\sigma)(X, e_i) - (\nabla_X\sigma)(e_i, e_i)\big) &= \sum_{i=1}^{3} g(R(e_i, X)e_i, N)\\ &= \sum_{i=1}^{3} g(R(e_i, X)e_i, N) + g(R(N, X)N, N)\\ &= \overline{\mathrm{Ric}}(X, N)\\ &= \tfrac{\bar R}{4}\, g(X, N)\\ &= 0.\end{aligned}$$

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Thus,

$$\begin{aligned}\sum_{i=1}^{3}(\nabla_{e_i}\sigma)(X,e_i) &= \sum_{i=1}^{3}(\nabla_X\sigma)(e_i,e_i)\\ &= \sum_{i=1}^{3}\nabla_X(\sigma(e_i,e_i)) - \sigma(\nabla_X e_i,e_i) - \sigma(e_i,\nabla_X e_i)\\ &= 3\nabla_X H - 2\sum_{i=1}^{3} g(\nabla_X e_i, Ae_i)\\ &= 3g(X,\nabla H) - 2\sum_{i=1}^{3}\lambda_i g(\nabla_X e_i,e_i)\\ &= 3g(X,\nabla H).\end{aligned}$$

We now have

$$\begin{aligned}\Delta C &= \sum_{i=1}^{3}\nabla^2 C(e_i,e_i)\\ &= -2\sum_{i=1}^{3}\big((\nabla_{e_i}\sigma)(X,e_i) + Cg(Ae_i,Ae_i) - g(PAe_i,Ae_i)\big)\\ &= -6g(X,\nabla H) - 2\sum_{i=1}^{3}\big(\lambda_i^2 C - \lambda_i^2 g(Pe_i,e_i)\big),\end{aligned}$$

and this completes the proof. □

Let R, R_{ij}, R_{ijkl} be, respectively, the scalar curvature, the Ricci tensor, and the curvature tensor of the metric g_+ induced on Σ and let $\bar{R}, \bar{R}_{ij}, \bar{R}_{ijkl}$ be, respectively, the scalar curvature, the Ricci tensor, and the curvature of the ambient metric g_+.

Using the Gauss equation, we get (for simplicity, we omit the subscript $+$):

$$\begin{aligned}R &= g^{ij}R_{ij}\\ &= g^{ij}g^{kl}(\bar{R}_{kilj} + \sigma_{ij}\sigma_{kl} - \sigma_{il}\sigma_{kj})\\ &= g^{ij}g^{kl}\bar{R}_{kilj} + 9H^2 - |\sigma|^2.\end{aligned}$$

The fact the g_+ is Einstein implies

$$\begin{aligned}g^{ij}g^{kl}\bar{R}_{kilj} &= g^{ij}\bar{R}_{ij} - g^{NN}\overline{\mathrm{Ric}}(NN)\\ &= (\bar{R}_+ - g^{NN}\overline{\mathrm{Ric}}(NN)) - g^{NN}\overline{\mathrm{Ric}}(NN)\\ &= \bar{R}_+ - 2\overline{\mathrm{Ric}}(NN)\\ &= \bar{R}_+ - 2(\bar{R}_+/4)g_+(N,N)\\ &= \bar{R}_+/2.\end{aligned}$$

We then have

$$R_+ = \tfrac{1}{2}\bar{R}_+ + 9H_+^2 - |\sigma_+|^2. \tag{2.4}$$

We then have

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Proposition 3. *Assume (M, g_+) has positive (resp. negative) scalar curvature and Σ is a totally geodesic hypersurface. Then the metric g_+ induced on Σ has positive (resp. negative) scalar curvature.*

3. Null hypersurfaces

Definition 1. *A null hypersurface in a pseudo-Riemannian manifold is an oriented hypersurface where the induced metric is indefinite and the normal vector field is null.*

In this section, when we refer to a null hypersurface we simply mean a hypersurface that is null with respect to the neutral metric of g_-.

Proposition 4. *Suppose Σ is an oriented hypersurface of M. Then, the following statements hold:*

1. $|C_+| \leq 1$, *and* $C_- > 0$.
2. $C_+ = 0$, *if and only if Σ is a null hypersurface.*
3. *If Σ is a null hypersurface, then PN_+ is a principal direction with zero corresponding principal curvature.*

Proof.

1. It is not hard to confirm that $|X|^2 = 1 - (C_+)^2 \geq 0$. Also,
$$C_- = g_+(N_-, N_-) > 0.$$
2. Assuming $C_+ = 0$, we have that $g_+(PN_+, N_+) = 0$ and using the fact that g_+ is Riemannian then, $PN_+ \in T\Sigma$. This implies
$$g_-(PN_+, N_-) = 0,$$
or,
$$g_+(N_+, N_-) = 0.$$
But this tells us that $N_- \in T\Sigma$, and therefore
$$g_-(N_-, N_-) = 0,$$
which means that Σ is null. Conversely, assume that Σ is null and consider the nonzero normal vector field N_-. Then, $g_-(N_-, N_-) = 0$. On the other hand, $g_-(N_-, T\Sigma) = 0$, which means $g_+(PN_-, T\Sigma) = 0$. Therefore, $PN_- = \lambda N_+$, where $\lambda \neq 0$, since N_- is nonzero vector field. Thus,
$$\begin{aligned} C_+ &= g_-(N_+, N_+) \\ &= \lambda^{-2} g_-(N_-, N_-) \\ &= 0, \end{aligned}$$
and this completes the proof.
3. Since Σ is null then $C_+ = 0$ and therefore,
$$X_+ = PN_+ - C_+N_+ = PN_+ \in T\Sigma.$$
Note that
$$0 = \nabla C_+ = -2A_+X_+,$$
which implies
$$A_+PN_+ = 0,$$
and therefore PN_+ is a principal direction. □

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For a null hypersurface Σ, we study the geometric properties of the metric g_+ induced on Σ and for this reason we omit the $+$ subscripts unless it is necessary.

3.1. Examples of null hypersurfaces

Example 3.1. *We now describe the almost paracomplex structure defined in the spaces of oriented geodesics of 3-manifolds of constant curvature using their (para) Kähler structures (see [1, 6, 8, 10] for more details).*

For $p \in \{0, 1, 2, 3\}$, consider the (pseudo-) Euclidean 4-space $\mathbb{R}^4_p := (\mathbb{R}^4, \langle .,. \rangle_p)$, where

$$\langle .,. \rangle_p = -\sum_{i=1}^{p} dX_i^2 + \sum_{i=p+1}^{4} dX_i^2,$$

and let $\mathbb{S}^3_p$ be the quadric

$$\mathbb{S}^3_p = \{x \in \mathbb{R}^4 \,|\, \langle x, x \rangle_p = 1\}.$$

The quadric $\mathbb{S}^3_0$ is the 3-sphere $\mathbb{S}^3$, $\mathbb{S}^3_3 \cap \{x \in \mathbb{R}^4 \,|\, X_4 > 0\}$ is anti-isometric to the hyperbolic 3-space $\mathbb{H}^3$, $\mathbb{S}^3_1$ is the de Sitter 3-space $d\mathbb{S}^3$, and $\mathbb{S}^3_2$ is anti-isometric to the anti-de Sitter 3-space $Ad\mathbb{S}^3$.

Let g_p be the metric $\langle .,. \rangle_p$ induced on $\mathbb{S}^3_p$ by the inclusion map. The space of oriented geodesics in $\mathbb{S}^3_p$ is a four-dimensional manifold and is identified with the following Grasmmannian spaces of oriented planes on $\mathbb{R}^4_p$:

$$\mathbb{L}^{\pm}(\mathbb{S}^3_p) = \{x \wedge y \in \Lambda^2(\mathbb{R}^4_p) \,|\, y \in T_x\mathbb{S}^3_p,\ g_p(y, y) = \pm 1\}.$$

Let $\iota : \mathbb{L}^{\pm}(\mathbb{S}^3_p) \to \Lambda^2(\mathbb{R}^4_p)$ be the inclusion map and $\langle\langle , \rangle\rangle_p$ be the flat metric in the 6-manifold $\Lambda^2(\mathbb{R}^4_p)$ defined by:

$$\langle\langle u_1 \wedge v_1, u_2 \wedge v_2 \rangle\rangle_p := \langle u_1, u_2 \rangle_p \langle v_1, v_2 \rangle_p - \langle u_1, v_2 \rangle_p \langle u_2, v_1 \rangle_p .$$

The metric $G_p = \iota^* \langle\langle , \rangle\rangle_p$ on $\mathbb{L}^{\pm}(\mathbb{S}^3_p)$ is Einstein [2].

It was shown in [5], that the Hodge star operator $*$ on the space of bivectors $\Lambda^2(\mathbb{R}^4_p)$ in $\mathbb{R}^4_p$, restricted to the space of oriented geodesics $\mathbb{L}^{\pm}(\mathbb{S}^3_p)$ defines an almost paracomplex structure $\mathbb{J}^*$ that is parallel and isometric with respect to the Einstein metric G_p. In particular, for $x \wedge y \in \mathbb{L}^{\pm}(\mathbb{S}^3_p)$, the almost paracomplex structure is defined by:

$$\mathbb{J}^*_{x\wedge y} = *|_{T_{x\wedge y}\mathbb{L}^{\pm}(\mathbb{S}^3_p)} .$$

The metric $G'_p := G_p(\mathbb{J}^*.,.)$, is of neutral signature, locally conformally flat and scalar flat in $\mathbb{L}^{\pm}(\mathbb{S}^3_p)$.

Let $\phi : S \to \mathbb{S}^3_p$ be a non-totally geodesic smooth surface and (e_1, e_2) be the principal directions of ϕ with corresponding eigenvalues κ_1 and κ_2. Then,

$$\Phi : S \times \mathbb{S}^1 \to \mathbb{L}(\mathbb{S}^3_p) : (x, \theta) \mapsto \phi(x) \wedge (\cos\theta\, e_1(x) + \sin\theta\, e_2(x)),$$

is the immersion of the tangential congruence $\Sigma = \Phi(S \times \mathbb{S}^1)$ in the space of oriented geodesics $\mathbb{L}(\mathbb{S}^3_p)$. It can be shown that if ϕ is a totally geodesic immersion, the mapping Φ is not an immersion. Also, Σ is a null hypersurface with respect to the locally conformally flat neutral metric g_- [6].

The eigenvalues of the tangential hypersurface Σ are $0, \lambda_+$ and λ_-, where

$$\lambda_+ = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \qquad \lambda_- = -\kappa_1 \sin^2\theta - \kappa_2 \cos^2\theta,$$

and therefore the mean curvature is

$$H = \tfrac{1}{3}(\kappa_1 - \kappa_2)\cos 2\theta.$$

This yields

Proposition 5. *If S is a totally umbilic surface in the non-flat three-dimensional real space form, then the corresponding tangential congruence Σ is a null hypersurface in $(\mathbb{L}(\mathbb{S}^3_p), G'_p)$ and is minimal in $(\mathbb{L}(\mathbb{S}^3_p), G_p)$.*

Example 3.2. *Consider the Cartesian product of the 2-spheres $\mathbb{S}^2\times\mathbb{S}^2$ endowed with the product metric:*

$$g_+ = g\oplus g,$$

where g is the round metric of $\mathbb{S}^2$. It is well known that g_+ is Einstein with scalar curvature $R=4$.

Define the almost paracomplex structure P on $\mathbb{S}^2\times\mathbb{S}^2$ by:

$$P(u,v)=(u,-v),$$

where $(u,v)\in T(\mathbb{S}^2\times\mathbb{S}^2)$. Then, P is G^+-parallel and isometric. For $t\in(-1,1)$, consider the homogeneous hypersurfaces:

$$\Sigma_t=\{(x,y)\in\mathbb{S}^2\times\mathbb{S}^2\subset\mathbb{R}^3\times\mathbb{R}^3)\mid\langle x,y\rangle=t\}.$$

In fact, Σ_t is a tube of radius $\cos^{-1}(t/\sqrt{2})$ over the diagonal surface $\Delta=\{(x,x)\in\mathbb{S}^2\times\mathbb{S}^2\}$. It was shown in [11] that Σ_t is null for every t with respect to the neutral metric:

$$g_- = g_+(P.,.)=g\oplus(-g)$$

and the principal curvatures are

$$\lambda_1=\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{1+t}{1-t}},\qquad \lambda_2=-\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{1-t}{1+t}},\qquad \lambda_3=0.$$

Thus, Σ_t is a CMC null hypersurface for any $t\in(-1,1)$ and is minimal only when $t=0$ as the mean curvature H is

$$H=\tfrac{1}{3\sqrt{2}}\left(\sqrt{\tfrac{1+t}{1-t}}-\sqrt{\tfrac{1-t}{1+t}}\right).$$

Similarly, we have the following example.

Example 3.3. *Consider the Cartesian product of the 2-spheres $\mathbb{H}^2\times\mathbb{H}^2$ endowed with the product metric:*

$$g_+ = g\oplus g,$$

where g is the standard hyperbolic metric of $\mathbb{H}^2$. It is not hard for one to see that g_+ is Einstein with scalar curvature $R=-4$. As before, the almost paracomplex structure P on $\mathbb{H}^2\times\mathbb{H}^2$ is given by:

$$P(u,v)=(u,-v),$$

where $(u,v)\in T(\mathbb{H}^2\times\mathbb{H}^2)$. Again, P is g_+-parallel and isometric and for $t\in(-\infty,-1)$, consider the homogeneous hypersurfaces:

$$\Sigma_t=\{(x,y)\in\mathbb{H}^2\times\mathbb{H}^2\subset\mathbb{R}^3\times\mathbb{R}^3)\mid\langle x,y\rangle=t\}.$$

In fact, Σ_t is a tube of radius $\cosh^{-1}(t/\sqrt{2})$ over the diagonal surface $\Delta=\{(x,x)\in\mathbb{H}^2\times\mathbb{H}^2\}$. It was shown in [4] that Σ_t is null for every t with respect to the neutral metric:

$$g_- = g_+(P.,.)=g\oplus(-g)$$

and the principal curvatures are

$$\lambda_1=\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{1+t}{1-t}},\qquad \lambda_2=\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{1-t}{1+t}},\qquad \lambda_3=0.$$

Thus, Σ_t is a CMC, non-minimal null hypersurface for any $t\in(-1,1)$ with mean curvature:

$$H=\tfrac{1}{3\sqrt{2}}\left(\sqrt{\tfrac{1+t}{1-t}}+\sqrt{\tfrac{1-t}{1+t}}\right).$$

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3.2. Main results

Consider the principal orthonormal frame $(e_1, e_2, e_3 = PN)$ of the null hypersurface Σ so that

$$Ae_i = \lambda_i e_i.$$

It is easily shown that there is an angle $\theta \in [0, 2\pi)$ such that

$$Pe_1 = \cos\theta e_1 + \sin\theta e_2 \qquad Pe_2 = \sin\theta e_1 - \cos\theta e_2.$$

We call the angle θ *the principal angle* of the null hypersurface Σ.

We now have the following result for totally geodesic null hypersurfaces:

Theorem 1. *Every totally geodesic null hypersurface is scalar flat. If M admits a totally geodesic null hypersurface, then (M, g_+) is Ricci-flat.*

Proof. Let $\{e_1, e_2.e_3\}$ be an orthonormal frame of Σ such that

$$Ae_i = \lambda_i e_i,$$

where $e_3 = PN$ and therefore, $\lambda_3 = 0$. The almost paracomplex structure P is

$$P = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ \sin\theta & -\cos\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

with respect to the orthonormal frame (e_1, e_2, e_3, N).

Let $\overline{\nabla}, \nabla$ be the Levi-Civita connections for the metrics g and the induced metric of g on Σ, respectively. For $i, j = 1, 2, 3$, we have

$$\overline{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + \lambda_i \delta_{ij} N,$$

and if we let $\omega_{ij}^k = g(\nabla_{e_i} e_j, e_k)$ then

$$\omega_{ij}^k = -\omega_{ik}^j.$$

Defining

$$k = \omega_{11}^2, \qquad \mu = \omega_{21}^2, \qquad \nu = \omega_{31}^2. \tag{3.1}$$

A brief calculation gives

$$g(R(e_2, e_1)e_1, e_2) = -e_1(\mu) + e_2(k) + \lambda_1\lambda_2 - k^2 - \mu^2 + \nu(\lambda_1 - \lambda_2)\sin\theta.$$

$$g(R(e_3, e_1)e_1, e_3) = -\lambda_1\nu\sin\theta - \lambda_2\nu\sin\theta + e_3(\lambda_1\cos\theta) - \lambda_1^2\cos^2\theta - \lambda_1\lambda_2\sin^2\theta.$$

$$g(R(e_3, e_2)e_2, e_3) = \lambda_1\nu\sin\theta + \lambda_2\nu\sin\theta - e_3(\lambda_2\cos\theta) - \lambda_2^2\cos^2\theta - \lambda_1\lambda_2\sin^2\theta.$$

Therefore, we deduce

$$\mathrm{Ric}(e_1, e_1) = -e_1(\mu) + e_2(k) + e_3(\lambda_1\cos\theta) + \lambda_1\lambda_2\cos^2\theta - \lambda_1^2\cos^2\theta - k^2 - \mu^2 - 2\nu\lambda_2\sin\theta.$$

$$\mathrm{Ric}(e_2, e_2) = -e_1(\mu) + e_2(k) - e_3(\lambda_2\cos\theta) + \lambda_1\lambda_2\cos^2\theta - \lambda_2^2\cos^2\theta - k^2 - \mu^2 + 2\nu\lambda_1\sin\theta.$$

$$\mathrm{Ric}(e_3, e_3) = e_3[(\lambda_1 - \lambda_2)\cos\theta] - 2\lambda_1\lambda_2\sin^2\theta - (\lambda_1^2 + \lambda_2^2)\cos^2\theta.$$

The scalar curvature R of Σ is

$$\begin{aligned} R = {} & -2e_1(\mu) + 2e_2(k) + 2\lambda_1\lambda_2\cos 2\theta - 2(\lambda_1^2 + \lambda_2^2)\cos^2\theta \\ & + 2e_3[(\lambda_1 - \lambda_2)\cos\theta] - 2k^2 - 2\mu^2 + 2\nu(\lambda_1 - \lambda_2)\sin\theta. \end{aligned} \tag{3.2}$$

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Using the fact that P is parallel, namely

$$P\overline{\nabla}_{e_i}e_j = \overline{\nabla}_{e_i}Pe_j,$$

we have

$$\begin{aligned}
\omega_{12}^3 &= \lambda_1 \sin\theta, & \omega_{11}^3 &= \lambda_1 \cos\theta, & \omega_{12}^1 &= e_1(\theta/2),\\
\omega_{21}^3 &= \lambda_2 \sin\theta, & \omega_{22}^3 &= -\lambda_2 \cos\theta, & \omega_{22}^1 &= e_2(\theta/2),\\
\omega_{13}^1 &= -\lambda_1 \cos\theta & \omega_{13}^2 &= -\lambda_1 \sin\theta & \omega_{23}^1 &= -\lambda_2 \sin\theta,\\
\omega_{31}^2 &= -e_3(\theta/2), & \omega_{31}^3 &= \omega_{32}^3 = 0,
\end{aligned} \tag{3.3}$$

and thus,

$$\begin{aligned}
\nabla_{e_1}e_1 &= -e_1(\theta/2)e_2 + \lambda_1 \cos\theta e_3 & \nabla_{e_1}e_2 &= e_1(\theta/2)e_1 + \lambda_1 \sin\theta e_3\\
\nabla_{e_1}e_3 &= -\lambda_1 \cos\theta e_1 - \lambda_1 \sin\theta e_2 & \nabla_{e_2}e_1 &= -e_2(\theta/2)e_2 + \lambda_2 \sin\theta e_3\\
\nabla_{e_2}e_2 &= e_2(\theta/2)e_1 - \lambda_2 \cos\theta e_3 & \nabla_{e_2}e_3 &= -\lambda_2 \sin\theta e_1 + \lambda_2 \cos\theta e_2
\end{aligned}$$

$$\nabla_{e_3}e_1 = -e_3(\theta/2)e_2 \qquad \nabla_{e_3}e_2 = e_3(\theta/2)e_1 \qquad \nabla_{e_3}e_3 = 0.$$

The relations (3.1) and (3.3) yield

$$\mu = -e_2(\theta/2), \qquad k = -e_1(\theta/2),$$

and therefore,

$$-e_1(\mu) + e_2(k) = [e_1, e_2](\theta/2).$$

On the other hand,

$$\begin{aligned}
[e_1, e_2] &= e_1(\theta/2)e_1 + \lambda_1 \sin\theta e_3 - (-e_2(\theta/2)e_2 + \lambda_2 \sin\theta e_3)\\
&= e_1(\theta/2)e_1 + e_2(\theta/2)e_2 + (\lambda_1 - \lambda_2)\sin\theta\, e_3.
\end{aligned}$$

Thus,

$$\begin{aligned}
-e_1(\mu) + e_2(k) &= [e_1, e_2](\theta/2)\\
&= e_1(\theta/2)e_1(\theta/2) + e_2(\theta/2)e_2(\theta/2) + (\lambda_1 - \lambda_2)\sin\theta\, e_3(\theta/2)\\
&= k^2 + \mu^2 - \nu(\lambda_1 - \lambda_2)\sin\theta
\end{aligned}$$

The scalar curvature given in (3.2) now becomes

$$R = 2e_3[(\lambda_1 - \lambda_2)\cos\theta] + 2\lambda_1\lambda_2 \cos 2\theta - 2(\lambda_1^2 + \lambda_2^2)\cos^2\theta. \tag{3.4}$$

Assuming that Σ is totally geodesic, we can see easily that $R = 0$. In this case, the Gauss equation implies also that (M, g) is scalar flat since

$$\frac{\bar{R}}{2} = R - 9H^2 + |\sigma|^2 = 0.$$

The Ricci flatness of (M, g) follows from the fact g is Einstein. □

If Σ is a null hypersurface, the principal curvature corresponding to the principal direction PN will be called *trivial*. The following theorem explores null hypersurfaces where the nontrivial eigenvalues are equal.

Theorem 2. *Suppose (M, g) has nonnegative scalar curvature and Σ is a null hypersurface with equal nontrivial principal curvatures. Then, g is Ricci-flat and Σ is totally geodesic.*

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Proof. Using the scalar curvature R in (3.4), the Gauss equation for Σ becomes

$$\frac{\bar{R}}{2} + (\lambda_1 + \lambda_2)^2 = -(\lambda_1 - \lambda_2)^2 \cos 2\theta.$$

Since $\lambda_1 = \lambda_2$, we have

$$\frac{\bar{R}}{2} + (\lambda_1 + \lambda_2)^2 = 0,$$

implying $\bar{R} = 0$ and $\lambda_1 + \lambda_2 = 0$. This means that $\lambda_1 = \lambda_2 = 0$ and thus, Σ is totally null. □

We now have the following theorem about CMC null hypersurfaces:

Theorem 3. *Let Σ be a CMC, non-minimal null hypersurface in (M, g). Then, all principal curvatures and the scalar curvature of Σ are constant. Furthermore, the scalar curvature of g is given by:*

$$\bar{R} = -8\lambda_1\lambda_2, \tag{3.5}$$

where λ_1, λ_2, denote the nontrivial principal curvatures of Σ.

Proof. We recall the principal orthonormal frame $\{e_1, e_2.e_3 = PN\}$ of the null hypersurface Σ. The Laplacian of the function C with respect to the induced metric is

$$\Delta C = -6\, g(X, \nabla H) - 2C|\sigma|^2 + 2\mathrm{Tr}(P^T A^2).$$

Since $C = 0$ and $\nabla H = 0$, we have

$$\mathrm{Tr}(P^T A^2) = 0,$$

which ensures

$$\sum_{i=1}^{3} g(PA^2 e_i, e_i) = 0.$$

It follows

$$\sum_{i=1}^{2} \lambda_i^2 g(Pe_i, e_i) = 0,$$

and therefore,

$$(\lambda_1^2 - \lambda_2^2)\cos\theta = 0.$$

Note that Σ is non-minimal and therefore, $\lambda_1 + \lambda_2 \neq 0$.

If $\lambda_1 = \lambda_2$, we have that $H = \frac{2}{3}\lambda_1$ is constant and considering the scalar curvature in (3.4), we find

$$\begin{aligned} \tfrac{1}{2}R &= \lambda_1^2 \cos 2\theta - 2\lambda_1^2 \cos^2\theta \\ &= -\lambda_1^2. \end{aligned}$$

Using the Gauss equation (2.4), we obtain

$$\begin{aligned} -2\lambda_1^2 &= R \\ &= \tfrac{1}{2}\bar{R} + 9H^2 - |\sigma|^2 \\ &= \tfrac{1}{2}\bar{R} + (2\lambda_1)^2 - 2\lambda_1^2, \end{aligned}$$

which implies that $\bar{R} = -8\lambda_1^2$.

If $\cos\theta = 0$, then either $\theta = \pi/2$ or $\theta = 3\pi/2$. The scalar curvature of Σ given (3.4) becomes

$$R = -2\lambda_1\lambda_2.$$

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On the other hand, the scalar curvature in (2.4) yields

$$\begin{aligned}-2\lambda_1\lambda_2 &= R \\ &= \tfrac{1}{2}\bar{R} + 9H^2 - |\sigma|^2 \\ &= \tfrac{1}{2}\bar{R} + (\lambda_1+\lambda_2)^2 - \lambda_1^2 - \lambda_2^2,\end{aligned}$$

and therefore, $\bar{R} = -8\lambda_1\lambda_2$. Note that $\bar{R}$ is constant and as such $\lambda_1\lambda_2$ is constant. However, $\lambda_1+\lambda_2$ is also constant and thus both λ_1 and λ_2 are constant.

All principal curvatures are constant, and therefore the Gauss equation, given in (2.4), tells us that the scalar curvature R must also be constant. □

Theorem 5 can no longer be extended to minimal null hypersurfaces, since the relation (3.5) does not necessarily hold. To see this, consider the minimal, null hypersurfaces $M_{a,b} \subset \mathbb{S}^2 \times \mathbb{S}^2$, for $a, b \in \mathbb{S}^2 \subset \mathbb{R}^3$:

$$M_{a,b} = \{(x,y) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle x,a\rangle + \langle y,b\rangle = 0\}.$$

In [11], Urbano showed that the principal curvatures are nonconstant and in particular, if $(x,y) \in M_{a,b}$ then:

$$\lambda_1(x,y) = \frac{\langle x,a\rangle}{\sqrt{2(1-\langle x,a\rangle^2)}}, \qquad \lambda_2(x,y) = -\frac{\langle x,a\rangle}{\sqrt{2(1-\langle x,a\rangle^2)}}, \qquad \lambda_3(x,y) = 0.$$

As such

$$-8\lambda_1\lambda_2 = \frac{4\langle x,a\rangle^2}{1-\langle x,a\rangle^2} \neq 4 = \bar{R}.$$

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